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NATURAL TRANSFORMATIONS OF 2-QUASIJETS

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ABSTRACT. This paper deals with a concrete application of the theory of prolongation functors. We describe explicitly all natural transformations of quasijets of the second order into themselves.

The notion of quasijets of the second order was introduced by Pradines, [8]. In the case of the higher order we refer to [1]. As to the theory of prolongation functors we use methods developed by many authors mainly in [5], [6], [7], [8].

Let M be a smooth manifold and $p_M : TM \to M$, $p_{TM} : TTM \equiv T_2M \to TM$ be tangent bundles. A chart (x^i) on M induces the charts (x^i, x^i_1), $(x^i, x^i_{10}, x^i_{01}, x^i_{11})$ on TM, T_2M, respectively. On T_2M there is a canonical involution, see [2], with the following coordinate form $i_2 : (x^i, x^i_{10}, x^i_{01}, x^i_{11}) \mapsto (x^i, x^i_{01}, x^i_{10}, x^i_{11})$.

Let Tf denote the differential of a map: $f : M \to \bar{M}$. Let (p_{TM}), (T_{p_M}) shortly denote the vector bundles $p_{TM} : T(TM) \to TM$, $T_{p_M} : T_2M \to TM$, respectively. Let $(p_{TM})_0$, $(T_{p_M})_0$ be the sets of zero-vectors on (p_{TM}), (T_{p_M}), respectively, and let V_0TM be the set of vertical vectors on TM along the set of zero-vectors on M. There exist three canonical embedding $E_i : TM \to TTM$, $i = 1, 2, 3$:

$$E_1(TM) = (p_{TM})_0, \quad E_1(x^i, x^i_1) = (x^i, x^i_1, 0, 0)$$
$$E_2(TM) = (T_{p_M})_0, \quad E_2(x^i, x^i_1) = (x^i, 0, x^i_1, 0)$$
$$E_3(TM) = V_0TM, \quad E_3(x^i, x^i_1) = (x^i, 0, 0, x^i_1)$$

Let M, N be smooth manifolds. A quasijet of the second order with source $x \in M$ and target $y \in N$ is map $\Phi : (T_2M)_x \to (T_2N)_y$ which is linear with respect to both vector bundle structures (p_{TM}) and (T_{p_M}).

Denote by $QJ^2_x(M, N)_y$ the set of all 2-quasijets with source x and target y. Let $QJ^2(M, N)$ indicate the set of all quasijets from M into N.

Let (x^i), (y^p) be charts on M, N, respectively. In the induced charts on T_2M and T_2N a quasijet from M to N has the following form:

$$y^p_{10} = b^p_i x^i_{10}, \; y^p_{01} = c^p_i x^i_{01}, \; y^p_{11} = e^p_{ij} x^i_{10} x^j_{01} + d^p_i x^i_{11}.$$

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It induces the chart $(x^i, y^p, b_i^p, c_i^p, d_i^p, e_{ij}^p)$ on $QJ^2(M,N)$.

Let us recall that the manifold $J^1(M,N)$ of all 1-jets from M into N can be identified with the set $\bigcup_{x\in M, y\in N} L(T_xM, T_yN)$ of all linear maps from T_xM into T_yN, for all $x \in M$, $y \in N$.

The embeddings E_i, $i = 1,2,3$, determine three different submersions π_i : $QJ^2(M,N) \to J^1(M,N)$, $i = 1,2,3$, as follows:

$$\pi_1 z(u) = p_{TN}(zE_1(u)),\ \ \pi_1 z = (x^i, y^p, b_i^p)$$
$$\pi_2 z(u) = T_{p_N}(zE_2(u)),\ \ \pi_2 z = (x^i, y^p, c_i^p)$$
$$\pi_3 z(u) = p_2(zE_3(u)),\ \ \pi_3 z = (x^i, y^p, d_i^p)$$

where p_2 $VTN \to TN$ denotes the projection on the second factor of the identification $VTN \equiv TN \times_N TN$, $z \in QJ_x^2(M,N)$, $u \in T_xM$.

