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METRICALLY REGULAR SQUARE OF METRICALLY REGULAR BIGRAPHS I.

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ABSTRACT. The present paper deals with the spectra of powers of metrically regular graphs. A necessary condition for G to have the square G^2 metrically regular is found and the problem of the construction such graphs G is solved for metrically regular bipartite graphs with 4 and 5 distinct eigenvalues (these eigenvalues can have various multiplicities)

1. INTRODUCTION AND NOTATION

The theory of metrically regular graphs originates from the theory of association schemes first introduced by R.C. Bose and Shimamoto [2]. All graphs will be undirected, without loops and multiple edges.

1.1. Definition [1]. Let X be a finite set, $n := |X| \geq 2$. For an arbitrary natural number D let $R = \{R_0, R_1, \ldots, R_D\}$ be a system of binary relations on X. A pair (X, R) will called an association scheme with n classes if and only if it satisfies the axioms $A1 - A4$:

A1. The system R forms a partition of the set X^2 and R_0 is the diagonal relation, i.e. $R_0 = \{(x,x); x \in X\}$.

A2. For each $i \in \{0, 1, \ldots, D\}$ it holds $R_i^{-1} \in R$.

A3. For each $i, j, k \in \{0, 1, \ldots, D\}$ it holds

$$(x,y) \in R_k \wedge (x_1, y_1) \in R_k \Rightarrow p_{ij}(x,y) = p_{ij}(x_1, y_1),$$

where $p_{ij}(x,y) = |\{z; (x,z) \in R_i \wedge (z,y) \in R_j\}|$.

Then define $p_{ij}^k := p_{ij}(x,y)$ where $(x,y) \in R_k$.

A4. For each $i, j, k \in \{0, 1, \ldots, D\}$ it holds $p_{ij}^k = p_{ji}^k$.

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The set X will be called the carrier of the association scheme (X, R). Especially, $p_{i0}^k = \delta_{ik}$, $p_{ij}^0 = v_i\delta_{ij}$, where δ_{ij} is the Kronecker-Symbol and $v_i := p_{ii}^0$, and define $P_j := (p_{ij}^k)$, $0 \le i, j, k \le D$.

Given a graph $G = (X, E)$ of diameter D we may define $R_k = \{(x, y); d(x, y) = k\}$, where $\mathrm{d}(x, y)$ is the distance from the vertex x to the vertex y in the standard graph metric. If (X, R), $R = \{R_0, R_1, \dots, R_D\}$, gives rise to an association scheme, the graph is called metrically regular and the p_{ij}^k are said to be its parameters or its structural constants. Especially, metrically regular graphs with the diameter $D = 2$ are called strongly regular.

1.2. Definition. Let $G = (X, E)$ be an undirected graph without loops and multiple edges. The second power (or the square) of G is the graph $G^2 = (X, E)$ with the same vertex set X and in which different vertices are adjacent if and only if there is at least one path of the length 2 or 1 in G between them.

1.3. Definition. Let G be a graph with an adjacency matrix A. The characteristic polynomial $|\lambda I - A|$ of the adjacency matrix A is called the characteristic polynomial of G and denoted by $P_G(\lambda)$. The eigenvalues of A and the spectrum of A are called the eigenvalues and the spectrum of G, respectively. If $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ are the eigenvalues of G, the whole spectrum is denoted by $S_p(G)$ and λ_1 is called the index of G.

Define $(0,1)$-matrices $A_0, \dots, A_D$ by $A_0 = I$ and $(A_i)_{jk} = 1$ if and only if the distance from the vertex j to the vertex k in G is $\mathrm{d}(j, k) = i$. Using these notations it follows:

1.4. Theorem [4]. *For a metrically regular graph G with diameter D and any real numbers $r_1, \dots, r_D$ the distinct eigenvalues of $\sum_{i=1}^D r_i A_i$ and $\sum_{i=1}^D r_i P_i$ are the same. In particular the distinct eigenvalues of a metrically regular graph are the same as those of P_1.*

1.5. Theorem [9]. *For a graph G with an adjacency matrix A there exists a polynomial $P(x)$, such that $P(A) = J$, if and only if G is regular and connected. In this case we have*

$$P(x) = \frac{n(x - \lambda_2)\dots(x - \lambda_m)}{(r - \lambda_2)\dots(r - \lambda_m)}$$

where n is the number of vertices, r is the index of the graph G and $\lambda_1 = r$, $\lambda_2, \dots, \lambda_m$ are all distinct eigenvalues of G. ($J = (\varepsilon_{ij})$ with $\varepsilon_{ij} = 1$.)

1.6. Theorem [13]. *A metrically regular graph with a diameter D has $D + 1$ distinct eigenvalues.*

1.7. Theorem [4]. *Let $\lambda_1 = r, \lambda_2, \dots, \lambda_n$ be the spectrum of a graph G, r being the index of G. G is regular if and only if*

$$\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = r.$$

Then G is regular of the degree r.

1.8. Theorem [11]. *The number of components of a regular graph G is equal to the multiplicity of its index.*

1.9. Theorem [6, p.161]. *Let G be a regular connected graph with n vertices whose spectrum is $S_p(G)$ and whose set of distinct eigenvalues is T. Suppose $|T| \leq 4$. Then the following statements are equivalent:*
(i) *H is cospectral with G,*
(ii) *H is regular, connected, has n vertices, and has T for its set of distinct eigenvalues.*

1.10. Theorem [6, p.87]. *A graph containing at least one edge is bipatite if and only if its spectrum, considered as a set of points the real axis, is symmetric with respect to the zero point.*

1.11. Theorem [6, p.82]. *A strongly connected digraph G with the greates eigenvalue r has no odd cycles if and only if $-r$ is also an eigenvalue of G.*

1.12. Theorem [11]. *If G is a regular graph of degree r with n vertices, then for the complementary graph $\bar{G}$ it holds*

$$P_{\bar{G}}(\lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} P_G(-\lambda - 1)$$

i.e., if the spectrum of G contains $\lambda_1 = r$, $\lambda_2, \ldots, \lambda_n$, then the spectrum of $\bar{G}$ contains $n - 1 - r$, $\lambda_2 - 1, \ldots, \lambda_n - 1$.

