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HEXAGONAL QUASIGROUPS

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Abstract. Hexagonal quasigroups are defined and it is shown that a hexagonal quasigroup is a special idempotent medial quasigroup. In hexagonal quasigroups a geometrical terminology and methods are introduced. A characterization of hexagonal quasigroups by commutative groups is obtained.

Key words. Hexagonal quasigroup.

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1. INTRODUCTION

We have obviously:

Lemma 1. *In every quasigroup* $(Q, .)$ *any two of the following three statements* (*one equivalence and two identities*) *are equivalent:*

$$ab = c \Leftrightarrow a = bc, \tag{1}$$

$$ab \,.\, a = b, \qquad a \,.\, ba = b. \tag{2}, (2)'$$

A quasigroup $(Q, .)$ which has the properties (1), (2) and (2)' is said to be *semi-symmetric*. Let us prove some more statements.

Lemma 2. *In a semisymmetric quasigroup* $(Q, .)$ *any two of the following three identities are equivalent;*

$$ab \,.\, cd = ac \,.\, bd \qquad (\textit{mediality}), \tag{3}$$

$$a(bc \,.\, d) = b(ac \,.\, d), (a \,.\, bc)\, d = (a \,.\, bd)\, c. \tag{4}, (4)'$$

Proof. The identities (4) and (4)' are mutually dual with respect to the exchange of the left and right factors in every product, while the identity (3) is dual to itself. Therefore, it is enough to prove the equivalence (3) $\Leftrightarrow$ (4). We shall use this kind of facilitations several times. According to (1), the equality $a(bc \,.\, d) = e$ is equivalent to $bc \, . \, d = ea$ and then to $ea \, . \, bc = d$. Similarly, the equality $b(ac \, . \, d) = e$ is equivalent to $eb \, . \, ac = d$. Therefore, (4) is valid iff it holds $ea \, . \, bc = eb \, . \, ac$. But, this is mediality.

Lemma 3. *A quasigroup satisfying one of the identities* (4), (4)′ *and the identity*

$$aa = a \qquad (\textit{idempotency}) \tag{5}$$

is semisymmetric.

Proof. Suppose the identities (4) and (5) are valid. From (4) with $c = d = a$ it follows, according to (5), the identity $a(ba \, . \, a) = ba$. By the substitution $ba \to b$ we obtain the identity (2)′.

From Lemma 2 and Lemma 3 we have immediately:

Corollary 1. *For an idempotent quasigroup the identities* (4) *and* (4)′ *are equivalent.*

A quasigroup $(Q, .)$ is said to be *hexagonal* if it is idempotent and if it has one of the properties (4) and (4)′, and then necessarily both properties.

Now from Lemma 2 and Lemma 3 it follows:

Corollary 2. *A quasigroup is hexagonal iff it is idempotent, medial and semisymmetric.*

Moreover we have:

Corollary 3. *A hexagonal quasigroup has all mentioned properties* (1) – (5), (2)′, (4)′.

Remark. A hexagonal quasigroup may be defined by only one identity. Such an identity is $a(bc \, . \, dd) = b(ac \, . \, d)$. It is obvious that this identity follows from (4) and (5). Conversely, from this identity with $b = a$, it follows at once $dd = d$, i.e. the property (5), and by (5) the considered identity implies (4).

2. EXAMPLES

Example 1. Let $(G, +)$ be a commutative group with an automorphism φ such that for every $a \in G$ it holds

$$(\varphi \circ \varphi)(a) - \varphi(a) + a = 0. \tag{6}$$

If . is a binary operation on the set G defined by

$$ab = a + \varphi(b - a), \tag{7}$$

then $(G, .)$ is a hexagonal quasigroup. Let us prove this statement! For every

$a, b \in G$ the equations $ax = b$ and $ya = b$ are equivalent (because of (7)) to the equations $a + \varphi(x - a) = b$ and $y + \varphi(a) - \varphi(y) = b$. First of these equations has the unique solution $x = a + \varphi^{-1}(b - a)$ and, owing to (6), the second equation can be written in the form $(\varphi \circ \varphi)(y) = \varphi(a) - b$. Therefore, it has the unique solution $y = \varphi^{-1}(\varphi^{-1}(\varphi(a) - b))$. The idempotency of the quasigroup $(G, .)$ is obvious. By (7) we obtain after some simplifications

$$a(bc \,.\, d) = a - \varphi(a) + \varphi(bc) + (\varphi \circ \varphi)(d) - (\varphi \circ \varphi)(bc).$$