Lemma 1. *Let $h \in J_x^1(M,N)_y$. Then there exists unique $h_1, h_2 \in QJ_x^2(M,N)_y$ such that*

$$\pi_1(h_1) = h,\quad h_1 : (p_{TM})_x \to 0 \subset (p_{TN})_y,$$
$$\pi_2(h_2) = h,\quad h_2 : (T_{p_M})_x \to 0 \subset (T_{p_M})_y.$$

Proof. Let $h = (x^i, y^p, h_i^p)$. Consider $h_1 = (x^i, y^p, b_i^p, c_i^p, d_i^p, e_{ij}^p)$. By the condition $\pi_1(h_1) = h$, $b_i^p = h_i^p$. The coordinate form of the condition $h_1(p_{TM})_x = 0$ is the following one

$$c_i^p x_{01}^i = 0,\ \ e_{ij}^p x_{10}^i x_{01}^j + d_i^p x_{11}^i = 0 \text{ for any } x_{01}^i, x_{10}^i, x_{11}^i.$$

It holds $c_i^p = 0$, $d_i^p = 0$, $e_{ij}^p = 0$. Analogously, $h_2 = (x^i, y^p, 0, h_i^p, 0, 0)$. □

Corollary. *There are two embeddings $\kappa_1, \kappa_2 : J(M,N) \to QJ^2(M,N)$; $\kappa_1(h) = h_1$, $\kappa_2(h) = h_2$.*

This immediately gives

Proposition 1. *Let $u \in QJ^2(M,N)$, $c_1, c_2, c_3 \in R$. Then by the rules*

$$u \mapsto \kappa_i(c_1\pi_1(u) + c_2\pi_2(u) + c_3\pi_3(u)),\ \ i = 1,2$$

are determined two families of maps from $QJ^2(M,N)$ into $QJ^2(M,N)$ of the following coordinate forms

$$\kappa_1(c_1\pi_1(u) + c_2\pi_2(u) + c_3\pi_3(u)) = (x^i, y^p, c_1b_i^p + c_2c_i^p + c_3d_i^p, 0, 0, 0)$$
$$\kappa_2(c_1\pi_1(u) + c_2\pi_2(u) + c_3\pi_3(u)) = (x^i, y^p, 0, c_1b_i^p + c_2c_i^p + c_3d_i^p, 0, 0).$$

The canonical involution i_2 from T_2 into T_2 induces the involution $I_2 : u \mapsto i_2 u i_2$ on $QJ^2(M,N)$, $I_2(x^i, y^p, b_i^p, c_i^p, d_i^p, e_{ij}^p) = (x^i, y^p, c_i^p, b_i^p, d_i^p, e_{ij}^p)$.

Let $u \in QJ^2_x(M,N)_y$. Denote by u_1, u_2 the linear maps $(p_{TM})_x \mapsto (p_{TN})_y$, $(T_{p_M})_x \mapsto (T_{p_N})_y$, respectively, which are determined by u. Every $t \in R$ states the only two elements $U, \bar{U} \in QJ^2_x(M,N)$ such that $\pi_1 U \doteq \pi_1 u$, $U_1 = tu_1$ and $\pi_2 \bar{U} = \pi_2 u$, $\bar{U}_2 = tu_2$. In coordinates, $U = (x^i, y^p, b^p_i, tc^p_i, td^p_i, te^p_{ij})$ $\bar{U} = (x^i, y^p, tb^p_i, c^p_i, td^p_i, te^p_{ij})$. So t determines two mappings $\tau_1 : u \to U$, $\tau_2 : u \to \bar{U}$ indicated by the corresponding greek letters. For example if $t, c \in R$ then

$$\gamma_2(\tau_1(u)) = (x^i, y^p, cb^p_i, tc^p_i, ctd^p_i, cte^p_{ij}).$$

We get

Proposition 2. *In general, two real numbers t, c determined two transformations*

$$u \mapsto \gamma_2\tau_1(u)$$
$$u \mapsto \gamma_2\tau_1(I_2(u))$$

from $QJ^2(M,N)$ into $QJ^2(M,N)$.

We are interested in finding all so called natural transformations on $QJ^2(M,N)$. We will state that the only natural transformations on $QJ^2(M,N)$ are the ones described in Proposition 1 and 2.