1.13. Theorem [7]. *The spectrum of a graph G determines whether or not it is a regular connected line graph except of 17 cases. In these cases G has the spectrum of the line graph $L(H)$ of H where H is one of the 3-connected regular on 8-vertices or H is a connected semiregular bipartite graph on $6 + 3$ vertices.*

1.14. Theorem [7]. *The line graphs of the following 17 graphs are cospectral with an exceptional graph (a graph that is cospectral to a regular connected line graph but is not itself a line graph):*
(i) $K_{4,4}, K_{3,6}$.
(ii) *The coctail party graph* CP(4) *on 8 vertices.*
(iii) K_8.
(iv) $\bar{C}_8$.
(v) $\overline{C_m \cup C_n}$, $\{m, n\} = \{3, 5\}$, $\{4, 4\}$.
(vi)-a *H where H is regular, connected and cubic graph on 8 vertices (four graphs in all).*
(vi)-b *$\bar{H}$ where H is regular, connected graph on 8 vertices (five graphs in all).*
(vii) *The semiregular bipartite graph with the parameters* $(m, n, r_1, r_2) = = (6, 3, 2, 4)$.

1.15 Theorem [12]. *If G is a regular graph of degree r with n vertices and m $(=\frac{1}{2}nr)$ edges, then the following relation holds:*

$$P_{L(G)}(\lambda) = (\lambda+2)^{m-n} P_G(\lambda - r + 2). \tag{1.1}$$

A multigraph (i.e. multiple edges are allowed) G is called semiregular of degrees r_1, r_2 if
it is bipartite having a representation $G = (X_1, X_2, E)$ with $|X_i| = n_i$, $n_1+n_2 = n$, where each vertex $x \in X_i$ has valency r_i $(i = 1,2)$.

1.16. Theorem [5]. *Let G be a semiregular multigraph with $n_1 \geq n_2$. Then for the line graph of G the relation*

$$P_{L(G)}(\lambda) = (\lambda+2)^{\beta} \sqrt{\left(\frac{-\alpha_1}{\alpha_2}\right)^{n_1-n_2} P_G(\sqrt{\alpha_1\alpha_2}) P_G(-\sqrt{\alpha_1\alpha_2})}$$

holds where $\alpha_i = \lambda - r_i + 2$ $(i = 1,2)$, $\beta = n_1 r_1 - n_1 - n_2$.

1.17. Theorem [10]. *Let G, G' be connected graphs, $L(G) \cong L(G')$. Then $G \cong G'$ exceptly the case $G = K_3$, $G' = K_{1,3}$.*

2. Metrically regular graphs with 4 distinct eigenvalues

For metrically regular graphs with 4 distinct eigenvalues we get by Theorem 1.4.:

$$\begin{aligned}
|\lambda I - P_1| = \\
&= \lambda^4 - \lambda^3(p_{11}^1 + p_{12}^2 + p_{13}^3) + \\
&+ \lambda^2(-\lambda_1 + p_{11}^1 p_{13}^3 + p_{11}^1 p_{12}^2 + p_{12}^2 p_{13}^3 - p_{12}^3 p_{13}^2 - p_{11}^2 p_{12}^1) - \\
&- \lambda(p_{11}^1 p_{12}^2 p_{13}^3 - p_{11}^1 p_{12}^3 p_{13}^2 - p_{11}^2 p_{12}^1 p_{13}^3 - \lambda_1 p_{12}^2 - \lambda_1 p_{13}^3) + \\
&+ \lambda_1(p_{12}^3 p_{13}^2 - p_{12}^2 p_{13}^3).
\end{aligned}$$

By simple calculations we obtain

$$\lambda_2 + \lambda_3 + \lambda_4 = p_{11}^1 + p_{12}^2 + p_{13}^3 - \lambda_1 \tag{2.1}$$

$$\begin{aligned}
\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 &= -\lambda_1 + p_{11}^1(p_{13}^3 + p_{12}^2) + p_{12}^2 p_{13}^3 - p_{12}^3 p_{13}^2 - \\
&- p_{11}^2 p_{12}^1 - \lambda_1(p_{11}^1 + p_{12}^2 + p_{13}^3 - \lambda_1) = \\
&= -\lambda_1 + p_{11}^1(p_{13}^3 + p_{12}^2 - \lambda_1) + p_{12}^2(p_{13}^3 - \lambda_1) + \\
&+ p_{12}^3(\lambda_1 - p_{13}^2) - p_{11}^2(\lambda_1 - 1 - p_{11}^1) = \\
&= -\lambda_1 + p_{11}^1(p_{13}^3 - p_{13}^2) + p_{11}^2(1 - \lambda_1) + \\
&+ p_{12}^3(\lambda_1 - p_{13}^2 - p_{12}^2) = \\
&= -\lambda_1 + p_{11}^1(p_{13}^3 - p_{13}^2) + p_{11}^2(1 - \lambda_1 + p_{12}^3) = \\
&= -\lambda_1 + p_{11}^1(p_{13}^3 - p_{13}^2) + p_{11}^2(1 - p_{13}^3)
\end{aligned} \tag{2.2}$$

(2.3) $$\lambda_2\lambda_3\lambda_4 = p_{12}^3 p_{13}^2 - p_{12}^2 p_{13}^3$$

We use some of known relations from the theory of the association scheme [1]:

(2.4) $$v_i = \sum_j p_{ij}^k$$

(2.5) $$v_i v_j = \sum_{k=0}^{3} v_k p_{ij}^k$$

(2.6) $$v_i p_{jk}^i = v_j p_{ik}^j$$

2.1. Condition for G to have the square G^2 strongly regular

If A denotes the adjacency matrix of a metrically regular graph G and A_2 the adjacency matrix of G^2 it is easy to see that

(2.7) $$A_2 = \frac{1}{p_{11}^2}A^2 + \frac{p_{11}^2 - p_{11}^1}{p_{11}^2}A - \frac{\lambda_1}{p_{11}^2}I$$

So, if the eigenvalues of G are $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ with respective multiplicities $m_1 = 1, m_2, m_3, m_4$, the eigenvalues of G^2 are

$$\mu_i = \frac{\lambda_i^2 + (p_{11}^2 - p_{11}^1)\lambda_i - \lambda_1}{p_{11}^2}$$

with the same "formal" multiplicities as λ_i (for $i = 1, 2, 3, 4$), i.e. the multiplicity of μ_i is $\sum_{j \in M_i} m_j$ with $M_i = \{j : \mu_j = \mu_i\}$.

If μ_i is the index of G^2 it holds $\mu_i = v_1 + v_2$, where v_1, v_2 are the parameters of G. Because of $v_1 = \lambda_1$ from (2.6) ($i = 1$, $j = 2$, $k = 1$) we get $\lambda_1 p_{12}^1 = v_2 p_{11}^2$; thus we obtain ($p_{12}^1 = \lambda_1 - p_{11}^1 - 1$)

$$\mu_i = \lambda_1 + \frac{\lambda_1(\lambda_1 - p_{11}^1 - 1)}{p_{11}^2} = \frac{\lambda_1^2 + (p_{11}^2 - p_{11}^1)\lambda_1 - \lambda_1}{p_{11}^2} = \mu_1.$$

As the multiplicity of the index of a graph is equal to 1, the index of G^2 must be μ_1. Because of Theorem 1.6 G^2 has diameter 2. Hence, if G^2 is metrically regular then G^2 is strongly regular and because of Theorem 1.6 one of the following cases a), b), c) occurs

(2.8) a) $\mu_2 = \mu_3$, then $\lambda_2 + \lambda_3 = p_{11}^1 - p_{11}^2$

(2.9) b) $\mu_2 = \mu_4$; then $\lambda_2 + \lambda_4 = p_{11}^1 - p_{11}^2$

(2.10) c) $\mu_3 = \mu_4$; then $\lambda_3 + \lambda_4 = p_{11}^1 - p_{11}^2$

On the other hand if G^2 is strongly regular, the parameters of G^2 are

(2.11) $$^2p_{11}^1 = p_{11}^1 + 2p_{12}^1 + p_{22}^1 = p_{11}^2 + 2p_{12}^2 + p_{22}^2$$

(2.12) $$^2p_{12}^1 = p_{23}^1 = p_{13}^2 + p_{23}^2$$

(2.13) $$^2p_{22}^1 = p_{33}^1 = p_{33}^2$$

$$^2p_{11}^2 = 2p_{12}^3 + p_{22}^3, \; ^2p_{12}^2 = p_{13}^3 + p_{23}^3, \; ^2p_{22}^2 = p_{33}^3.$$

2.2. Lemma. *For metrically regular graphs with 4 distinct eigevalues the conditions (2.11), (2.12), (2.13) are equivalent.*

Proof. (2.11) $\Rightarrow$ (2.12). From (2.4) $i = 2$, $k = 1,2$ it follows

$$v_2 = p^1_{12} + p^1_{22} + p^1_{23} = 1 + p^2_{12} + p^2_{22} + p^2_{23}. \tag{2.14}$$

If we substitute (2.4) $i = 1$, $k = 1,2$ in (2.11) we get

$$\lambda_1 - 1 + p^1_{12} + p^1_{22} = \lambda_1 - p^2_{13} + p^2_{12} + p^2_{22}.$$

So from (2.14) we obtain (2.12).
(2.12) $\Rightarrow$ (2.11). From (2.14) and (2.4) $i = 1$, $k = 1,2$ we get

$$p^1_{11} + 2p^1_{12} + p^1_{22} = \lambda_1 - 1 + p^1_{12} + p^1_{22} =$$
$$= \lambda_1 + p^2_{12} + p^2_{22} - p^2_{13} = p^2_{11} + 2p^2_{12} + p^2_{22}.$$

(2.12) $\Leftrightarrow$ (2.13). From (2.4) $i = 3$, $k = 1,2$ it follows

$$v_3 = p^1_{23} + p^1_{33} = p^2_{13} + p^2_{23} + p^2_{33}$$

and equivalence is easy to see. □

2.3. Theorem. *Let G be a metrically regular graph with 4 distinct eigenvalues and G^2 be a strongly regular graph. Then condition (2.9) holds and $\lambda_3 = -1$, $\lambda_2 > 0$. The conditions (2.8) and (2.10) cannot set in.*

