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ON NONOSCILLATORY SOLUTIONS OF A LINEAR DEVIATED SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. The paper contains a certain sufficient condition under which there exists a T such that all nontrivial solutions of a linear deviated system of n ordinary differential equations have a bounded number of zeros of their components in the interval $[T, \infty)$. For this purpose we transform the considered system into a system of $n-1$ differential-integral equations.

Key words. Linear deviated system of differential equations, system of differential-integral equations, transformation of systems of equations, nonoscillatory solution.

MS Classification. 34 C 10.

1. INTRODUCTION

Nonoscillation of systems of ordinary differential equations is investigated by many authors. In particular, Z. Ratajczak [6] has given some criteria of nonoscillation of the linear systems. In other papers, there are considered some sufficient conditions in order that components of all nontrivial solutions of systems of differential equations have a bounded number of their zeros. Z. Butlewski [1, 2] has given sufficient conditions under which components of all nontrivial solutions of the linear homogeneous system of two and three differential equations have at most one and two zeros, respectively. In [5], these results have been extended to the linear homogeneous system of four differential equations. A generalization of this problem for the linear homogeneous system of n differential equations and an extension to the linear nonhomogeneous system of n differential equations are contained in [3] for $n = 2$, and in [4] for $n \geqq 3$.

The present paper contains an extension of these results to the linear system of n differential equations with deviating argument

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}(t)\, x_j(g(t)) + f_i(t), \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where

$$a_{ij}, f_i, g \in C^1[t_0, \infty) \qquad \text{for } i, j = 1, 2, \dots, n, \tag{1.2}$$

$$a_{1n}(t) \neq 0 \qquad \text{for } t \in [t_0, \infty), \tag{1.3}$$

$$\underset{t_1 \geq t_0}{\exists} \ \underset{t \geq t_1}{\forall} \ g(t) \geq t_0. \tag{1.4}$$

Let

$$T = \min \{t_1 : g(t) \geq t_0 \text{ for } t \geq t_1\}.$$

We give a sufficient condition under which the nontrivial first components of all solutions of the considered system have a bounded number of zeros in the interval $[T, \infty)$. Analogous conditions for the remaining components can be given.

A function f: $[\tau, \infty) \to R$ is called nonoscillatory if

$$\underset{\tau_1 \geq \tau}{\exists} \ \underset{t > \tau_1}{\forall} \ f(t) \neq 0.$$

2. Lemma. *Let the conditions* (1.2)–(1.4) *hold. Assume that there exists a solution* $(\xi_1, \xi_2, \dots, \xi_n)$ *of the linear homogeneous system*

$$y_i'(t) = \sum_{j=1}^{n} a_{ij}(t)\, y_j(g(t)), \qquad i = 1, 2, \dots, n, \tag{2.1}$$

corresponding to the system of differential equations (1.1) *such that* $\xi_1(t) \neq 0$ *for* $t \in [T, \infty)$. *Then the substitution*

$$\begin{aligned} x_1(t) &= \xi_1(t) \int_T^t z_1(s)\, ds, \\ x_i(t) &= \xi_i(t) \int_T^t z_1(s)\, ds + z_i(t), \qquad i = 2, 3, \dots, n, \end{aligned} \tag{2.2}$$

brings the system (1.1) *on the interval* $[T, \infty)$ *to the system of equations*

$$\begin{aligned} z_1'(t) = {} & b_{11}(t)\, z_1(t) + c_1(t)\, z_1(g(t)) + \sum_{j=2}^{n-1} [b_{1j}(t)\, z_j(g(t)) + c_j(t)\, z_j(g(g(t)))] + \\ & + h_1(t) + A_1(t) \int_t^{g(t)} z_1(s)\, ds + B(t) \int_{g(t)}^{g(g(t))} z_1(s)\, ds, \\ z_i'(t) = {} & b_{i1}(t)\, z_1(t) + \sum_{j=2}^{n-1} b_{ij}(t)\, z_j(g(t)) + h_i(t) + \\ & + A_i(t) \int_t^{g(t)} z_1(s)\, ds, \qquad i = 2, 3, \dots, n-1, \end{aligned} \tag{2.3}$$

where

$$b_{11} = \frac{1}{\xi_1} \left[\frac{a_{1n}'}{a_{1n}} \xi_1 - \sum_{k=1}^{n} a_{1k}(\xi_k \circ g) - \xi_1' \right],$$

