Nikolaos S. Papageorgiou

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ON THE EXISTENCE OF ψ–MINIMAL VIABLE SOLUTIONS FOR A CLASS OF DIFFERENTIAL INCLUSIONS

NIKOLAS S. PAPAGEORGIOU

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ABSTRACT. In this paper we establish the existence of ψ-minimal viable solutions for a class of differential inclusions with a Hausdorff continuous orientor field defined on a general Banach space and satisfying a compactness hypothesis and a strong Nagumo type condition (theorem 3.1 and 3.2). When the space is finite dimensional, we show that the strong Nagumo condition can be weakened to a regular Nagumo type (tangential) condition.

1. INTRODUCTION

In a recent paper Falcone-Saint Pierre [6] established sufficient conditions for the existence of slow viable solutions for a class of differential inclusions defined on a finite dimensional Banach space.

In this note we generalize the results of Falcone-Sait Pierre [6], by relaxing some of their hypotheses and by proving an existence result for finite dimensional differential inclusions.

Consider the following multivalued Cauchy problem on a Banach space X:

$$\left\{ \begin{array}{c} \dot{x}(t) \in F(x(t)) \quad \text{a.e.} \\ x(0) = x_0 \in K \subseteq X \\ x(t) \in K, \ t \in T = [0, b] \end{array} \right\} \tag{*}$$

In their works, Deimling [4] and the author [13], proved that under some compactness type hypothesis on the orientor field $F(x)$, a necessary and sufficient condition for existence of solutions of (*), is that for all $x \in K$, $F(x) \cap T_K(x) \neq \emptyset$ (Nagumo type condition). Here $T_K(x)$ denotes the Bouligand tangent cone to K at x.

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In this paper we will be looking for a special type of viable solutions, namely solutions with velocity which is minimal with respect to a certain criterion $\psi(\cdot)$ ($\psi(\cdot)$-minimal solutions). So let $\psi : X \to \mathbb{R}$ be a continuous, convex function. We say that a trajectory $x(\cdot)$ of (*) is "ψ-minimal, viable" if and only if

$$\psi(\dot{x}(t)) = \inf\{\psi(z) : z \in F(x(t)) \cap T_K(x(t))\} \quad \text{a.e.}$$

Note that when $\psi(x) = \|x\|$ (the norm function), we recover the notion of slow solution, which is important in mathematical economics and control theory (see Aubin [1] and Henry [7]). In this case $\psi(\cdot)$ is nothing else but the metric projection on the set $R(x) = F(x) \cap T_K(x)$. Recall that if the underlying state space X is a strictly convex, reflexive Banach space and $K \subseteq X$ is nonempty, closed, convex, then the metric projection function $x \to \operatorname{proj}(x, K)$ is single valued.

2. Preliminaries

Let X be a Banach space. Throught this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed, (convex)}\}$$

and

$$P_{k(c)}(X) = \{A \subseteq X : \text{ nonempty, compact, (convex)}\}.$$

On $P_f(X)$ we can define a generalized metric $h(\cdot,\cdot)$, known as the Hausdorff metric, by setting

$$h(A,B) = \max\{\sup_{a\in A}(\inf \|a-b\| : b \in B),\ \sup_{b\in B}(\inf \|b-a\| : a \in A)\}.$$

Recall that $(P_f(X), h)$ is a complete metric space.

A multifunction $F : X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous as a function from X into the metric space $(P_f(X), h)$.

More generally if Y, Z are Hausdorff topological spaces, a multifunction $F : Y \to 2^Z \setminus \{\emptyset\}$ is said to be lower semicontinuous (l.s.c.), if for all $U \subseteq Z$ open, $F^-(U) = \{y \in Y : F(y) \cap U \neq \emptyset\}$ is open in Y. If Y, Z are metric spaces, this definition is equivalent to saying that for any $y_n \to y$ in Y, we have $F(y) \subseteq \lim F(y_n) = \{z \in Z : \lim d(z, F(y_n)) = 0\}$, where $d(z, F(y_n)) = \inf\{\|z - z'\| : z' \in F(y_n)\}$. We will say that $F : Y \to 2^Z \setminus \{\emptyset\}$ is upper semicontinuous (u.s.c.), if for all $U \subseteq Z$ open $F^+(U) = \{y \in Y : F(y) \subseteq U\}$ is open in Y (see Delahaye-Denel [5]).