Proof. a) Substituting (2.8) in (2.1) we obtain from (2.4) $i = 1$, $k = 2$

$$\lambda_4 = p^2_{11} + p^2_{12} - p^3_{12} = p^3_{13} - p^2_{13} \tag{2.15}$$

The relations (2.2), (2.8) and (2.15) imply

$$\lambda_2\lambda_3 + (p^3_{13} - p^2_{13})(p^1_{11} - p^2_{11}) = p^1_{11}(p^3_{13} - p^2_{13}) + p^2_{11}(1 - p^3_{13}) - \lambda_1$$

and we obtain

$$\lambda_1 + \lambda_2\lambda_3 = p^2_{11}(1 - p^2_{13}). \tag{2.16}$$

By the substitution (2.15) and (2.16) in (2.3) we get

$$[p^2_{11}(1 - p^2_{13}) - \lambda_1][p^2_{11} + p^2_{12} - p^3_{12}] = p^3_{12}p^2_{13} - p^2_{12}p^3_{13}$$

and by a simple calculation using (2.15) and (2.4) ($i = 1$, $k = 3$) we obtain $-p^2_{11}p^2_{13}(p^2_{11} + p^2_{12} - p^3_{12} + 1) = 0$. As the diameter of G is $D = 3$ we obtain $p^2_{11} \neq 0$, $p^2_{13} \neq 0$, so

$$\lambda_4 = p^2_{11} + p^2_{12} - p^3_{12} = -1.$$

e smallest eigenvalue of a graph is equal to -1 if and only if the components re complete graphs [8] which is a contradiction to $D = 3$.
f the condition (2.9) holds the procedure is equal as above in a) if we ute λ_3 for λ_4. So we get

$$\lambda_3 = p_{11}^2 + p_{12}^2 - p_{12}^3 = -1$$

ondition (2.10) gives

$$\lambda_2 = p_{11}^2 + p_{12}^2 - p_{12}^3 = -1.$$

graph has exactly one positive eigenvalue if and only if its non-isolated s form a complete multipartite graph [13], which is a contradiction with by Theorem 1.6. This completes the proof. □

3. Bipartite graphs with 4 distinct eigenvalues

heorem. *For every $k \in N$, $k \geq 2$ there is one and only one metrically bipartite graph $G = (X, E)$ with diameter $D = 3$, $n = |X| = 2k + 2$, so ^{2} is a strongly regular graph. Its structure constants are:*

$p_{10}^1 = 1$	$p_{20}^2 = 1$	$p_{30}^3 = 1$	$v_0 = 1$	$\lambda_1 = k$
$p_{11}^1 = 0$	$p_{11}^2 = k - 1$	$p_{11}^3 = 0$	$v_1 = k$	$\lambda_2 = 1$
$p_{12}^1 = k - 1$	$p_{12}^2 = 0$	$p_{12}^3 = k$	$v_2 = k$	$\lambda_3 = -1$
$p_{13}^1 = 0$	$p_{13}^2 = 1$	$p_{13}^3 = 0$	$v_3 = 1$	$\lambda_4 = -k$
$p_{22}^1 = 0$	$p_{22}^2 = k - 1$	$p_{22}^3 = 0$	$m_1 = 1$	$m_4 = 1$
$p_{23}^1 = 1$	$p_{23}^2 = 0$	$p_{23}^3 = 0$	$m_2 = k$	$m_3 = k$
$p_{33}^1 = 0$	$p_{33}^2 = 0$	$p_{33}^3 = 0$		

As G is a bipartite graph, $p_{ij}^k = 0$ must be fulfilled for any $i, j, k \in \{1, 2, 3\}$, $k \equiv 1 \pmod 2$. Thus it follows

$$p_{11}^1 = p_{13}^1 = p_{22}^1 = p_{33}^1 = p_{12}^2 = p_{23}^2 = p_{11}^3 = p_{13}^3 = p_{22}^3 = p_{33}^3 = 0$$

cording to Theorems 1.10 and 2.3 we get $\lambda_1 = -\lambda_4$, $m_1 = m_4$, $\lambda_2 = -\lambda_3 = 1$, m_3. With respect to (2.4) $i = 1$, $k = 1, 2, 3$ it holds $p_{12}^1 = \lambda_1 - 1$, $p_{13}^2 = p_{11}^2$, $p_{12}^3 = \lambda_1$.
(2.3) gives $\lambda_1 \lambda_2^2 = \lambda_1(\lambda_1 - p_{11}^2)$. This implies $p_{11}^2 = \lambda_1 - 1$, $p_{13}^2 = 1$. (2.11) $p_{22}^2 = \lambda_1 - 1$. Using relations (2.6) we obtain

$$v_2 = \lambda_1 \frac{p_{12}^1}{p_{11}^2} = \lambda_1, \quad v_3 = v_2 \frac{p_{13}^2}{p_{12}^3} = 1,$$

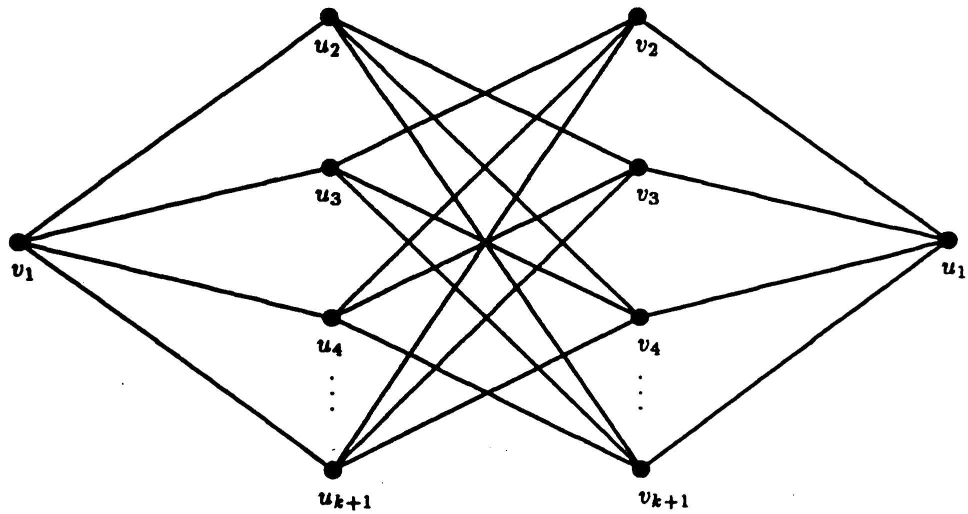
yields $p^2_{33} = 0$ and from (2.4) ($i = 2$, $k = 3$) $p^3_{23} = 0$. According to (2.7) the eigenvalues of G^2 are

$$\begin{aligned} \mu_1 &= 2\lambda_1 \quad \text{multiplicity} \quad m_1^{(2)} = 1 = m_1; \\ \mu_2 &= 0 \quad \text{multiplicity} \quad m_2^{(2)} = \lambda_1 + 1 = m_2 + m_4, \\ \mu_3 &= -2 \quad \text{multiplicity} \quad m_3^{(2)} = \lambda_1 = m_3. \end{aligned}$$