But, because of (6) and (7) we have

$$\varphi(bc) - (\varphi \circ \varphi)(bc) = bc = b + \varphi(c) - \varphi(b),$$

and finally we obtain

$$a(bc \,.\, d) = a + b - \varphi(a) - \varphi(b) + \varphi(c) + (\varphi \circ \varphi)(d).$$

The symmetry of the right side of this equality in the variables a and b proves the identity (4).

In this paper we shall prove that this example is a characteristic example for hexagonal quasigroups, i.e. that every hexagonal quasigroup can be derived from a commutative group as in Example 1.

Example 2. Let $(F, +, .)$ be a field in which the equation

$$q^2 - q + 1 = 0 \tag{8}$$

has a solution q and let $*$ be the operation in the set F defined by

$$a * b = (1 - q)\, a + qb. \tag{9}$$

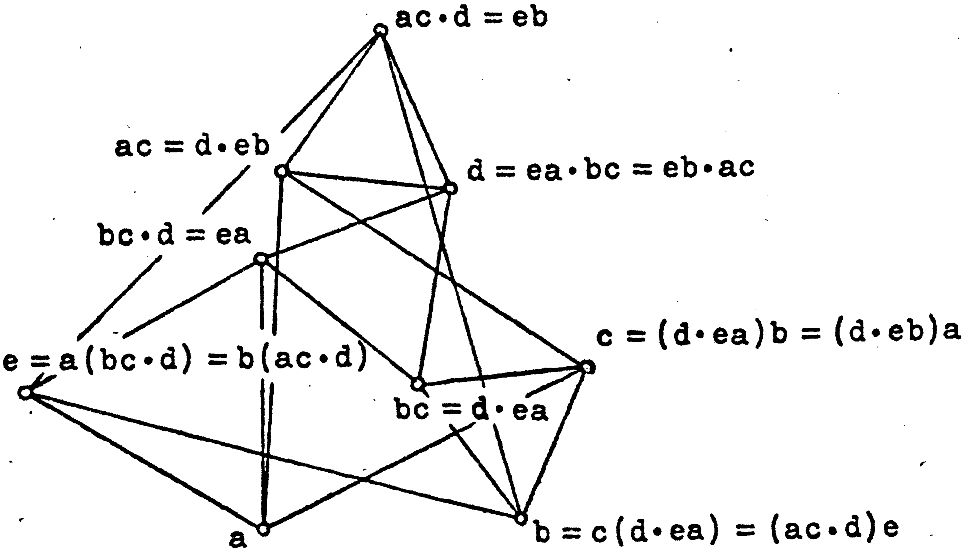
Then the identity $\varphi(a) = qa$ defines obviously an automorphism φ of the commutative group $(F, +)$. Because of (8) the identity (6) holds. The equality (9) can be written in the form $a * b = a + \varphi(b - a)$ and because of Example 1 it follows that $(F, *)$ is a hexagonal quasigroup.

Example 3. Let $(C, +, .)$ be the field of complex numbers and $*$ the operation on C defined by (9), where $q = e^{i\pi/3}$. Then the equality (8) holds and because of Example 2 it follows that $(C, *)$ is a hexagonal quasigroup. This quasigroup has a beautiful geometrical interpretation which motivates the study of hexagonal quasigroups. Let us regard the complex numbers as points of the Euclidean plane. For any point a we obviously have $a * a = a$, and for every two different points a, b the equality (9) can be written in the form

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0},$$

which means that the points a, b, $a * b$ are the vertices of a triangle directly similar to the triangle with the vertices 0, 1, q, i.e. the vertices of a positively oriented equilateral triangle. We can say that $a * b$ is the centre of the positively oriented regular hexagon with two adjacent vertices a and b, which justifies the name "hexagonal quasigroup". The hexagonal quasigroup $(C, *)$ was investigated in [2].