Now let us turn to X being a Banach space, let $K \subseteq X$ be nonempty and let $x \in \bar{K}$. The "Bouligand or contingent cone" to K at x is defined by:

$$T_K(x) = \left\{h \in X : \underline{\lim}_{\lambda \downarrow 0} \frac{d_K(x + \lambda h)}{\lambda} = 0\right\}$$

where for any $z \in X$, $d_K(z) = \inf\{\|z - x'\| : x' \in K\}$ (see Aubin-Cellina [2]). It is clear that this cone is closed, $T_K(x) = T_{\bar{K}}(x)$ and if $x \in \operatorname{int} K$, then $T_K(x) = X$.

Note that unfortunately $T_K(x)$ in general is not convex. Also note that $int\ K \neq \emptyset$, then for all $x \in \bar{K}$, $int\ T_K(x) \neq \emptyset$ (see Aubin-Ekeland [3], p.169).

By $\alpha(\cdot)$. We will denote the "Kuratowski measure of noncompactness" which is defined on the nonempty, bounded subsets of X. So if A is such a set we have:

$$\alpha(A) = \inf\{d > 0 : A \subseteq \bigcup_{k=1}^{m} A_k, \text{ for some } m \text{ and } A_k's \text{ s.t. } \operatorname{diam}(A_k) \leq d\}.$$

Finally given a multifunction $F : Y \to 2^Z \setminus \{\emptyset\}$, by "graph of F" we will mean the set $GrF = \{(y, z) \in Y \times Z : z \in F(y)\}$.

3. Main results

Let X be a Banach space and $\psi : X \to \mathbb{R}$ a continuous, convex function. We will be looking for ψ-minimal viable trajectories of (*). Recall that $x : T = [0, b] \to X$ is a "ψ-minimal viable trajectory", if there exists $f \in S^1_{F(x(\cdot))} = \{g \in L^1(X) : g(t) \in F(x(t)) \text{ a.e.}\}$ s.t. $x(t) = x_0 + \int_0^t g(s)\,ds$ for all $t \in T$, $x(t) \in K$ and $\psi(\dot{x}(t)) = \inf\{\psi(z) : z \in F(x(t)) \cap T_K(x(t))\}$ a.e.

In our first theorem we will establish the existence of such solutions for a large class of infinite dimensional differential inclusions. For this we will need the following simple lemma.

Lemma. *If Y, Z are Hausdorff topological spaces, $F : Y \to 2^Z \setminus \{\emptyset\}$ is l.s.c., $G : Y \to 2^Z \setminus \{\emptyset\}$ has an open graph and for all $y \in Y$, $F(y) \cap G(y) \neq \emptyset$, then $y \to L(y) = F(y) \cap G(y)$ is l.s.c.*

Proof. We need to show that given $V \subseteq Z$ open, $L^-(V) = \{y \in Y : L(y) \cap V \neq \emptyset\} = \{y \in Y : F(y) \cap G(y) \cap V \neq \emptyset\}$ is open. Let $y \in L^-(V)$ and $z \in F(y) \cap G(y) \cap V$. Then $(y, z) \in GrG \cap (Y \times V)$. Note that since by hypothesis $G(\cdot)$ has an open graph, $GrG \cap (Y \times V)$ is open. So we can find $U_1(y)$ an open neighborhood of y and $W_1(z)$ an open neighborhood of z.s.t. $U_1(y) \times W_1(z) \subseteq GrG \cap (Y \times V)$. Observe that $F(y) \cap W_1(z) \neq \emptyset$ since it contains z and because by hypothesis $F(\cdot)$ is l.s.c., we can find $U_2(y)$ and open neighborhood of y s.t. $F(y')\varepsilon W_1(z) \neq \emptyset$ for all $y' \in U_2(y)$. Set $U(y) = U_1(y) \cap U_2(y)$. Then for all $y' \in U(y)$ we have $F(y') \cap W_1(z) \neq \emptyset$ while $U(y) \times W_1(z) \subseteq GrG \cap (Y \times V)$. So for all $y' \in U(y)$, $F(y') \cap G(y') \cap V = L(y') \cap V \neq \emptyset \Rightarrow L^-(V)$ is open $\Rightarrow L(\cdot)$ is l.s.c. □