Construction of $G = (X_1, X_2, E)$:

$$X_1 = \{v_1, v_2, \dots, v_{k+1}\}; \qquad X_2 = \{u_1, u_2, \dots, u_{k+1}\};$$
$$E = \{(v_i, u_j);\ i, j = 1, 2, \dots, k+1;\ i \neq j\}.$$

This graph is shown in Fig.



It remains to prove the characterization of G by its spectrum. According to Theorem 1.12. this is equivalent to the assertion that the complement of G is characterized by its spectrum. □

As $\bar{G} \cong K_{k+1} \dot{+} K_2 \cong L(K_{k+1,2})$ we must prove that the line graph of $K_{k+1,2}$ is characterized by its spectrum. But we prove the following theorem.

3.2. Theorem. *Let $G \cong L(K_{n_1,n_2})$, $n_1 \geq n_2$. Then G is characterized by its spectrum unless:*

(i) $\{n_1, n_2\} = \{4, 4\}$,
(ii) $\{n_1, n_2\} = \{6, 3\}$,
(iii) $\{n_1, n_2\} = \{t(2t+1), t(2t-1)\}$, $t \geq 2$.

In the case (i) *and* (ii) *there is a cospectral mate that is not itself a line graph. In the case* (iii) *for $t = 2$ it holds*

$$L(K_{10,6}) \not\cong L(L(K_6)),$$

but $L(K_{10,6})$ is cospectral with $L(L(K_6))$.

Proof. The cases (i) and (ii) follow from Theorems 1.13. and 1.14. Because of Theorem 1.13. it is enough to consider the case G is cospectral with $L(H)$.

With respect to [6, p.72] the spectrum of K_{n_1,n_2} contains the numbers $\sqrt{n_1 n_2}$, $-\sqrt{n_1 n_2}$, and $n_1 + n_2 - 2$ numbers all equal to 0 and we obtain

$$P_{K_{n_1,n_2}}(\lambda) = (\lambda^2 - n_1 n_2)\lambda^{n_1+n_2-2}$$

By Theorem 1.16. it holds

$$P_{L(K_{n_1,n_2})}(\lambda) = (\lambda + 2)^{n_1 n_2 - n_1 - n_2}(\alpha_1\alpha_2 - n_1 n_2)\alpha_1^{n_1-1}\alpha_2^{n_2-1}.$$

As $\alpha_1\alpha_2 - n_1 n_2 = (\lambda - n_2 + 2)(\lambda - n_1 + 2) - n_1 n_2 = (\lambda + 2)[\lambda - (n_1 + n_2 - 2)]$ we get

$$\begin{aligned} P_{L(K_{n_1,n_2})}(\lambda) &= (\lambda + 2)^{n_1 n_2 - n_1 - n_2 + 1}\cdot \\ &\cdot[\lambda - (n_1 + n_2 - 2)][\lambda - (n_2 - 2)]^{n_1-1}[\lambda - (n_1 - 2)]^{n_2-1} \end{aligned} \tag{3.1}$$

So we get a connected regular graph of degree $n_1 + n_2 - 2$ with $n_1 n_2$ vertices. By Theorem 1.9. the line graph $L(H)$ cospectral with $L(K_{n_1,n_2})$ must be regular, connected and it has $n_1 n_2$ vertices and the same set of the distinct eigenvalues. It follows, that H must be either a semiregular or a regular graph.

A. Let $H = (X_1, X_2, E)$ be a semiregular graph of degrees r_1, r_2 with $|X_i| = m_i$, $d_H(x_i) = r_i$, $x_i \in X_i$ $(i = 1,2)$, $m_1 \geq m_2$. According to [6, p.31] we get $\pm\sqrt{r_1 r_2} \in S_p(H)$ with the multiplicity 1 and from Theorem 1.16. it follows that the multiplicity of -2 in $L(H)$ is $m_1 r_1 - m_1 - m_2 + 1$ and that $r_1 - 2$ is an eigenvalue of $L(H)$. Hence from (3.1) we obtain $n_2 = r_1$. So, it follows

$r_1 = n_2$	
$m_1 r_1 = m_2 r_2$	- the necessary condition for H.
$m_1 r_1 = n_1 n_2$	- the numbers of vertices of $L(H)$, G.
$m_1 r_1 - m_1 - m_2 + 1 =$	- the multiplicity of -2 in the spectrum
$= n_1 n_2 - n_1 - n_2 + 1$	of $L(H)$ and G.

But these equations give $H \cong K_{n_1,n_2}$.

B. Let H be a regular graph of a degree r with n vertices and $m = \frac{1}{2}nr$ edges. Denote $S_p(L(H)) = \{\lambda_i\} = S_p(G)$, $\lambda_1 \geq \lambda_2 \geq \ldots$. According to Theorem 1.15. we obtain comparing the degrees of $L(H)$ and G, $\lambda_1 = 2r - 2 = n_1 + n_2 - 2$, so $r = \frac{n_1+n_2}{2}$. Moreover, $m = n_1 n_2$.