$$
\begin{aligned}
b_{1j} &= \frac{1}{\xi_1}\left(a'_{1j} - \frac{a_{1j}a'_{1n}}{a_{1n}}\right),\\
b_{i1} &= \frac{a_{in}}{a_{1n}}\xi_1 - \xi_i,\\
b_{ij} &= a_{ij} - \frac{a_{1j}a_{in}}{a_{1n}},\\
c_1 &= \frac{(\xi_1 \circ g)\, g'}{(a_{1n} \circ g)\, \xi_1} \sum_{k=1}^{n} a_{1k}(a_{kn} \circ g),\\
c_j &= \frac{g'}{\xi_1} \sum_{k=2}^{n} a_{1k}\left[a_{kj} \circ g - \frac{(a_{1j} \circ g) \cdot (a_{kn} \circ g)}{a_{1n} \circ g}\right],\\
h_1 &= \frac{1}{\xi_1}\left\{-\frac{a'_{1n}}{a_{1n}} f_1 + g' \sum_{k=2}^{n} a_{1k}\left[f_k \circ g - \frac{a_{kn} \circ g}{a_{1n} \circ g}(f_1 \circ g)\right] + f'_1\right\},\\
h_i &= -\frac{a_{in}}{a_{1n}} f_1 + f_i,\\
A_1 &= \frac{1}{\xi_1} \sum_{k=1}^{n}\left[\left(a'_{1k} - \frac{a_{1k}a'_{1n}}{a_{1n}}\right) \cdot (\xi_k \circ g) + a_{1k}g'(\xi'_k \circ g)\right],\\
A_i &= \sum_{k=1}^{n-1}\left(a_{ik} - \frac{a_{1k}a_{in}}{a_{1n}}\right) \cdot (\xi_k \circ g),\\
B &= \frac{g'}{\xi_1} \sum_{k=2}^{n} a_{1k} \sum_{p=1}^{n-1}\left[a_{pk} \circ g - \frac{(a_{1p} \circ g) \cdot (a_{kn} \circ g)}{a_{1n} \circ g}\right] \cdot (\xi_p \circ g \circ g),
\end{aligned} \tag{2.4}
$$

for $i, j = 2, 3, \ldots, n-1$. *Moreover*

$$
\begin{aligned}
z_n(g(t)) = \frac{1}{a_{1n}(t)}\Big[&\xi_1(t)\, z_1(t) - \sum_{j=2}^{n-1} a_{1j}(t)\, z_j(g(t)) - f_1(t) +\\
&- \sum_{j=1}^{n} a_{1j}(t)\, \xi_j(g(t)) \int_t^{g(t)} z_1(s)\, ds\Big].
\end{aligned} \tag{2.5}
$$

Proof. Let $(\xi_1, \xi_2, \ldots, \xi_n)$ be the solution of the system (2.1) appearing in the assumption of the lemma. In the system of differential equations (1.1) we change variables according to the formulae (2.2) on the interval $[T, \infty)$. We get

$$
\begin{aligned}
\xi_1(t)\, z_1(t) &= \sum_{j=2}^{n} a_{1j}(t)\, z_j(g(t)) + f_1(t) +\\
&\quad + \sum_{j=1}^{n} a_{1j}(t)\, \xi_j(g(t)) \int_t^{g(t)} z_1(s)\, ds,\\
z'_i(t) &= -\xi_i(t)\, z_1(t) + \sum_{j=2}^{n} a_{ij}(t)\, z_j(g(t)) + f_i(t) +\\
&\quad + \sum_{j=1}^{n} a_{ij}(t)\, \xi_j(g(t)) \int_t^{g(t)} z_1(s)\, ds, \qquad i = 2, 3, \ldots, n.
\end{aligned} \tag{2.6}
$$

Hence the equation (2.5) follows. Differentiating the first equation of (2.6) and eliminating the variable z_n, we obtain the system