Let us recall that the manifolds of all holonomic, semiholonomic, non-holonomic 2-jets from M into N are submanifolds of $QJ^2(M,N)$. For instance the equations of the submanifold in the holonomic case are of the form: $d^p_i = c^p_i = b^p_i, e^p_{ij} = e^p_{ji}$. The composition rule for jets extends on quasijets by the composition of maps.

Let H^2M, H^2N be the principal fibre bundle of all frames of the second order on M, N, respectively. Let G^2_m, G^2_n be the structure groups of H^2M, H^2N, respectively. For example, H^2M is the space of all 2-jets of local diffeomorphisms from $\mathbf{R}^m$ into M with target $0 \in \mathbf{R}^m$ and G^2_m is the set of all 2-jets $j^2_0\varphi$ of local diffeomorphisms $\varphi : \mathbf{R}^m \to \mathbf{R}^m$, $\varphi(0) = 0$. Then $H^2M \times H^2N$ is a principal fibre bundle with the structure group $G^2_n \times G^2_m$. It can be shown that $QJ^2(M,N) \to M \times N$ is associated with $H^2M \times H^2N$ with standard fibre $QJ^2_0(\mathbf{R}^m, \mathbf{R}^n)_0$.

Let $f : M \to \bar{M}$, $g : N \to \bar{N}$ be local diffeomorphisms. Let $u \in TT_xM$, $h \in QJ^2_x(M,N)_y$. Then by the rule

$$QJ^2(f \times g)(h)(u) := TTg \cdot h \cdot TTf^{-1}(u)$$

is defined the map $QJ^2(f \times g)$ from $QJ^2(M,N)$ into $QJ^2(\bar{M},\bar{N})$ which has the following coordinate form:

$$\begin{aligned}
&\bar{y}^p_{10} = g^p_q b^q_j f^j_i \bar{x}^i_{10}, \quad \bar{y}^p_{01} = g^p_q c^q_j f^j_i \bar{x}^i_{01},\\
(1)\qquad &\bar{y}^p_{11} = [g^p_{qr} b^q_k c^r_l f^k_i f^l_j + g^p_q(e^q_{kl} f^k_i f^l_j + d^q_k f^k_{ij})]\bar{x}^i_{10}\bar{x}^j_{01} +\\
&+ g^p_q d^q_j f^j_i \bar{x}^i_{11}.
\end{aligned}$$

Analogously as in [3] it can be proved that QJ^2 is a prolongation functor from category $\mathcal{M}_m \times \mathcal{M}_n$ into category of fibre bundles. Here $\mathcal{M}_n$ denotes category of all n-dimensional manifolds and their local diffeomorphisms.

Now in our case, a natural transformation from QJ^2 to QJ^2 can be formulated as a family of maps A such that if $h \in QJ^2(M,N)$ then

$$QJ^2(f \times g)A_{M\times N}(h) = A_{\bar{M}\times\bar{N}}QJ^2(f \times g)(h)$$

for any local diffeomorphisms $f : M \to \bar{M}$, $g : N \to \bar{N}$.

By the Krupka procedure [7] it can be shown that there is a bijection between the set of all natural transformation from QJ^2 to QJ^2 and the set of all $G^2_m \times G^2_n$-equivariant maps on $QJ^2_0(\mathbf{R}^m, \mathbf{R}^n)_0$.

Let $(b^p_i, c^p_i, d^p_i, e^p_{ij}) \in QJ^2_0(\mathbf{R}^m,\mathbf{R}^n)_0$, $(f^i_j, f^i_{jk}) \in G^2_m$, $(q^p_q, q^p_{qr}) \in G^2_n$. The equations (1) implies the following rule of an action of the group $G^2_m \times G^2_n$ on $QJ^2_0(\mathbf{R}^m,\mathbf{R}^n)_0$:

$$\begin{aligned} \bar{b}^p_i &= g^p_q b^q_j f^j_i, \ \bar{c}^p_i = g^p_q c^q_j f^j_i, \ \bar{d}^p_i = g^p_q d^q_j f^j_i, \\ \bar{e}^p_{ij} &= g^p_{qr} b^q_k c^r_l f^k_i f^l_j + g^p_q e^q_{kl} f^k_i f^l_j + g^p_q d^q_k f^k_{ij}. \end{aligned} \tag{2}$$