1) Let H be a bipartite regular graph (X_1, X_2, E) with $|X_i| = m_i$, $i = 1,2$, $m_1 \geq m_2$. Then $m_1 r = m_2 r$, i.e. $m_1 = m_2 = \frac{1}{2}n$. By Theorem 1.10., $-r \in S_p(H)$, and from (3.1) and (1.1) it follows $m_1 r = n_1 n_2$ and that the multiplicity of -2 is

$$m - n + 1 = n_1 n_2 - n_1 - n_2 + 1 \tag{3.2a}$$

As $m = \frac{1}{2}nr$ we obtain

$$n = 4\left(\frac{n_1 n_2 - 4}{n_1 + n_2 - 4} - 1\right), \text{ if } n_1 + n_2 - 4 \neq 0.$$

Comparing (3.1) and (1.1) we get that the spectrum of H contains the numbers

$\mu_1 = \frac{n_1+n_2}{2}$ (multiplicity 1)
$\mu_2 = \frac{n_1-n_2}{2}$ (multiplicity $n_2 - 1$)
$\mu_3 = -\frac{n_1-n_2}{2}$ (multiplicity $n_1 - 1$)
$\mu_4 = -\frac{n_1+n_2}{2}$ (multiplicity 1)

As H is a bipartite graph we obtain by Theorem 1.10.

$$n_1 = n_2, \text{ so } n = 2n_1 \text{ and } m_1 = m_2 = n_1 = n_2 = r,$$

hence

$$H \cong K_{n_1,n_2}.$$

If $n_1 + n_2 - 4 = 0$ we get from (3.2a)

$$\frac{n}{4}(n_1 + n_2 - 4) = n_1 n_2 - n_1 - n_2 = 0.$$

So we obtain $n_1 + n_2 = 4$, $n_1 n_2 = 4$. This implies $n_1 = 2$, $n_2 = 2$, $r = 2$, $m_1 = m_2 = 2$, $m = n = 4$ and $H \cong C_4 \cong K_{2,2}$.

2) Let H be a nonbipartite regular graph. Comparing (1.1) and (3.1) we obtain ($-r$ is not eigenvalue of H - Theorem 1.11)

$$\begin{aligned} &m - n = n_1 n_2 - n_1 - n_2 + 1, \\ &\frac{n}{4}(n_1 + n_2 - 4) = n_1 n_2 - n_1 - n_2 + 1. \end{aligned} \tag{3.2}$$

a) $n_1 + n_2 - 4 = 0$. It implies $n_1 n_2 = 3$. So we obtain

$$n_1 = 3,\ n_2 = 1,\ r = 2,\ m = 3 = n \text{ and } H \cong K_3.$$

But $L(K_3) \cong L(K_{3,1})$.

b) $n_1 + n_2 > 4$. So we get from (3.2)

$$n = \frac{4n_1 n_2 - 12}{n_1 + n_2 - 4} - 4 \tag{3.3}$$

Comparing (3.1) and (1.1) we get in this case that the spectrum of H contains the numbers

$\mu_1 = \frac{n_1+n_2}{2}$ (multiplicity 1)
$\mu_2 = \frac{n_1-n_2}{2}$ (multiplicity $n_2 - 1$)
$\mu_3 = \frac{n_2-n_1}{2}$ (multiplicity $n_1 - 1$)

if $n_1 \neq n_2$; if $n_1 = n_2$ then $\mu_2 = \mu_3 = 0$ has the multiplicity $2n_1 - 2$. As H has no loop the sum of all eigenvalues $\mu_1, \ldots, \mu_n$ $\sum_{i=1}^{n} \mu_i = 0$ and we obtain

$$(n_1 - n_2)^2 - (n_1 + n_2) = 0.$$

If we denote $n_1 - n_2 = t$, we get

$$t^2 - t - 2n_2 = 0.$$

As $t \in N$ we obtain $n_1 > n_2$

$$n_1 = \frac{(k+1)(k+2)}{2}, n_2 = \frac{k(k+1)}{2}, n = k(k+2); k \in N$$

. As $r = \frac{n_1+n_2}{2} = \frac{(k+1)^2}{2} \in N$, we obtain $k = 2j - 1$, $j \geq 1$. So we get $r = 2j^2$, $n_1 = j(2j+1)$, $n_2 = j(2j-1)$, $n = 4j^2 - 1$, $m = \frac{1}{2}nr = j^2(4j^2 - 1)$.

It is easy to see that H is a strongly regular graph with the following table of the structural constants:

$p_{10}^1 = 1$	$p_{20}^2 = 1$	$\mu_1 = 2j^2$	(multiplicity 1)
$p_{11}^1 = j^2$	$p_{11}^2 = j^2$	$\mu_2 = j$	(multiplicity $2j^2 - j - 1$)
$p_{12}^1 = j^2 - 1$	$p_{12}^2 = j^2$	$\mu_3 = -j$	(multiplicity $2j^2 - j - 1$)
$p_{22}^1 = j^2 - 1$	$p_{22}^2 = j^2 - 3$	$v_1 = 2j^2$	$v_2 = 2(j^2 - 1)$ $j \in N \setminus \{0\}$.

So for $j = 1$ we get $H = K_3$, $n_1 = 3$, $n_2 = 1$, but $L(K_{3,1}) \cong L(K_3)$. For $j = 2$ it is easy to see that $H\ congL(K_6)$. According to Theorem 1.17. we obtain $S_p(L(K_{10,6})) = S_p(L(K_6)))$ but $L(K_{10,6}) \ncong L(L(K_6))$, because $K_{10,6} \ncong L(K_6)$.