Now we are ready for the theorem establishing the existence of ψ-minimal viable solutions for (*). Our result extends theorem 4.1 of Falcone-Saint Pierre [6], since our state space is infinite dimensional, the growth hypothesis on the orientor field $F(\cdot)$ is more general and $\psi(\cdot)$ need not be inf-compact as in [6]. Note that this last fact is very important, because it allows $\psi(\cdot)$ to be the norm of an infinite dimensional Banach space and so our existence theorem incorporates the results on the existence of slow solutions (see Aubin-Cellina [2]).

Theorem 3.1. *If $K \in P_{fc}(X)$ with int $K \neq \emptyset$, $\psi : X \to \mathbb{R}$ is continuous, convex and $F : K \to P_{fc}(X)$ is a multifunction s.t.*
(1) $F(\cdot)$ is h-continouous,
(2) $|F(x)| \leq c(1 + \|x\|)$, $c > 0$,
(3) $\alpha(F(B)) \leq k\,\alpha(B)$ for all $B \subseteq K$ nonempty bounded, $k > 0$,
(4) $F(x) \cap$ int $T_K(x) \neq \emptyset$ for all $x \in K$,
then (*) admits a ψ-minimal viable solution $x(\cdot)$.

Proof. Let $G : K \to 2^X$ be defined by

$$G(x) = \{v \in Y : \psi(y) \leq \inf(\psi(z) : z \in R(x)) = \lambda(x)\}$$

where $R(x) = F(x) \cap T_K(x)$. Since $F(\cdot)$ is h-continuous (hence l.s.c. too) and $x \to$ int $T_K(x)$ has an open graph (see Aubin-Ekeland [3], proposition 7, p. 169), from the lemma we deduce that $x \to F(x) \cap$ int $T_K(x)$ l.s.c.. Hence $x \to \overline{F(x) \cap \text{ int } T_K(x)} = F(x) \cap T_K(x) = R(x)$ is l.s.c. (see Klein-Thompson [9], proposition 7.3.3, p. 85). Also since $R(\cdot)$ is compact valued ($F(\cdot)$ being compact valued because of hypothesis (3)), there exists a $\hat{z} \in R(x)$ (depending on x) s.t. $\psi(\hat{z}) = \lambda(x) \Rightarrow G(x) \neq \emptyset$ and in fact, since $\psi(\cdot)$ is also convex, it is easy to see that $G(x) \in P_{kc}(X)$.

We claim that $G(\cdot)$ has closed graph. To this end let $(x_n, y_n) \in GrG$ $(x_n, y_n) \xrightarrow{s} (x, y)$ in $K \times X$. We have $\psi(y_n) \leq \lambda(x_n)$ for all $n \geq 1$. Since $R(\cdot)$ is l.s.c. from theorem 4, p. 51, in Aubin-Cellina [2], we have that $\lambda(\cdot)$ is u.s.c.. So by passing to the limit we get:

$$\lim \psi(y_n) = \psi(y) \leq \overline{\lim} \lambda(x_n) \leq \lambda(x)$$
$$\Longrightarrow (x, y) \in GrG$$
$$\Longrightarrow GrG \text{ is closed in } KxL.$$