A map $\Phi : QJ^2_0(\mathbf{R}^m,\mathbf{R}^n)_0 \to QJ^2_0(\mathbf{R}^m,\mathbf{R}^n)_0$, $\bar{b}^p_i = \beta^p_i(b,c,d,e)$, $\bar{c}^p_i = \gamma^p_i(b,c,d,e)$, $\bar{d}^p_i = \delta^p_i(b,c,d,e)$, $\bar{e}^p_{ij} = \eta^p_{ij}(b,c,d,e)$, where for example b is a shortened denoting of b^q_j, is $G^2_m \times G^2_n$-equivariant if $g\Phi(h) = \Phi g(h)$ for every $h \in QJ^2_0(\mathbf{R}^m,\mathbf{R}^n)_0$ and every $g \in G^2_m \times G^2_n$. In the case of the functions β^p_i, this condition is of the form, (use (2)):

$$\begin{aligned} &g^p_q \beta^q_j(b,c,d,e) f^j_i = \\ &= \beta^p_i(g^p_q b^q_k f^k_j, g^p_q c^q_k f^k_j, g^p_q d^q_k f^k_j, g^p_{qr} b^q_k c^r_l f^k_i f^l_j + g^p_q e^q_{kl} f^k_i f^l_j + g^p_q d^q_k f^k_{ij}). \end{aligned} \tag{3}$$

Analogously in the case of γ^p_i, δ^p_i, η^p_{ij}.

In what follows it will be useful a modification of some Kolář and Janyška results on homogeneous functions, [4].

Lemma 2. *Let $f(x^i, y^p, \dots, z^s)$ be a smooth function defined on $\mathbf{R}^N$ such that for every positive real number k a homogenity condition*

$$k^d f(x^i, y^p, \dots, z^s) = f(k^a x^i, k^b y^p, \dots, k^c z^s) \tag{4}$$

where $a, b, \dots, c$ are positive real numbers, $d \in \mathbf{R}$, are satisfied. Then non-zero functions of this property are sums of homogeneous polynomials of degrees $(m,n,\dots,q)$ in variables $(x^i, y^p, \dots, z^s)$ such that

$$am + bn + \dots + cq = d. \tag{5}$$

If there is no positive integer solution $(m,n,\dots,q)$ of (5) then only $f = 0$ satisfies (4).

Using the canonical injective group homomorphism $G^1_m \times G^1_n \to G^2_m \times G^2_n$ and restricting to the subgroup of homotheties in G^1_m, ($f^i_j = k\delta^i_j, g^p_g = \delta^q_p, f^i_{jk} = 0, g^p_{qr} = 0$), we get for (3):

$$k\beta^p_i(b, c, d, e) = \beta^p_i(kb, kc, kd, k^2e).$$

By Lemma 2, the functions β^p_i are linear according to the variables b^p_i, c^p_i, d^p_i and independent on e^p_{ij}. It holds analogously in the cases of the functions γ^p_i, η^p_i. We have

$$\begin{aligned} \bar{b}^p_i &= k_1 b^p_i + k_2 c^p_i + k_3 d^p_i \\ \bar{c}^p_i &= k_4 b^p_i + k_5 c^p_i + k_6 d^p_i \\ \bar{d}^p_i &= k_7 b^p_i + k_8 c^p_i + k_9 d^p_i. \end{aligned} \tag{6}$$

By (2) the condition of the $G^2_m \times G^2_n$-equivariance gives for the functions η^p_{ij}

$$\begin{aligned} g^p_{qr}\beta^q_k\gamma^r_l f^k_i f^l_j + g^p_q\eta^q_{kl}f^k_i f^l_j + g^p_q\delta^q_k f^k_{ij} = \eta^p_{ij}(g^p_q b^q_k f^k_j, g^p_q c^q_k f^k_j, \\ g^p_q d^q_k f^k_j, g^p_{qr} b^q_k c^r_l f^k_i f^l_j + g^p_q e^q_{kl} f^k_i f^l_j + g^p_q d^q_k f^k_{ij}). \end{aligned} \tag{7}$$