Every identity in the hexagonal quasigroup $(C, *)$ can be interpreted as a geometrical theorem which, of course, can be proved directly, but the theory of hexagonal quasigroups gives a better insight into the mutual relations of such theorems. For example, Figure 1 gives an illustration of the proof of Lemma 2 in the case of the quasigroup $(C, *)$ (here and in all the other figures we shall use the sign . instead of the sign $*$). In the same Figure 1 the identity (4)′ is also illustrated in the form $(d \,.\, ea)\, b = (d \,.\, eb)\, a$.



3. PARALLELOGRAMS

From now on let $(Q, .)$ be any hexagonal quasigroup.
At first let us prove the following theorem.

Theorem 1. *In a hexagonal quasigroup* $(Q, .)$ *the identities of left and right distributivity*

(10) $\qquad a \,.\, bc = ab \,.\, ac, \qquad ab \,.\, c = ac \,.\, bc \qquad$ (10)′

and the identities

$$(ab\,.\,c)\,d = b(c\,.\,da), \tag{11}$$

$$(ab\,.\,c)\,d = (a\,.\,bd)\,.\,ca, \qquad b(c\,.\,da) = ac\,.\,(bd\,.\,a) \tag{12}$$ (12)′

hold.

Proof. If we put $b = a$ in (3), then by (5) it follows $a\,.\,cd = ac\,.\,ad$, i.e. the identity (10). Now, let $(ab\,.\,c)\,d = e$. Because of (1) we obtain successively $ab\,.\,c = de$, $ab = c\,.\,de$, $b = (c\,.\,de)\,a$ and by (4)′ it follows $b = (c\,.\,da)\,e$. Owing to (1) we finally have $b(c\,.\,da) = e$, which proves (11). The identity (12) can be proved as follows:

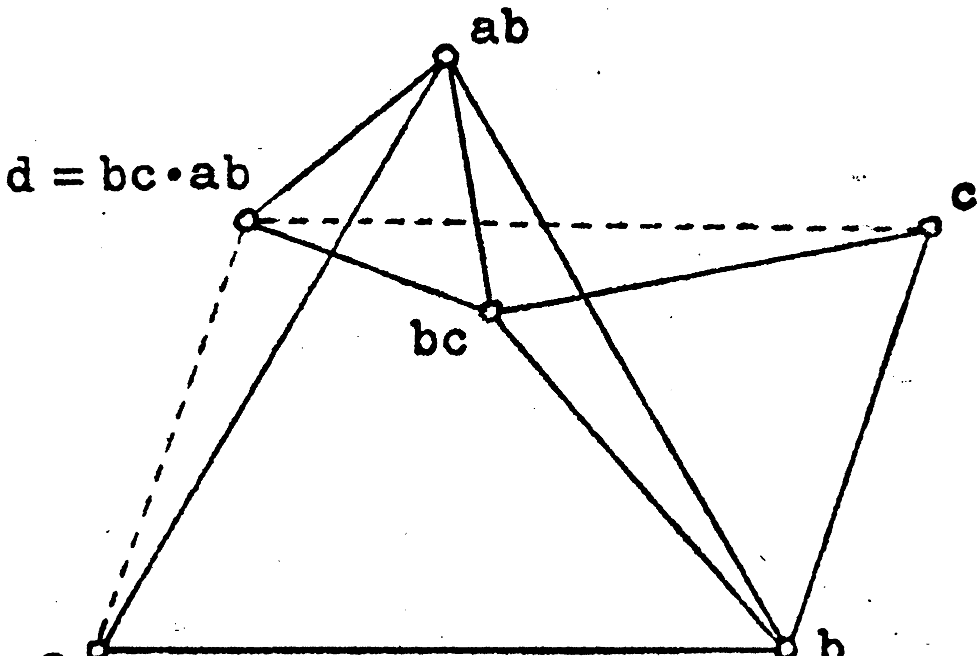
$$(ab\,.\,c)\,d \overset{(2)}{=} (ab\,.\,c)\,(ad\,.\,a) \overset{(3)}{=} (ab\,.\,ad)\,.\,ca \overset{(10)}{=} (a\,.\,bd)\,.\,ca.$$

Figure 1 also illustrates the proof of the identity (11) in the form $(ac\,.\,d)\,e = c(d\,.\,ea)$, where we have successively the equalities $(ac\,.\,d)\,e = b$, $ac\,.\,d = eb$, $ac = d\,.\,eb$, $c = (d\,.\,eb)\,a$, $c = (d\,.\,ea)\,b$, $c(d\,.\,ea) = b$.