The fact that H is isomorphic with $L(K_6)$ we can obtain by the same procedure as before. At first, $H = L(H')$ with some H' holds because of Theorem 1.13.; then we get:

A. H' - semiregular graph of the type (m_1, m_2, r_1, r_2). Comparing $L(H')$ and H we obtain: $m_1r_1 = m_2r_2$, $m_1r_1 = 15$, $r_1 + r_2 = 8$ (the degree of $L(H')$ and H), $m_1r_1 - m_1 - m_2 + 1 = 9$ (the multiplicity of -2). But there are no m_1, m_2, r_1, r_2 satisfying these conditions.

B. H' - a regular connected graph of a degree r'. So we obtain:

1) H' - a bipartite graph. It implies $2r' - 2 = 8$, $r' = 5$, $m - n + 1 = 9$, so $\frac{3}{2}n = 8$.

2) H' - a nonbipartite graph. It implies $m - n = 9$ and as $r' = 5$, $n = 6$. So $H' \cong K_6$.

The table of the association scheme with the parameters p_{ij}^k for this case (i.e. for the graph H) is the following

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2
2	1	0	2	2	1	1	1	2	1	1	1	1	2	2	2
3	1	2	0	1	2	1	1	1	2	2	2	1	1	1	2
4	1	2	1	0	2	2	1	1	1	1	2	2	2	1	1
5	1	1	2	2	0	1	2	1	1	2	1	2	1	2	1
6	1	1	1	2	1	0	2	2	1	2	2	1	1	1	2
7	1	1	1	1	2	2	0	1	2	1	1	1	2	2	2
8	1	2	1	1	1	2	1	0	2	2	1	2	1	2	1
9	1	1	2	1	1	1	2	2	0	1	2	2	2	1	1
10	2	1	2	1	2	2	1	2	1	0	1	1	2	1	1
11	2	1	2	2	1	2	1	1	2	1	0	1	1	2	1
12	2	1	1	2	2	1	1	2	2	1	1	0	1	1	2
13	2	2	1	2	1	1	2	1	2	2	1	1	0	1	1
14	2	2	1	1	2	1	2	2	1	1	2	1	1	0	1
15	2	2	2	1	1	2	2	1	1	1	1	2	1	1	0

Substituting 2 by 0 we obtain the adjacency matrix of H for $j = 2$. □

4. Metrically regular bipartite graphs with 5 distinct eigenvalues

Let $T = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be the set of the distinct eigenvalues of the graph G and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5$. As G is a bipartite graph we obtain from Theorem 1.10.

$$\lambda_1 = -\lambda_5, \quad \lambda_2 = -\lambda_4, \quad \lambda_3 = 0.$$

According to Theorem 1.4., λ_i $(i = 1, 2, 3, 4, 5)$ is the solution of the equation

$$|\lambda I - P_1| = 0.$$

As G is bipartite it holds

$$p^i_{jk} = 0 \text{ for } i, j, k \in \{0, 1, 2, 3, 4\}, \quad i + j + k \equiv 1 (\mathrm{mod}\ 2)$$

and we get

$$\begin{aligned}\lambda^5 - \lambda^3[\lambda_1 + p^1_{12}p^2_{11} + p^2_{13}p^3_{12} + p^3_{14}p^4_{13}]+ \\ +\lambda[p^1_{12}p^2_{11}p^3_{14}p^4_{13} + \lambda_1(p^2_{13}p^3_{12} + p^3_{14}p^4_{13})] = 0.\end{aligned} \tag{4.1}$$

From the condition for G to have the square G^2 strongly regular we get for the structural constants ${}^2p^k_{ij}$ of G^2:

$$\begin{aligned}
{}^2p^1_{11} &= p^1_{11} + 2p^1_{12} + p^1_{22} = p^2_{11} + 2p^2_{12} + p^2_{22},\\
{}^2p^1_{12} &= p^1_{23} = p^2_{13} + p^2_{23} + p^2_{24},\\
{}^2p^1_{22} &= p^1_{33} + 2p^1_{34} + p^1_{44} = p^2_{33} + 2p^2_{34} + p^2_{44},\\
{}^2p^2_{11} &= 2p^3_{12} + p^3_{22} = p^4_{22},\\
{}^2p^2_{22} &= p^3_{33} + 2p^3_{34} + p^3_{44} = p^4_{33} + 2p^4_{34} + p^4_{44}.\\
{}^2p^2_{12} &= p^3_{13} + p^3_{14} + p^3_{23} + p^3_{24} = p^4_{13} + p^4_{14} +\\
&\quad + p^4_{23} + p^4_{24},
\end{aligned}$$

As G is a bipartite graph we obtain

$$^2p^1_{11} = 2p^1_{12} = p^2_{11} + p^2_{22} \tag{4.2}$$

$$^2p^1_{12} = p^1_{23} = p^2_{13} + p^2_{24} \tag{4.3}$$

$$^2p^1_{22} = 2p^1_{34} = p^2_{33} + p^2_{44} \tag{4.4}$$

$$^2p^2_{11} = 2p^3_{12} = p^4_{22} \tag{4.5}$$

$$^2p^2_{12} = P^3_{14} + P^3_{23} = P^4_{13} + P^4_{24} \tag{4.6}$$

$$^2p^2_{22} = 2p^3_{34} = p^4_{33} + p^4_{44} \tag{4.7}$$

According to the form of the matrix A_2 of G^2 we get the eigenvalues of G^2 in the form

$$\mu_i = \frac{\lambda_i^2 + p^2_{11}\lambda_i - \lambda_1}{p^2_{11}}; \quad i \in \{1,2,3,4,5\} \tag{4.8}$$

As for a bipartite graph it holds $p^2_{11}(\mu_1 - \mu_i) = p^2_{11}(\lambda_1 - \lambda_i)(\lambda_1 + \lambda_i + p^2_{11}) > 0$, μ_1 is the index of G^2. As a strongly regular graph has 3 distinct eigenvalues it must hold (for distinct numbers i,j,k,m; $i,j,k,m \neq 1$) either $\mu_i = \mu_j = \mu_k$ or $\mu_i = \mu_j$ and $\mu_k = \mu_m$.