In the case of the subgroup of homothetic maps on $\mathbf{R}^m$ or $\mathbf{R}^n$, respectively, (7) has the simple form

$$k\eta^p_{ij}(b, c, d, e) = \eta^p_{ij}(kb, kc, kd, ke)$$

or

$$k^2\eta^p_{ij}(b, c, d, e) = \eta^p_{ij}(kb, kc, kd, k^2e),$$

respectively. Then by Lemma 2 η^p_{ij} is linear according to e^p_{ij}, i.e. $e^p_{ij} = A^{pjk}_{ijq} e^q_{jk}$ and does not depend on b, c, d. Then (7), with respect to the subgroup $G^1_m \times G^1_n$, leads to finding all $G^1_m \times G^1_n$-equivariant linear maps on $\mathbf{R}^n \otimes \mathbf{R}^{m*} \otimes \mathbf{R}^{m*}$. By [10] such a map is of the form

$$\eta^p_{ij} = \alpha e^p_{ij} + \beta e^p_{ji}.$$

Let $\pi^2_1 : G^2_m \times G^2_n \to G^1_m \times G^1_n$ be the group homomorphism determined by the jet projection. Then (7) on Ker π^2_1 gives

$$\begin{aligned} g^p_{qr}(k_1 b^q_i + k_2 c^q_i + k_3 d^q_i)(k_4 b^r_j + k_5 c^r_j + k_6 d^r_j) + \\ + e^p_{ij} + e^p_{ji} + (k_7 b^p_k + k_8 c^p_k + k_9 d^p_k) f^k_{ij} = \\ = \alpha(g^p_{qr} b^q_i c^r_j + e^q_{ij} + d^p_k f^k_{ij}) + \beta(g^p_{qr} b^q_j c^r_i + e^q_{ji} + d^p_k f^k_{ij}). \end{aligned}$$

Comparing these two polynomials we get

$$\begin{aligned} &k_9 = \alpha + \beta,\ \alpha = k_1 k_5,\ \beta = k_2 k_4,\ k_1 k_4 = 0,\ k_1 k_6 = 0, \\ &k_2 k_5 = 0,\ k_2 k_6 = 0,\ k_3 k_4 = 0,\ k_3 k_5 = 0,\ k_3 k_6 = 0, \\ &k_7 = k_8 = 0. \end{aligned}$$

There are the following four different cases of all solutions of these equations

(i) $k_4 = k_5 = k_6 = 0$,

i.e. $\bar{b}_i^p = k_1 b_i^p + k_2 c_i^p + k_3 d_i^p$, $\bar{c}_i^p = 0$, $\bar{d}_i^p = 0$, $\bar{e}_{ij}^p = 0$

(ii) $k_1 = k_2 = k_3 = 0$,

i.e. $\bar{b}_i^p = 0$, $\bar{c}_i^p = k_4 b_i^p + k_5 c_i^p + k_6 d_i^p$, $\bar{d}_i^p = 0$, $\bar{e}_{ij}^p = 0$

(iii) $k_2 = k_3 = k_4 = k_6 = 0$,

i.e. $\bar{b}_i^p = k_1 b_i^p$, $\bar{c}_i^p = k_5 c_i^p$, $\bar{d}_i^p = k_1 k_5 d_i^p$, $\bar{e}_{ij}^p = k_1 k_5 e_{ij}^p$

(iiii) $k_1 = k_3 = k_5 = k_6 = 0$,

i.e. $\bar{b}_i^p = k_2 c_i^p$, $\bar{c}_i^p = k_4 b_i^p$, $\bar{d}_i^p = k_2 k_4 d_i^p$, $\bar{e}_{ij}^p = k_2 k_4 e_{ij}^p$.

Comparing (i) - (iiii) and Propositions 1 and 2 we verify the following theorem

Theorem. *There are the only following types of natural transformations of the functor QJ^2 into itself*

$$u \mapsto \kappa_i(k_1\pi_1(u) + k_2\pi_2(u) + k_3\pi_3(u)),\ i = 1,2$$
$$u \mapsto \gamma_2\tau_1(u)$$
$$u \mapsto \gamma_2\tau_1(I_2(u))$$

where γ_2, τ_1 are maps determined by real numbers c, t by the procedure described at Proposition 2.

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