Now, we shall introduce a geometrical terminology for the hexagonal quasigroup (Q, .), which is motivated by Example 3.

The elements of the set Q are said to be *points*.

Because of (3) we can apply all results of [4].



We shall say that the points a, b, c, d form a *parallelogram* and shall write $\operatorname{Par}(a, b, c, d)$ iff there are two points p and q such that $ap = bq$ and $dp = cq$ [4, Corollary 1]. In [4] it was proved that (Q, Par) is a *parallelogram space*, i.e. the quaternary relation $\operatorname{Par} \subset Q^4$ has the following properties:

1° *For any three points a, b, c there is one and only one point d such that* $\operatorname{Par}(a, b, c, d)$.

2° *If (e, f, g, h) is any cyclic permutation of (a, b, c, d) or of (d, c, b, a), then* $\operatorname{Par}(a, b, c, d)$ *implies* $\operatorname{Par}(e, f, g, h)$.

3° *From* $\operatorname{Par}(a, b, c, d)$ *and* $\operatorname{Par}(c, d, e, f)$ *it follows* $\operatorname{Par}(a, b, f, e)$.

Let us prove:

Theorem 2. $\operatorname{Par}(a, b, c, bc \cdot ab)$ *for any points a, b, c* (Fig. 2).

Proof. It is sufficient to prove the equalities $ap = bq$, $(bc \cdot ab)\, p = cq$ with $p = ba$, $q = b$. We have successively

$$a \cdot ba \overset{(2)'}{=} b \overset{(5)}{=} bb,$$

$$(bc \cdot ab) \cdot ba \overset{(3)}{=} (bc \cdot b)\,(ab \cdot a) \overset{(2)}{=} cb.$$

In Figure 2 we can see, by the way, how the fourth vertex of a parallelogram can be constructed (when three vertices are given) by means of the compasses only, where the compasses are used only for the drawing of circles and not for transfer of segments.

Because of 1° Theorem 2 gives an alternative definition of parallelograms:

$$\operatorname{Par}(a, b, c, d) \Leftrightarrow d = bc \cdot ab. \tag{13}$$

On the other hand, we can start with this definition (13) and prove the properties 1–3°. The property 1° is obvious. Further, let $\operatorname{Par}(a, b, c, d)$, i.e. $d = bc \cdot ab$. For the proof of 2° it suffices to prove $\operatorname{Par}(b, c, d, a)$ and $\operatorname{Par}(d, c, b, a)$, i.e. $cd \cdot bc = a$ and $cb \cdot dc = a$. Because of (3) it is necessary to prove only one of these two equalities. But, we have successively

$$cd \cdot bc = c(bc \cdot ab) \cdot bc \overset{(10)'}{=} (c \cdot bc) \cdot (bc \cdot ab)\,(bc) \overset{(2)}{=}$$
$$= (c \cdot bc) \cdot ab \overset{(2)'}{=} b \cdot ab \overset{(2)'}{=} a.$$

Now, let $\operatorname{Par}(a, b, c, d)$ and $\operatorname{Par}(c, d, e, f)$, i.e. $d = bc \cdot ab$ and $f = de \cdot cd$. It follows successively

$$ab \cdot ea \overset{(2)}{=} (bc \cdot ab)\,(bc) \cdot ea = (d \cdot bc) \cdot ea \overset{(3)}{=}$$
$$= de \cdot (bc \cdot a) \overset{(2)'}{=} de \cdot (bc)\,(b \cdot ab) \overset{(10)}{=} de \cdot b(c \cdot ab) \overset{(2)'}{=}$$
$$= de \cdot (c \cdot bc)\,(c \cdot ab) \overset{(10)}{=} de \cdot c(bc \cdot ab) = de \cdot cd = f,$$

i.e. $\operatorname{Par}(e, a, b, f)$, and by 2° we obtain $\operatorname{Par}(a, b, f, e)$.

