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ON THE NATURAL OPERATORS OF BIANCHI TYPE

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Abstract. In this paper we determine all natural operators $J^1 Y \to VY \otimes \wedge^3 T^*X$ for a fibred manifold $Y \to X$. We prove that the only operator of this type is the zero operator. This gives another proof of the Bianchi identity for generalized connections.

Key words. Natural operator, first jet prolongation, fibred manifold, generalized connection.

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In this paper we determine all natural operators $J^1 Y \to VY \otimes \wedge^3 T^*X$ for a fibred manifold $Y \to X$. We prove that the only operator of this type is the zero operator. This gives another proof of the Bianchi identity for generalized connections.

In the paper we use a form of Bianchi identity for generalized connections described in [1] by I. Kolař and his method for finding all natural operators of certain types elaborated in [2], [3].

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1. Let $p : Y \to X$ be a fibred manifold, $\dim Y = n + m$, $\dim X = n$, and let (x^i, y^p) be a fibre chart on Y. A generalized connection Γ on Y is a section $\Gamma : Y \to J^1 Y$ of the first jet prolongation with respect to target jet projection $\beta : J^1 Y \to Y$. In local fibred coordinates (x^i, y^p, y^p_i) on $J^1 Y$, the equations of Γ are:

$$\Gamma : y^p_i = F^p_i(x, y) \qquad \text{or} \qquad \mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i \tag{1}$$

with arbitrary smooth functions $F^p_i(x, y)$ on Y.

Let $\Gamma\xi$ denotes the horizontal lift of a vector field ξ on X. In local coordinates, if $\xi = \xi^i(x)\dfrac{\partial}{\partial x^i}$ then its horizontal lift is of the form:

$$\Gamma\xi = \xi^i(x)\frac{\partial}{\partial x^i} + F^p_i(x, y)\,\xi^i(x)\frac{\partial}{\partial y^p}. \tag{2}$$

The curvature of a connection Γ on Y is a map $\Omega_\Gamma : Y \to VY \otimes \wedge^2 T^*X$ determined by the difference: $[\Gamma\xi, \Gamma\zeta] - \Gamma([\xi, \zeta])$ for any vector fields ξ, ζ on X, [2]. In local coordinates, the curvature Ω_Γ of Γ is of the form:

(3) $$\Omega_\Gamma = \Omega^p_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \frac{\partial}{\partial y^p} = \left(\frac{\partial F^p_j}{\partial x^i} + F^q_i \frac{\partial F^p_j}{\partial y^q}\right)\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \frac{\partial}{\partial y^p}.$$

Curvature Ω_Γ determines a natural operator

$$\Omega : J^1 Y \to VY \otimes \wedge^2 T^*X, \qquad \Gamma \to \Omega_\Gamma.$$

Consider the flow prolongation $V\Gamma\xi$ on VY of the horizontal lift $\Gamma\xi$:

(4) $$V\Gamma\xi = \xi^i \frac{\partial}{\partial x^i} + F^p_i \xi^i \frac{\partial}{\partial y^p} + \frac{\partial F^p_i}{\partial y^q} Y^q \xi^i \frac{\partial}{\partial Y^p},$$

where (x^i, y^p, Y^p) are the induced coordinates on VY. The vector field $V\Gamma\xi$ on VY defines a horizontal lift with respect to a unique connection $V\Gamma$ on $VY \to X$ of the form:

(5) $$V\Gamma : \mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i, \qquad \mathrm{d}Y^p = \frac{\partial F^p_i}{\partial y^q} Y^q \mathrm{d}x^i.$$

We use a construction of the exterior differential of curvature Ω_Γ: $Y \to VY \otimes \otimes \wedge^2 T^*X$ with respect to the vertical lift $V\Gamma$, given in [1] in the form:

(6) $$\mathrm{d}_{V\Gamma}\Omega_F \colon Y \to VY \otimes \wedge^3 T^*X,$$

$$\mathrm{d}_{V\Gamma}\Omega_\Gamma = \left(\frac{\partial \Omega^p_{ij}}{\partial x^k} + F^r_k \frac{\partial \Omega^p_{ij}}{\partial y^r} - \frac{\partial F^p_k}{\partial y^q}\Omega^q_{ij}\right)\mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \frac{\partial}{\partial y^p}.$$

Evaluating (6), we obtain the following:

Proposition 1. [1] *(Bianchi identity) It holds*: $\mathrm{d}_{V\Gamma}\Omega_F = 0$.
The rule $\Gamma \to \mathrm{d}_{V\Gamma}\Omega_\Gamma$ *is a natural operator*

(7) $$A : J^1 Y \to VY \otimes \wedge^3 T^*X.$$

The Bianchi identity says that A *is the zero operator.*
The following Proposition determines all natural operators of Bianchi type:

Proposition 2. *The only natural operator* $J^1 Y \to VY \otimes \wedge^3 T^*X$ *is the zero operator.*

Proof: I. The second order natural operators $A : J^1 Y \to VY \otimes \wedge^3 T^*X$ are in bijection with the natural transformations $A : J^2(J^1 Y) \to VY \otimes \wedge^3 T^*X$ o with $G^3_{n,m}$ – equivariant maps of standard fibres

r(8) $$(J^1(\to J^m{:}\ _{+}fRR^n) \to R^{n+m}) \to R^m \otimes \wedge^3 R^{n*},$$

where $G^3_{n,m}$ is the group of all 3-jets at origin of the diffeomorphisms of R^{n+m}: $\bar{x}^i = \bar{x}^i(x)$, $\bar{y}^p = \bar{y}^p(x, y)$ preserving origin and the fibration p: $R^{n+m} \to R^n$.

Any section $\sigma : R^{n+m} \to J^1(R^{n+m} \to R^n)$ is of the form:

$$\sigma : (x^i, y^p) \to (x^i, y^p, y^p_i = \sigma^p_i(x, y)). \tag{9}$$

The canonical coordinates on the standard fibre $J^2_0 J^1$ of the second jet prolongation $J^2_0(J^1(R^{n+m} \to R^n) \to R^{n+m})$ are:

$$y^p_i,\ y^p_{ij},\ y^p_{iq},\ y^p_{ijk},\ y^p_{iqr},\ y^p_{ijq}. \tag{10}$$

The coördinates on $G^3_{n,m}$, which correspond to the values of the partial derivatives of functions $\bar{x}^i(x)$, $\bar{y}^p(x, y)$ at the origin are:

$$a^i_j,\ a^i_{jk},\ a^i_{jkl},\ a^p_i,\ a^p_{ij},\ a^p_{ijk},\ a^p_q,\ a^p_{ql},\ a^p_{qr},\ a^p_{qrl},\ a^p_{qlj},\ a^p_{qrs}. \tag{11}$$

Using standard evaluations we find the following action of $G^3_{n,m}$ on the standard fibre $J^2_0 J^1$:

$$\begin{aligned}
\bar{y}^p_i &= a^p_r y^r_l \tilde{a}^l_i + a^p_l \tilde{a}^l_i, \\
\bar{y}^p_{iq} &= a^p_r y^r_{ls} \tilde{a}^l_i \tilde{a}^s_q + a^p_{rs} y^r_l \tilde{a}^s_q \tilde{a}^l_i + a^p_{ls} \tilde{a}^l_i \tilde{a}^s_q, \\
\bar{y}^p_{ij} &= a^p_r y^r_{lm} \tilde{a}^l_i \tilde{a}^m_j + a^p_r y^r_{ls} \tilde{a}^l_i \tilde{a}^s_j + a^p_r y^r_l \tilde{a}^l_{ij} + \\
&\quad + a^p_{rs} y^r_l \tilde{a}^l_i \tilde{a}^s_j + a^p_{rm} y^r_l \tilde{a}^l_i \tilde{a}^m_j + a^p_l \tilde{a}^l_{ij} + \\
&\quad + a^p_{lr} \tilde{a}^l_i \tilde{a}^r_j + a^p_{lm} \tilde{a}^m_j \tilde{a}^l_i, \\
\bar{y}^p_{iqr} &= a^p_t y^t_{lsu} \tilde{a}^s_q \tilde{a}^u_r \tilde{a}^l_i + a^p_t y^t_{ls} \tilde{a}^s_{qr} \tilde{a}^l_i + \\
&\quad + a^p_{tu} y^t_{ls} \tilde{a}^u_r \tilde{a}^s_q \tilde{a}^l_i + a^p_{ts} y^t_l \tilde{a}^s_{qr} \tilde{a}^l_i + \\
&\quad + a^p_{ts} y^t_{lu} \tilde{a}^u_r \tilde{a}^s_q \tilde{a}^l_i + a^p_{tsu} y^t_l \tilde{a}^u_r \tilde{a}^s_q \tilde{a}^l_i + \\
&\quad + a^p_{ts} y^t_l \tilde{a}^s_{qr} \tilde{a}^l_i + a^p_{lst} \tilde{a}^l_i \tilde{a}^t_r \tilde{a}^s_q, \\
\bar{y}^p_{ijq} &= a^p_r y^r_{lms} \tilde{a}^s_q \tilde{a}^l_i \tilde{a}^m_j + a^p_r y^r_{lst} \tilde{a}^l_i \tilde{a}^t_q \tilde{a}^s_j + \\
&\quad + a^p_{rs} y^r_{lm} \tilde{a}^s_q \tilde{a}^l_i \tilde{a}^m_j + a^p_{rs} y^r_{lt} \tilde{a}^s_j \tilde{a}^t_q \tilde{a}^l_i + \\
&\quad + a^p_{rt} y^r_{ls} \tilde{a}^l_i \tilde{a}^t_q \tilde{a}^s_j + a^p_r y^r_{ls} \tilde{a}^l_i \tilde{a}^s_{jq} + \\
&\quad + a^p_r y^r_{lt} \tilde{a}^t_q \tilde{a}^l_{ij} + a^p_{mr} y^r_{ls} \tilde{a}^s_q \tilde{a}^l_i \tilde{a}^m_j + \\
&\quad + a^p_{rt} y^r_l \tilde{a}^t_q \tilde{a}^l_{ij} + a^p_{rs} y^r_l \tilde{a}^s_{jq} \tilde{a}^l_i + \\
&\quad + a^p_{rst} y^r_l \tilde{a}^s_j \tilde{a}^t_q \tilde{a}^l_i + a^p_{mrs} \tilde{a}^s_q y^r_l \tilde{a}^l_i \tilde{a}^m_j + \\
&\quad + a^p_{ls} \tilde{a}^l_{ij} \tilde{a}^s_q + a^p_{lr} \tilde{a}^l_i \tilde{a}^r_{jq} + a^p_{lrs} \tilde{a}^l_i \tilde{a}^r_j \tilde{a}^s_q + \\
&\quad + a^p_{mlr} \tilde{a}^m_j \tilde{a}^l_i \tilde{a}^r_q, \\
\bar{y}^p_{ijk} &= a^p_r y^r_{lmn} \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k + \ldots
\end{aligned} \tag{12}$$