$$
\begin{aligned}
\xi_1(t)\, z_1'(t) = &\left[\frac{a_{1n}'(t)}{a_{1n}(t)}\xi_1(t) - \sum_{k=1}^{n} a_{1k}(t)\,\xi_k(g(t)) - \xi_1'(t)\right] z_1(t) + \\
&+ \left[\frac{\xi_1(g(t))}{a_{1n}(g(t))}\, g'(t) \sum_{k=1}^{n} a_{1k}(t)\, a_{kn}(g(t))\right] z_1(g(t)) + \\
&+ \sum_{j=2}^{n-1} \left\{\left[a_{1j}'(t) - \frac{a_{1j}(t)\, a_{1n}'(t)}{a_{1n}(t)}\right] z_j(g(t)) + \right. \\
&\left. + g'(t) \sum_{k=2}^{n} a_{1k}(t)\left[a_{kj}(g(t)) - \frac{a_{1j}(g(t))\, a_{kn}(g(t))}{a_{1n}(g(t))}\right] z_j(g(g(t)))\right\} - \\
&- \frac{a_{1n}'(t)}{a_{1n}(t)} f_1(t) + g'(t) \sum_{k=2}^{n} a_{1k}(t)\left[f_k(g(t)) - \frac{a_{kn}(g(t))}{a_{1n}(g(t))} f_1(g(t))\right] + f_1'(t) + \\
(2.7)\qquad &+ \sum_{k=1}^{n} \left\{\left[a_{1k}'(t) - \frac{a_{1k}(t)\, a_{1n}'(t)}{a_{1n}(t)}\right] \xi_k(g(t)) + a_{1k}'(t)\, g'(t)\, \xi_k'(g(t))\right\} \int_t^{g(t)} z_1(s)\, ds + \\
&+ g'(t) \sum_{k=2}^{n} a_{1k}(t) \sum_{p=1}^{n-1} \left[a_{kp}(g(t)) - \frac{a_{1p}(g(t))\, a_{kn}(g(t))}{a_{1n}(g(t))}\right] \xi_p(g(g(t))) \int_{g(t)}^{g(g(t))} z_1(s)\, ds, \\
z_i'(t) = &\left[\frac{a_{in}(t)}{a_{1n}(t)}\xi_1(t) - \xi_i(t)\right] z_1(t) + \\
&+ \sum_{j=2}^{n-1} \left[a_{ij}(t) - \frac{a_{1j}(t)\, a_{in}(t)}{a_{1n}}\right] z_j(g(t)) - \frac{a_{in}(t)}{a_{1n}(t)} f_1(t) + f_i(t) + \\
&+ \sum_{k=1}^{n-1} \left[a_{ik}(t) - \frac{a_{1k}(t)\, a_{in}(t)}{a_{1n}(t)}\right] \xi_k(g(t)) \int_t^{q(t)} z_1(s)\, ds, \qquad i = 2, 3, \dots, n-1,
\end{aligned}
$$

for $t \in [T, \infty)$. This is a system of the form (2.3) with the coefficients defined by the formulae (2.4).

3. Theorem. *Let $(u_1, u_2, \dots, u_n)$ be a solution of the system of differential equations* (1.1) *with the coefficients satisfying the conditions* (1.2)–(1.4). *Assume that there exists a solution $(\xi_1, \xi_2, \dots, \xi_n)$ of the system of differential equations* (2.1) *such that $\xi_1(t) \neq 0$ for $t \in [T, \infty)$. If for every solution $(v_1, v_2, \dots, v_{n-1})$ of the system of differential-integral equations* (2.7) *the function v_1 either is identically equal to zero or has at most m zeros in the interval (T, ∞), then the function u_1 either is identically equal to zero or has at most $m + 1$ zeros in the interval $[T, \infty)$.*

Proof. Take the solution $(u_1, u_2, \dots, u_n)$ of the system of differential equations (1.1) and the solution $(\xi_1, \xi_2, \dots, \xi_n)$ of the system (2.1) appearing in the assumptions of the theorem. Let $c = (u_1/\xi_1)(T)$ and

$$\eta_i = u_i - c\xi_i, \qquad i = 1, 2, \dots, n.$$

Hence $(\eta_1, \eta_2, \dots, \eta_n)$ is a solution of the system (1.1). Since

$$\eta_1(t) = \xi_1(t) \int_T^t (\eta_1/\xi_1)'(s)\, ds,$$

so, by Lemma 2, the function $(\eta_1/\xi_1)'$. is the first component of a solution of the system (2.7).

We have

$$u_1(t) = \xi_1(t)[c + \int_T^t (\eta_1/\xi_1)'(s)\, ds].$$

Denote

$$U(t) = c + \int_T^t (\eta_1/\xi_1)'(s)\, ds, \qquad t \in [T, \infty).$$

Differentiating the function U, we obtain $U' = (\eta_1/\xi_1)'$. Thus U' is the first component of a solution of the system (2.7). By the assumption, the function U' either is identically equal to zero or has at most m zeros in the interval (T, ∞). Hence the function U either is identically equal to zero or has at most $m + 1$ zeros in the interval $[T, \infty)$. Obviously, the function u_1 has the same property.

4. Corollary. *Let $(u_1, u_2, \dots, u_n)$ be a solution of the system of differential equations* (1.1) *with the coefficients satisfying the conditions* (1.2)–(1.4). *Assume that there exists a solution $(\xi_1, \xi_2, \dots, \xi_n)$ of the system of differential equations* (2.1) *such that the $\xi_1(t) \neq 0$ for $t \in [T, \infty)$. If for every solution $(v_1, v_2, \dots, v_{n-1})$ of the system of differential-integral equations* (2.7) *the function v_1 is either identically equal to zero or nonoscillatory, then the function u_1 has the same property.*

This corollary follows from the proof of Theorem 3, immediately.