Invoking theorem 1, p. 41 of Aubin-Cellina [2], we get $x \to L(x) = F(x) \cap G(x)$ is u.s.c.. Also because of hypothesis (4), $L(x) \cap T_K(x) \neq \emptyset$ for all $x \in K$. Furthermore we have $|L(x)| = \sup\{\|z\| : z \in L(x)\} \leq |F(x)| = \sup\{\|z'\| : z' \in F(x)\} \leq c(1 + \|x\|)$ (hypothesis (2)), while for $B \subseteq K$ nonempty bounded, since the Kuratowski measure of noncompactness is monotone, we have $\alpha(L(B)) \leq \alpha(F(B)) \leq k\,\alpha(B)$. So if we consider the following viability problem

$$\left\{ \begin{array}{c} \dot{x}(t) \in L(x(t)) \quad \text{a.e.} \\ x(0) = x_0 \in K \\ x(t) \in K, \ t \in T = [0, b] \end{array} \right\} \qquad (*)'$$

we see that all the hypotheses of theorem 1 of Deimling [4] are satisfied and so according to that theorem there exists solution $x(\cdot)$ of $(*)'$. It is easy to see that $x(\cdot)$ is the desired $\psi(\cdot)$-minimal viable solution of (*). □

We can also have an integral selection criterion.

So as before let $\psi : X \to \mathbb{R}$ be a continuous , convex function and set $I_\psi(v) = \int_0^b \psi(v(t))\,dt$, for all $v(\cdot) \in L^1(X)$ if the integral exists, permitting $\pm\infty$. We say that a trajectory $x(\cdot)$ of (*) is "I_ψ-minimal viable" if and only if $I_\psi(\dot{x}) = \inf\{I_\psi(v) : v \in S^1_{R(x(\cdot))}\}$, where $R(x) = F(x) \cap T_K(x)$ and $S^1_{R(x(\cdot))} = \{g \in L^1(X) : g(t) \in R(x(t))\}$ a.e.

Our existence result concerning I_ψ-minimal viable trajectories of (*), reads as follows:

Theorem 3.2. *If X is a separable Banach space, $K \in P_{fc}(X)$ with int $K \neq \emptyset$, $\psi : K \to \mathbb{R}$ is continuous, convex, for all $z(\cdot)$ viable trajectories of (*) and all $v \in S^1_{R(z(\cdot))}$, $I_\psi(v)$ is defined and finite for at least one such v and $F : K \to P_{fc}(X)$ is a multifunction satisfying hypothesis (1) $\to$ (4) of theorem 3.1, then (*) admits an I_ψ-minimal viable trajectory.*

Proof. From theorem 3.1 we know that there exists ψ-minimal viable trajectory $x(\cdot)$ of (*). So $\psi(\dot{x}(t)) = \inf\{\psi(z) : z \in R(x(t))\}$ a.e. Note that $R(\cdot)$ being l.s.c. is measurable. So we can apply theorem 2.2 of Hiai-Umegaki [8] and get that:

$$\inf\{I_\psi(v) : v \in S^1_{R(x(\cdot))}\} = \int_0^b \inf\{\psi(z) : z \in R(x(t))\}\, dt$$
$$= \int_0^b \psi(\dot{x}(t))\, dt = I_\psi(\dot{x}),$$

$\Longrightarrow x(\cdot)$ is an I_ψ-minimal viable trajectory of (*). □

If the underlying state space X is finite dimensional, then we can improve theorem 3.1 by replacing hypothesis (4), with a standard Nagumo type hypothesis. So we have the following existence result.

Theorem 3.3. *If* $\dim X < \infty$, *$K \in P_{fc}(X)$ with* int $K \neq \emptyset$, *$\psi : X \to \mathbb{R}$ is a continuous, strictly convex,* inf-compact *function and $F : K \to P_{fc}(X)$ is a multifunction s.t.*

(1) $F(\cdot)$ is h-continuous,
(2) $|F(x)| \leq c(1 + \|x\|)$, $c > 0$,
(3) $F(x) \cap T_X(x) \neq \emptyset$ for all $x \in K$,

then (*) admits a ψ-minimal viable trajectory $x(\cdot)$.