We can also give direct proofs of some statements on parallelograms given in [4] using the equivalence (13). Let us prove two statements:

4° *For any* $a, b \in Q$ Par (a, a, b, b) *holds.*

5° *From* Par (a, b, d, e) and Par (b, c, e, f) *it follows* Par (c, d, f, a).

In fact, the statement Par (a, a, b, b) is a consequence of (5) and (2). For the proof of 5° we must show that $e = bd \,.\, ab$ and $f = ce \,.\, bc$ imply $df \,.\, cd = a$. But, we have successively

$$f = c(bd \,.\, ab) \,.\, bc \overset{(11)}{=} (bc \,.\, bd)\, a \,.\, bc \overset{(10)}{=}$$
$$= (b \,.\, cd)\, a \,.\, bc \overset{(3)}{=} (b \,.\, cd)\, b \,.\, ac \overset{(2)}{=} cd \,.\, ac$$

and therefore

$$df \,.\, cd = d(cd \,.\, ac) \,.\, cd \overset{(11)}{=} (cd \,.\, cd)\, a \,.\, cd \overset{(5)}{=} (cd \,.\, a) \,.\, cd \overset{(2)}{=} a.$$

Let us prove two more statements about the parallelograms in the hexagonal quasigroup $(Q, .)$.

Theorem 3. *For any points* a, b *and* c *we have* Par $(a, ab, ab \,.\, c, bc)$, Par $(c, ab, a \,.\, bc, bc)$ *and* Par $(a, c, ab \,.\, c, a \,.\, bc)$.

Proof. Because of [4, Th. 25] and the identities (2), (2)′, (5) and (10)′ we have implications

$$\left.\begin{array}{l}\text{Par}\,(b, ab, ab, b)\\ \text{Par}\,(ab, ab, c, c)\end{array}\right\} \Rightarrow \text{Par}\,(b \,.\, ab, ab \,.\, ab, ab \,.\, c, bc) \Rightarrow \text{Par}\,(a, ab, ab \,.\, c, bc),$$

$$\left.\begin{array}{l}\text{Par}\,(bc, a, a, bc)\\ \text{Par}\,(b, b, bc, bc)\end{array}\right\} \Rightarrow \text{Par}\,(bc \,.\, b, ab, a \,.\, bc, bc \,.\, bc) \Rightarrow \text{Par}\,(c, ab, a \,.\, bc, bc),$$

$$\left.\begin{array}{l}\text{Par}\,(a, ac, ac, a)\\ \text{Par}\,(a, a, bc, bc)\end{array}\right\} \Rightarrow \text{Par}\,(aa, ac \,.\, a, ac \,.\, bc, a \,.\, bc) \Rightarrow \text{Par}\,(a, c, ab \,.\, c, a \,.\, bc),$$

and the assumptions of these implications are consequences of 4° and 2°.

Theorem 4. *From* Par $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ $(i = 1, 2, 3, 4)$ *and* Par $(a_{1j}, a_{2j}, a_{3j}, a_{4j})$ $(j = 1, 2, 3)$ *it follows* Par $(a_{14}, a_{24}, a_{34}, a_{44})$.

Proof. We have equalities $a_{i2}a_{i3} \,.\, a_{i1}a_{i2} = a_{i4}$ $(i = 1, 2, 3, 4)$ and $a_{2j}a_{3j} \,.\, a_{1j}a_{2j} = a_{4j}$ $(j = 1, 2, 3)$ and therefore successively