A. $\mu_i = \mu_j = \mu_k$.

According to (4.8) we obtain

$$p^1_{11} - p^2_{11} = \lambda_i + \lambda_j = \lambda_i + \lambda_k = \lambda_j + \lambda_k.$$

So we get the contradiction with $\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i$.

B. $\mu_i = \mu_j$, $\mu_k = \mu_m$.

Because of Theorem 1.4. we obtain for a strongly regular G^2 of a degree r $\mu_2\mu_3 = -(r - {}^2p^2_{11}) < 0$. As G^2 contains at least one edge $\mu_3 < 0$, so $\mu_2 > 0$. Because of $p^2_{11} > 0$ it remains $\mu_2 = \mu_5$, $\mu_3 = \mu_4$.
In this case we obtain from (4.8) for the bipartite graph G

$$\lambda_2 - \lambda_1 = -p^2_{11} = -\lambda_2$$

$$\text{so} \qquad \lambda_1 = 2p^2_{11} = 2\lambda_2. \tag{4.9}$$

From (4.1) we obtain

$$\lambda_1^2 + \lambda_2^2 = \lambda_1 + p_{12}^1 p_{11}^2 + p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 \tag{4.10}$$

and using (4.9), (4.10) and (2.4) ($i = 1$, $k = 1,2,3$) we get

$$\begin{gathered} p_{12}^1 = 2p_{11}^2 - 1,\ p_{13}^2 = p_{11}^2,\ p_{13}^4 = 2p_{11}^2, \\ p_{12}^3 = p_{11}^2 + 1,\ p_{14}^3 = p_{11}^2 - 1. \end{gathered} \tag{4.11}$$

As $p_{14}^3 > 0$ ($D = 4$) we get $p_{11}^2 > 1$.
From the relation (2.6) ($i = 1$, $j = 2$, $k = 1$ and $i = 2$, $j = 3$, $k = 1$) we get

$$v_2 = 2(2p_{11}^2 - 1) \tag{4.12}$$

$$\text{and} \qquad v_3 = 4p_{11}^2 - 6 + \frac{6}{p_{11}^2 + 1}. \tag{4.13}$$

As v_3 is an integer we get $p_{11}^2 \in \{2, 5\}$.
a) $p_{11}^2 = 5$.
Accroding to (4.9) and (4.10) - (4.13) we obtain

$$\begin{array}{llll} v_1 = 10 & v_2 = 18 & v_3 = 15 & p_{12}^1 = 9 \\ p_{13}^2 = 5 & p_{12}^3 = 6 & p_{14}^3 = 4 & p_{13}^4 = 10. \end{array}$$

By (2.6) ($i = 3$, $j = 4$, $k = 1$) we get $v_4 = 6$. (2.4) ($i = 2$, $k = 1$) implies $p_{23}^1 = 9$ and from (4.3) we obtain $p_{24}^2 = 4$. By (2.4) ($i = 4$, $k = 2$) we get $p_{44}^2 = 2$. As (2.6) ($i = 2$, $j = 4$, $k = 4$) implies $p_{24}^4 = 6$ and $v_4 > p_{24}^4$, which follows from (2.4) ($i = 4$, $k = 4$) we obtain a contradiction.
b) $p_{11}^2 = 2$.
According to (4.2) - (4.13) and (2.4) - (2.6) we obtain the following table:

$p_{10}^1 = 1$	$p_{20}^2 = 1$	$p_{30}^3 = 1$	$p_{40}^4 = 1$	$v_0 = 1$
$p_{11}^1 = 0$	$p_{11}^2 = 2$	$p_{11}^3 = 0$	$p_{11}^4 = 0$	$v_1 = 4$
$p_{12}^1 = 3$	$p_{12}^2 = 0$	$p_{12}^3 = 3$	$p_{12}^4 = 0$	$v_2 = 6$
$p_{13}^1 = 0$	$p_{13}^2 = 2$	$p_{13}^3 = 0$	$p_{13}^4 = 4$	$v_3 = 4$
$p_{14}^1 = 0$	$p_{14}^2 = 0$	$p_{14}^3 = 1$	$p_{14}^4 = 0$	$v_4 = 1$
$p_{22} = 0$	$p_{22}^2 = 4$	$p_{22}^3 = 0$	$p_{22}^4 = 6$	$\lambda_1 = 4 = -\lambda_5$
$p_{23}^1 = 3$	$p_{23}^2 = 0$	$p_{23}^3 = 3$	$p_{23}^4 = 0$	$\lambda_2 = 2 = -\lambda_4$
$p_{24}^1 = 0$	$p_{24}^2 = 1$	$p_{24}^3 = 0$	$p_{24}^4 = 0$	$\lambda_3 = 0$
$p_{33}^1 = 0$	$p_{33}^2 = 2$	$p_{33}^3 = 0$	$p_{33}^4 = 0$	$m_1 = 1 = m_5$
$p_{34}^1 = 1$	$p_{34}^2 = 0$	$p_{34}^3 = 0$	$p_{34}^4 = 0$	$m_2 = 4 = m_4$
$p_{44}^1 = 0$	$p_{44}^2 = 0$	$p_{44}^3 = 0$	$p_{44}^4 = 0$	$m_3 = 6$

The realization of this table is the 4-dimensional unit cube. So we have proved the following theorem:

4.1. Theorem. *There is only one table of the parameters of an association scheme so that the corresponding metrically regular bipartite graph with 5 distinct eigenvalues has the strongly regular square.*

4.2. Remark. Theorems 3.1. and 4.1. show that for $k = 3$ and $k = 4$ the k-dimensional unit cubes have the strongly regular square.

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