Proof. Let $F_n(x) = F(x) + \frac{1}{n}B_1$, where B_1 is the closed unit ball in X. Clearly $F_n(\cdot)$ is h-continuous, $|F_n(x)| \leq c(+\frac{1}{n}) + c\|x\|$ and $F_n(x) \cap$ int $T_K(x) \neq \emptyset$ for all $x \in K$. Consider the following approximating viability problems:

$$\left\{\begin{array}{ll} \dot{x}_n(t) \in F_n(x_n(t)) & \text{a.e.} \\ x_n(0) = x_0 \in K & \\ x_n(t) \in K,\ t \in T = [0, b] & \end{array}\right\} \qquad (*)_n$$

From theorem 3.1 (note that hypothesis (3) of that theorem is automatically satisfied with $k = 0$ because of the finite dimensionality of X), we know that for every $n \geq 1$, $(^*)_n$ admits a ψ-minimal viable solution $x_n(\cdot)$. Then for all $n \geq 1$ we have:

$$\|\dot{x}_n(t)\| \leq (c+1) + c\|x_n(t)\| \quad \text{a.e.}$$
$$\Rightarrow \|x_n(t)\| \leq \|x_0\| + (c+1)b + \int_0^t c\|x_n(s)\|\, ds.$$

So from Gronwall's inequality we get that for all $n \geq 1$ and all $t \in T$

$$\|x_n(t)\| \leq (\|x_0\| + (c+1)b)\exp(cb) = M.$$

Thus $\|\dot{x}_n(t)\| \leq (c+1) + cM = \overline{M}$ a.e. Therefore $\{\dot{x}_n(\cdot)\}_{n\geq 1}$ is uniformly integrable in $L^1(X)$ and so $\{x_n(\cdot)\}_{n\geq 1}$ is equicontinuous in $C(T, X)$. It is also bounded. So from the Arzela-Ascoti theorem, we deduce that $\overline{\{x_n\}_{n\geq 1}}$ is compact in $C(T, X)$. Hence by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, X)$.

Note that $F_n(x) \xrightarrow{K} F(x)$ (convergence in the sense of Kuratowski [10], p. 339). Because int $T_K(x) \neq \emptyset$, from lemma 1.4 of Mosco [11], we have that $F_n(x) \cap T_K(x) = R_n(x) \xrightarrow{K} F(x) \cap T_K(x) = R(x)$ for all $x \in K$.

Now we claim that the minmization problem $\min\{\varphi(z) : z \in R(x)\}$ is Tihonov well-posed i.e. it admits a unique solution $\hat{z}_n \in R(x)$ and every minimizing sequence converges to it. That a solution exists, follows from the continuity of $\psi(\cdot)$ and the compactness of $R(x)$. That is unique, is a consequence of the strict convexity of $\psi(\cdot)$. Finally let $\{z_n\}_{n\geq 1}$ be a minimizing sequence i.e. $\psi(z_n) \downarrow \lambda(x)$ where $\lambda(x)$ is the value of the minimization problem. Without any loss of generality, we may assume that for all $n \geq 1$, $\psi(z_n) \leq \lambda(x) + 1$. Since $\psi(\cdot)$ is inf-compact, $\{z_n\}_{n\geq 1}$ is relatively compact and so we may assume that $z_n \to \hat{z}$. Then $\psi(z_n) \to \psi(\hat{z}) = \lambda(x)$ i.e. $\hat{z}$ is the unique solution of the minimization problem. Therefore $\min\{\psi(z) : z \in R(x)\}$ is Tihonov well-posed. Without any loss of generality assume $\psi(0) = 0$.

Set $\lambda_n(x) = \min\{\psi(z) : z \in R_n(x)\}$ and $\lambda(x) = \min\{\psi(z) : z \in R(x)\}$. Since $R_n(x) \xrightarrow{K} R(x)$ and the limit problem is Tihonov well-posed, we can apply theorem 3 of Zolezzi [14] and get that $\lambda_n(x) \uparrow \lambda(x)$. Now note that for every $n \geq 1$, we have $\lambda_n(x_n(t)) \leq \lambda(x_n(t))$.