where $\tilde{a} = a^{-1}$ means the inverse element in $G^3_{n,m}$.
Any $G^3_{n,m}$ – equivariant map $f : J^2_0 J^1 \to R^m \otimes \Lambda^3 R^{n*}$ is the composition of a $G^3_{n,m}$ – equivariant map

$g : J^2_0 J^1 \to R^m \otimes \otimes\, ^3R^{n*}$ and of the alternation alt: $R^m \otimes \otimes\, ^3R^{n*} \to R^m \otimes \wedge\, ^3R^{n*}$. $G^3_{n,m}$ acts on the standard fibre $R^m \otimes \otimes\, ^3R^{n*}$ by:

$$\bar{z}^p_{ijk} = a^p_q z^q_{lmn} \tilde{a}^l_i \tilde{a}^m_j \tilde{a}^n_k. \tag{13}$$

Let a map $g : J^2_0 J^1 \to R^m \otimes \otimes\, ^3R^{n*}$ have the coordinate expression:

$$z^p_{ijk} = g^p_{ijk}(y^p_i, y^p_{iq}, y^p_{ij}, y^p_{iqr}, y^p_{ijq}, y^p_{ijk}). \tag{14}$$

Consider first equivariancy of g with respect to the base homoteties: $\tilde{a}^i_j = k\delta^i_j$, $a^p_q = \delta^p_q$ and all others a's vanishing. This gives a homogeneity condition:

$$k^3 g^p_{ijk} = g^p_{ijk}(ky^p_i, ky^p_{iq}, k^2 y^p_{ij}, ky^p_{iqr}, k^2 y^p_{ijq}, k^3 y^p_{ijk}). \tag{15}$$

Since g^p_{ijk} are globally defined smooth functions, (15) implies that g^p_{ijk} is a polynomial which can consists of some expressions: linear in y^p_{ijk}, bilinear in: (y^p_i, y^p_{ij}), (y^p_i, y^p_{ijq}), (y^p_{iq}, y^p_{ij}), (y^p_{iq}, y^p_{ijq}), (y^p_{iqr}, y^p_{ij}), (y^p_{iqr}, y^p_{ijq}), trilinear in $(y^p_i, y^p_{iq}, y^p_{iqr})$, (y^p_i, y^p_i, y^p_{iq}), $(y^p_i, y^p_i, y^p_{iqr})$, $(y^p_i, y^p_{iq}, y^p_{iq})$, $(y^p_i, y^p_{iqr}, y^p_{iqr})$, $(y^p_{iq}, y^p_{iq}, y^p_{iqr})$, $(y^p_{iq}, y^p_{iqr}, y^p_{iqr})$.