$$a_{24}a_{34} \,.\, a_{14}a_{24} =$$
$$= (a_{22}a_{23} \,.\, a_{21}a_{22})(a_{32}a_{33} \,.\, a_{31}a_{32}) \,.\, (a_{12}a_{13} \,.\, a_{11}a_{12})(a_{22}a_{23} \,.\, a_{21}a_{22}) \overset{(3)}{=}$$
$$= (a_{22}a_{23} \,.\, a_{32}a_{33})(a_{21}a_{22} \,.\, a_{31}a_{32}) \,.\, (a_{12}a_{13} \,.\, a_{22}a_{23})(a_{11}a_{12} \,.\, a_{21}a_{22}) \overset{(3)}{=}$$
$$= (a_{22}a_{32} \,.\, a_{23}a_{33})(a_{21}a_{31} \,.\, a_{22}a_{32}) \,.\, (a_{12}a_{22} \,.\, a_{13}a_{23})(a_{11}a_{21} \,.\, a_{12}a_{22}) \overset{(3)}{=}$$
$$= (a_{22}a_{32} \,.\, a_{23}a_{33})(a_{12}a_{22} \,.\, a_{13}a_{23}) \,.\, (a_{21}a_{31} \,.\, a_{22}a_{32})(a_{11}a_{21} \,.\, a_{12}a_{22}) \overset{(3)}{=}$$

$$= (a_{22}a_{32} \cdot a_{12}a_{22})(a_{23}a_{33} \cdot a_{13}a_{23}) \cdot (a_{21}a_{31} \cdot a_{11}a_{21})(a_{22}a_{32} \cdot a_{12}a_{22}) =$$
$$= a_{42}a_{43} \cdot a_{41}a_{42} = a_{44}.$$

We can formulate:

Problem 1. Do the statements of 4° and of Theorem 4 hold in any parallelogram space, i.e. can these statements be proved by 1–3° only?

If a ternary operation ( ) on the set Q is defined by

$$(abc) = d \Leftrightarrow \mathrm{Par}\,(a, b, c, d), \tag{14}$$

then it is well-known (see [4]) that $(Q, (\,))$ is a laterally commutative heap after V. V. Vagner [3], i.e. the following identities hold:

$$((abc)\,de) = (a(bcd)\,e) = (ab(cde)), \tag{15}$$

$$(abc) = (cba), \tag{16}$$

$$(abb) = a. \tag{17}$$

Because of (14) Theorem 4 immediately implies:

Corollary 4. *For any points a_{ij} $(i, j = 1, 2, 3)$ we have*

$$((a_{11}a_{12}a_{13})(a_{21}a_{22}a_{23})(a_{31}a_{32}a_{33})) =$$
$$= ((a_{11}a_{21}a_{31})(a_{12}a_{22}a_{32})(a_{13}a_{23}a_{33})).$$

Problem 2. Does the statement of Corollary 4 hold in any laterally commutative heap, i.e. can this corollary be proved by the identities (15)–(17) only?

4. CHARACTERIZATION OF HEXAGONAL QUASIGROUPS

Let 0 be any given point. We define an addition of points by the equivalence

$$a + b = c \Leftrightarrow \mathrm{Par}\,(0, a, c, b).$$

Therefore, we have identically Par $(b, 0, a, a + b)$, which implies by (13)

$$a + b \Leftrightarrow 0a \,.\, b0. \tag{18}$$

In [4] it is proved that $(Q, +)$ is a commutative group with the neutral element 0. This fact can be proved directly by means of (18):

$$(a + b) + c \overset{(18)}{=} 0(0a \,.\, b0) \,.\, c0 \overset{(10)}{=} (0 \,.\, 0a)(0 \,.\, b0) \,.\, c0 \overset{(2)'}{=}$$
$$= (0 \,.\, 0a)\, b \,.\, 0(c0 \,.\, 0) \overset{(3)}{=} (0 \,.\, 0a)\, 0 \,.\, b(c0 \,.\, 0) \overset{(2)}{=}$$

$$= 0a\,.\,(0b\,.\,0)\,(c0\,.\,0) \overset{(10)'}{=} 0a\,.\,(0b\,.\,c0)0 \overset{(18)}{=} a + (b + c),$$

$$a + b \overset{(18)}{=} 0a\,.\,b0 \overset{(3)}{=} 0b\,.\,a0 \overset{(18)}{=} b + a,$$

$$a + 0 \overset{(18)}{=} 0a\,.\,00 \overset{(5)}{=} 0a\,.\,0 \overset{(2)}{=} a,$$

$$b = 0(0\,.\,0a) \overset{(1)}{\Leftrightarrow} b0 = 0\,.\,0a \overset{(1)}{\Leftrightarrow} 0a\,.\,b0 = 0 \overset{(18)}{\Leftrightarrow} a + b = 0.$$

Now, let λ_0, ϱ_0 be the translations of the quasigroup $(Q, .)$ defined by the point 0, i.e. the bijections defined by identities $\lambda_0(a) = 0a$, $\varrho_0(a) = a0$. We can prove:

Theorem 5. *The translations λ_0, ϱ_0 of the quasigroup $(Q, .)$ are two automorphisms of the group $(Q, +)$ such that $\varrho_0 \circ \lambda_0$ resp. $\lambda_0 \circ \varrho_0$ is the identity on the set Q and*

$$\lambda_0(a) + \varrho_0(a) = a \tag{19}$$

for every point a.