Recall that $\lambda(\cdot)$ is u.s.c.. So we get:

$$\overline{\lim}\lambda_n(x_n(t)) \leq \overline{\lim}\lambda(x_n(t)) \leq \lambda(x(t)).$$

Also from the Dunford-Pettis compactness criterion and by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1(X)$. Then for all $A \subseteq T$ Lebesgue measurable we have $\chi_A \dot{x}_n \xrightarrow{w} \chi_A \dot{x}$ in $L^1(X)$. Recalling that $I_\psi(\cdot)$ is weakly l.s.c. we get:

$$\begin{aligned}\int_0^b \psi(\chi_A(t)\dot{x}(t))\,dt &\le \underline{\lim}\int_0^b \psi(\chi_A(t)\dot{x}_n(t))\,dt \le \overline{\lim}\int_A \psi(\dot{x}_n(t))\,dt = \\ &= \overline{\lim}\int_A \lambda_n(x_n(t))\,dt \le \int_A \overline{\lim}\lambda_n(x_n(t))\,dt \text{ (Fatou's lemma)} \le \\ &\le \int_A \lambda(x(t))\,dt, \\ &\Longrightarrow \int_A \psi(\dot{x}(t))\,dt \le \int_A \lambda(x(t)) \\ &\Longrightarrow \psi(\dot{x}(t)) \le \lambda(x(t)) \quad \text{a.e.}\end{aligned}$$

Next we claim that $F_n(x_n(t)) \xrightarrow{h} F(x(t))$ as $n \to \infty$. To this end, note that for every $n \ge 1$, $x \to u_n(x) = h(F_n(x), F(x))$ is continuous and $u_n(x) \downarrow 0$. So from Dini's theorem we have $u_n(x) \to 0$ uniformly on compacta. Then note that:

$$\begin{aligned}h(F_n(x_n(t)), F(x(t))) &\le h(F_n(x_n(t)), F(x_n(t))) + h(F(x_n(t)), F(x(t))) \\ &= u_n(x_n(t)) + h(F(x_n(t)), F(x(t)))\end{aligned}$$

We see that $u_n(x_n(t)) \to 0$, while from hypothesis (1) we have $h(F(x_n(t)), F(x(t))) \to 0$ as $n \to \infty \Rightarrow F_n(x_n(t)) \xrightarrow{h} F(x(t))$ as $n \to \infty$.

Now observe that

$$\dot{x}_n(t) \in F_n(x_n(t)) \quad \text{a.e.}$$

and $x_n \to x$ in $C(T, X)$, while $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1(X)$. Invoking theorem 1, p. 60 of Aubin-Cellina [2] (see also theorem 3.1 of [12]), we get:

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e.}$$

Furthermore $x(t) \in K$ for all $t \in T$. Then for $\lambda > 0$ we have:

$$\frac{d_K(x(t) + \lambda\dot{x}(t))}{\lambda} = \frac{d_K(x(t+\lambda) - \lambda\varepsilon(\lambda))}{\lambda} \le \frac{\lambda\varepsilon(\lambda)}{\lambda} = \varepsilon(\lambda)$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to 0^+$. So we have:

$$\begin{aligned}&\underline{\lim}_{\lambda\downarrow 0}\frac{d_K(x(t)+\lambda\dot{x}(t))}{\lambda} = 0 \quad \text{a.e.} \\ &\Longrightarrow \dot{x}(t) \in T_K(x(t)) \quad \text{a.e.} \\ &\Longrightarrow \dot{x}(t) \in F(x(t)) \cap T_K(x(t)) = R(x(t)) \text{ a.e.}\end{aligned}$$

Because of the uniqueness of the solution of $\min\{\psi(z) : z \in R(x(t))\} = \lambda(x(t))$ and since as we saw above $\psi(\dot{x}(t)) \le \lambda(x(t))$ a.e., we conclude that $x(\cdot)$ is the desired ψ-minimal viable trajectory of (*). □

If in theorem 3.3, X is strictly convex and $\psi(z) = \|z\|$, then the result applies and we get slow viable solutions for (*). By the way, note that there is a minor inaccuracy in the work of Falcone-Saint Pierre [6]. The state space X has to be strictly convex, or otherwise the metric projection need not be single valued.

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Nikolas S. Papageorgiou
National Technical University
Department of Mathematics
Athens 15773, Greece