5. Remark. Let the conditions (1.2)–(1.4) hold. Assume that there exists a solution $(\xi_1, \xi_2, \dots, \xi_n)$ of the system of differential equations (2.1) such that $\xi_1(t) \neq 0$ for $t \in [T, \infty)$. If the system of differential-integral equations (2.7) has no solutions with the first component identically equal to zero, then the system of differential equations (1.1) has the same property.

6. Remark. In particular, if the function g is identity, then $T = t_0$ and the system of differential-integral equations (2.7) reduces to the linear system of $n - 1$ differential equations

$$z_i'(t) = \sum_{j=1}^{n-1} d_{ij}(t)\, z_j(t) + h_i(t), \qquad i = 1, 2, \dots, n - 1,$$

where

$$d_{11} = \frac{1}{\xi_1}\left[\sum_{k=2}^{n} a_{1k}\left(\frac{a_{kn}}{a_{1n}}\xi_1 - \xi_k\right) + \frac{a_{1n}'}{a_{1n}}\xi_1 - \xi_1'\right],$$

$$
\begin{aligned}
d_{1j} &= \frac{1}{\xi_1}\left[\sum_{k=2}^{n} a_{1k}\left(a_{kj} - \frac{a_{1j}a_{kn}}{a_{1n}}\right) - \frac{a_{1j}a'_{1n}}{a_{1n}} + a'_{1j}\right],\\
d_{i1} &= \frac{a_{in}}{a_{1n}}\xi_1 - \xi_i,\\
d_{ij} &= a_{ij} - \frac{a_{1j}a_{in}}{a_{1n}},\\
h_1 &= \frac{1}{\xi_1}\left[-\left(\sum_{k=2}^{n}\frac{a_{1k}a_{kn}}{a_{1n}} + \frac{a'_{1n}}{a_{1n}}\right)f_1 + \sum_{k=2}^{n} a_{1k}f_k + f'_1\right],\\
h_i &= -\frac{a_{in}}{a_{1n}} f_1 + f_i
\end{aligned}
$$

for $i, j = 2, 3, \ldots, n - 1$.

7. Remark. The assumption (1.2) of our paper can be weakened to the form

$$
a_{ij}, f_i, g \in C[t_0, \infty] \qquad \text{for } i, j = 1, 2, \ldots, n,
$$

$$
a_{1j}, f_1, g \in C^1[T, \infty) \qquad \text{for} \qquad j = \begin{cases} 1, 2, \ldots, n, & \text{if} \quad g \neq id, \\ 2, 3, \ldots, n, & \text{if} \quad g = id. \end{cases}
$$

8. Example. For the system of differential equations

$$
\begin{aligned}
x'_1(t) &= a[x_1(g(t)) - x_2(g(t))] + f_1(t), \qquad a \neq 0,\\
x'_2(t) &= b(t)\,[x_1(g(t)) - x_2(g(t))] + f_2(t)
\end{aligned}
\tag{8.1}
$$

and the solution

$$
\xi_1(t) = 1, \qquad \xi_2(t) = 1
$$

of the linear homogeneous system corresponding to this system, the system (2.7) is the equation

$$
\begin{aligned}
z'_1(t) = {} & [a - b(g(t))]\, g'(t)\, z_1(g(t)) + \\
& + [b(g(t))\, f_1(g(t)) - af_2(g(t))]\, g'(t) + f'_1(t).
\end{aligned}
$$

By Corollary 4, if the coefficients guarantee nonoscillation of this equation, then the first component of any solution of the system (8.1) is either identically equal to zero or nonoscillatory.

9. Example. For the system of differential equations

$$
\begin{aligned}
x'_1(t) &= (2 - e^t)\, e^t x_1(t) - 2x_2(t) + x_3(t) + 2t,\\
x'_2(t) &= (1 - t)\, e^t x_1(t) + t x_2(t) + 1,\\
x'_3(t) &= (2 - t)\, e^{2t} x_1(t) + t x_3(t) + 4t
\end{aligned}
\tag{9.1}
$$

and the solution

$$\xi_1'(t) = 1, \xi_2(t) = e^t, \xi_3(t) = e^{2t}$$

of the linear homogeneous system corresponding to this system, the first equation of the system (2.7) is

$$z_1'(t) = (t + 2e^t - e^{2t})\, z_1(t) + 4t - 2t^2. \tag{9.2}$$

Solving this equation, we get

$$z_1(t) = c(t) \exp\left(\frac{t^2}{2} + 2e^t - \frac{e^{2t}}{2}\right),$$

where

$$c'(t) = (4t - 2t^2) \exp\left(-\frac{t^2}{2} - 2e^t + \frac{e^{2t}}{2}\right).$$

The function c' has two zeros in the interval $[-1, \infty)$. Hence the function c has at most three zeros in this interval and all solutions of the equation (9.2) have the same property. By Theorem 3, the first component of any solution of the system (9.1) has at most four zeros in the interval $[-1, \infty)$.

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