Proof. For any points a, b we have

$$\lambda_0(a + b) = 0(a + b) \overset{(18)}{=} 0(0a\,.\,b0) \overset{(10)}{=} (0\,.\,0a)\,(0\,.\,b0) \overset{(2)'}{=}$$

$$= (0\,.\,0a)\,b \overset{(2)}{=} (0\,.\,0a)\,(0b\,.\,0) \overset{(18)}{=} 0a + 0b = \lambda_0(a) + \lambda_0(b).$$

Dually, ϱ_0 is also an automorphism. By (2) and (2)′ it follows that $\varrho_0 \circ \lambda_0$ resp. $\lambda_0 \circ \varrho_0$ is the identity on the set Q. For any point a we get

$$\varrho_0(a) + \lambda_0(a) = a0 + 0a \overset{(18)}{=} (0\,.\,a0)\,(0a\,.\,0) \overset{(2)}{=} (0\,.\,a0)\,a \overset{(2)'}{=} aa \overset{(5)}{=} a.$$

Because of (2)′, (2) and (18) we obtain successively

$$ab = (0\,.\,a0)\,(0b\,.\,0) = a0 + 0b = \varrho_0(a) + \lambda_0(b),$$

which agrees with the well-known Toyoda's theorem (see [1], p. 33). But, by Theorem 5 it follows further

$$ab = a - \lambda_0(a) + \lambda_0(b) = a + \lambda_0(b - a).$$

Moreover, (19) implies $(\lambda_0 \circ \lambda_0)\,(a) + (\lambda_0 \circ \varrho_0)\,(a) = \lambda_0(a)$, i.e. $(\lambda_0 \circ \lambda_0)\,(a) - \lambda_0(a) + a = 0$ owing to the fact that $\lambda_0 \circ \varrho_0$ is the identity. Therefore, every hexagonal quasigroup $(Q, .)$ can be obtained from a commutative group $(Q, +)$ as in Example 1, i.e. it holds

Theorem 6. *There is a hexagonal quasigroup $(Q, .)$ iff there is a commutative group $(Q, +)$ and an automorphism φ of this group satisfying* (6). *Each of two binary operations + and . is defined by means of the other by the identities* (7) *and* (18), *where 0 is the neutral element of the group $(Q, +)$.*

In [4] it is proved that Par (a, b, c, d) iff $a + c = b + d$. We can prove this statement directly by means of (13) and (18), i.e. we can prove the equivalence of the equalities $0a \,.\, c0 = 0b \,.\, d0$ and $d = bc \,.\, ab$. Because of (1), the first equality is equivalent to $d0 = (0a \,.\, c0) \,.\, 0b$. But,

$$(0a \,.\, c0) \,.\, 0b \overset{(3)}{=} (0a \,.\, 0)\,(c0 \,.\, b) \overset{(2)}{=} a(c0 \,.\, b)$$

and the obtained equality is equivalent to $d = 0 \,.\, a(c0 \,.\, b)$ owing to (1). Finally, we have successively

$$0 \,.\, a(c0 \,.\, b) \overset{(11)}{=} (b0 \,.\, a) \,.\, c0 \overset{(10)'}{=} (ba \,.\, 0a) \,.\, c0 \overset{(3)}{=} (ba \,.\, c)\,(0a \,.\, 0) \overset{(2)}{=}$$
$$= (ba \,.\, c)\, a \overset{(2)}{=} (ba \,.\, c)\,(ba \,.\, b) \overset{(10)}{=} ba \,.\, cb \overset{(3)}{=} bc \,.\, ab.$$

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