Equivariancy with respect to the fibres homoteties: $\tilde{a}^i_j = \delta^i_j$, $a^p_q = k\delta^p_q$, and with all others a's vanishing, gives:

$$kg^p_{ijk} = g^p_{ijk}\left(ky^p_i, y^p_{iq}, ky^p_{ij}, \frac{1}{k} y^p_{iqr}, y^p_{ijq}, ky^p_{ijk}\right). \tag{16}$$

Combining equivariancy with respect to fibres and base homoteties (16), (15), we get that g^p_{ijk} is a polynomial consisting some expressions: linear in y^p_{ijk}, bilinear in (y^p_i, y^p_{ijq}), (y^p_{iq}, y^p_{ij}), and trilinear in $(y^p_i, y^p_{iq}, y^p_{iq})$, $(y^p_i, y^p_i, y^p_{iqr})$. We shall use the fact that every $G^1_n \times G^1_m$ – invariant tensor P is a linear combination of the products $Q \otimes T$, where Q is G^1_n – invariant tensor and T is G^1_m – invariant tensor, [3]. By symmetry of the following expressions:

$$y^p_{ijk}, y^p_i y^q_{jq} y^r_{kr}, y^p_i y^r_{jq} y^q_{kr}, y^q_i y^r_j y^p_{kqr}, y^r_i y^q_j y^p_{kqr}, \tag{17}$$

their alternations are equal to zero. Thus, the map

$f : J^2_0 J^1 \to R^m \otimes \wedge\, ^3R^{n*}$ has the form:

$$\begin{aligned} f^p_{ijk} = {} & \beta_1 y^p_{[i} y^q_{jk]q} + \beta_2 y^q_{[i} y^p_{jk]q} + \gamma_1 y^p_{[iq} y^q_{jk]} + \\ & + \gamma_2 y^q_{[iq} y^p_{jk]} + \lambda_1 y^r_{[i} y^p_{jq} y^q_{k]r} + \lambda_2 y^r_{[i} y^q_{jq} y^p_{k]r} + \\ & + \mu y^p_{[i} y^q_j y^r_{k]qr}. \end{aligned} \tag{18}$$

Considering equivariancy of f with respect to the subgroup: $\tilde{a}^i_j = \delta^i_j$, $a^p_q = \delta^p_q$ and all others a's arbitrary, we get a sum including among others the following independent terms and the sum is equal zero:

$$\begin{gathered} \beta_1 a^p_{[i} y^q_{jk]q}, \beta_2 a^q_{[i} y^p_{jk]q}, \gamma_1 y^p_{[iq} a^q_{jk]}, \gamma_2 y^q_{[iq} a^p_{jk]}, \\ \lambda_1 y^r_{[i} a^p_{jq} y^q_{k]r}, \lambda_2 y^r_{[i} a^q_{jq} y^p_{k]r}, \mu y^p_{[i} a^q_j y^r_{k]qr}. \end{gathered} \tag{19}$$

Hence, we get: $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \lambda_1 = \lambda_2 = \mu = 0$. In this way we obtain the zero map $f : J_0^2 J^1 \to R^m \otimes \wedge^3 R^{n*}$ only.

II. Assume, we have an r-th order natural operator $A : J^1 Y \to VY \otimes \wedge^3 T^* X$. It corresponds to $G_{n,m}^{r+1}$ – equivariant maps between the standard fibres $f : J_0^r J^1 \to R^m \otimes \wedge^3 R^{n*}$. Denote by $y_{i\alpha\beta}^p$ the partial derivatives of y_i^p with respect to a multiindex α in x^i and a multiindex β in y^p. Any map $f : J_0^r J^1 \to R^m \otimes \otimes^3 R^{n*}$ is of the form:

$$z_{ijk}^p = f_{ijk}^p(y_{i\alpha\beta}^p), \qquad |\alpha| + |\beta| \leqq \nu.$$

Using base homoteties we obtain a homogeneity condition:

$$k^3 f_{ijk}^p = f_{ijk}^p(k^{1+|\alpha|} y_{i\alpha\beta}^p). \tag{20}$$

This implies that f_{ijk}^p is independent on $y_{i\alpha\beta}^p$ for $|\alpha| \geqq 3$ and is linear in $y_{ijk\beta}^p$, bilinear in $(y_{i\beta}^p, y_{ij\beta}^p)$ and trilinear in $y_{i\beta}^p$.

Using fibre homoteties, we get:

$$k f_{ijk}^p = f_{ijk}^p(k^{1-|\beta|} y_{i\alpha\beta}^p). \tag{21}$$

Hence, f_{ijk}^p is independet on $y_{i\alpha\beta}^p$ for $|\beta| \geqq 1$. Both (20), (21) homogeneity conditions implies that f_{ijk}^p is linear in y_{ijk}^p, bilinear in (y_i^p, y_{ijq}^p), (y_{iq}^p, y_{ij}^p) and trilinear in $(y_i^p, y_{iq}^p, y_{iq}^p)$, $(y_i^p, y_i^p, y_{iqr}^p)$. Hence the r-th order natural operators are reduced to the case I for every $r > 2$. By Slovak theorem [5] every operator of this type has finite order. This proves Proposition 2.

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