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THE HEAT SEMIGROUP ACTING ON TENSORS OR DIFFERENTIAL FORMS WITH VALUES IN VECTOR BUNDLE

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Abstract. We consider the heat semigroup $\{e^{-t\Delta}\}$ on noncompact complete Riemannian manifolds where Δ is acting on tensors or differential forms with values in a Riemannian vector bundle For $\frac{3}{2} \leqq p \leqq 3$ $e^{-t\Delta}$ is a strongly continuous contractive semigroup in L_p if Δ acts on tensors with values in E. The same holds for Δ on differential forms if the curvature endomorphism is nonnegative.

Key words. Laplace operator, heat operator, heat semigroup.

MS Clasification. 58 G 11, 58 G 25, 58 G 30.

1. INTRODUCTION

Let Δ denote the Laplace operator acting on q-forms on a complete Riemannian manifold (M^n, g). Then by the spectral theorem in L_2 the heat semigroup $\{e^{-t\Delta}\}$ is well defined. If M^n is compact then $e^{-t\Delta}$ is of trace class and is an integral operator with a smooth kernel $H(x, y, t)$. On open manifolds at the first instance these statements and many others break completely down. On the other hand, good properties of the heat operator imply good properties for Green's operator, and for many other purposes the heat operator plays an important role, for instance for trace formulas, L_2-index theory, probabilistic viewpoints etc. As a matter of fact, the theory of the heat operator and the heat semigroup on open manifolds is much more complicated than in the compact case. In [5] Strichartz presented a fundamental approach to the heat semigroup in L_p, as long as possible without curvature conditions. By this approach became clear that for the Laplace operator acting on functions there is a rich meaningful theory in L_p, for Δ acting on tensors the situation becomes much more complicated, and Δ acting on q-forms in [5] is completely excluded. The main purpose of this paper is to make a small step foreward, considering tensors and differential forms with values in a vector bundle E (as necessary for the sake of mathematical physics), where we impose in the case of differential forms curvature conditions. Then, for $\frac{3}{2} \leqq p \leqq 3$, $\{e^{-t\Delta}\}$ is again a strongly continuous contractive semigroup, $e^{-t\Delta}u$ satisfies the heat equation and uniqueness theorems holds. As a corollary, we get some results for usual differential forms on Riemannian manifolds. Our considerations and computations were essentially inspired by the paper of Strichartz.

2. THE HEAT SEMIGROUP $\{e^{-t}\}$

Suppose (M^n, g) open, complete, $(E, h) \to M$ a Riemannian vector bundle with an associated metric connection ∇^h. The Levi-Civita connection ∇^g and ∇^h define metric connections ∇ in all tensor bundles $T_q^r \otimes E$ over M. Denote by $\Omega^0(T_q^r \otimes E)$ tensors with values in E, in particular by $\Omega^0(\Lambda^q T^*M \otimes E) = \Omega^q(E)$ the q-forms with values in E, by $\Omega_0^0(T_q^r \otimes E)$ resp. $\Omega_0^q(E)$ those with compact support. If $1 \leqq p < \infty$ then ${}^p\Omega^0(T_q^r \otimes E)$ is the Banach space of all measurable (r, q)-tensor fields u with values in E such that

$$^p\| u \| = (\int_M | u |_x^p \operatorname{dvol}_x)^{1/p} < \infty.$$

Here $| u |_x$ is the fibre wise norm in $(T_q^r)_x \otimes E_x$. For $p = 2$ the ${}^2\Omega^0(T_q^r \otimes E)$ are Hilbert spaces with the scalar product

$$\langle u, u' \rangle = \int_M (u, u')_x \operatorname{dvol}_x.$$

If ∇^* is adjoint to ∇ with respect to $\langle , \rangle$ then the (trace) Laplace operator Δ: $\Omega^0(T_q \otimes E) \to \Omega^0(T_q^r \otimes E)$ is defined by $\Delta = \nabla^*\nabla$. As well known, for q-forms with values in E there is a second Laplace operator $\Delta = d\delta + \delta d$: $\Omega^q(E) \to \Omega^q(E)$, δ adjoint to the exterior differential d with respect to $\langle , \rangle$. The latter Laplace operator for q-forms is connected with the first by the Weitzenboeck identity

$$\Delta = \nabla^*\nabla + \varrho. \tag{2.1}$$

Here

$$\varrho(u)_{X_1, \dots, X_q} = \sum_{k=1}^{q} \sum_{j=1}^{n} (R_{e_j, X_k} u)_{X_1, \dots, X_{k-1}, e_j, X_{k+1}, \dots, X_q}, \tag{2.2}$$

$X_1, \dots, X_q \in T_xM$, $e_1, \dots, e_n$ an orthonormal base in T_xM, R the curvature endomorphism, i.e. $(R_{U,W}u)_{X_1, \dots, X_q} = R_{U,W}^E(u_{X_1, \dots, X_q}) - \sum_{k=1}^{q} u_{X_1}, \dots, R_{U,W}^M(X_k), \dots, X_q$, R^E resp. R^M the curvature tensors of E resp. M. For $q = 1$, $u \in \Omega^1(E)$, there holds

$$(\Delta u)_X = (\nabla^*\nabla u)_X + u_{\mathrm{Ric}(x)} + \mathscr{R}(u)_X, \tag{2.3}$$

where $\mathrm{Ric}(X) = -\sum_{j=1}^{n} R^M_{e_j,X}e_j$ equals to the Ricci transformation of X and $\mathscr{R}(u)_X =$ $= \sum_{j=1}^{n} R^E_{e_j,X}(u_{e_j})$.

On open, complete manifolds both Laplace operators for $p = 2$ are essentially self-adjoint on tensor fields resp. differential forms with compact support. In what follows we write again Δ instead of $\bar{\Delta}$. Therefore by the spectral theorem the operator

$$e^{-t\Delta} = \int_0^\infty e^{-t\lambda}\,\mathrm{d}E_\lambda$$

is well defined in $L_2(= {}^2\Omega^0(T^r_q \otimes E)$ or $= {}^2\Omega^q(E))$. $\{e^{-t\Delta}\}_{0\leq t<\infty}$ is a semigroup and it is not hard to show that $e^{-t\Delta}u$ satisfies the heat equation. We put the following questions.

1. Under which conditions is $\{e^{-t\Delta}\}_{0\leqq t<\infty}$ (if defined in L_p) a strongly continuous contractive semigroup?
2. Does $e^{-t\Delta}u$ satisfy in L_p the heat equation?
3. Are there several semigroups in L_p satisfying the heat equation (uniqueness problem)?
4. How is the heat semigroup related to the initial value problem for the heat equation?

Some partial answers to this questions extending [5] shall be given in section 4 of this paper.

3. DISSIPATIVITY OF THE LAPLACE OPERATORS

In this section we prepare the partial answers to the above questions establishing in some cases the dissipativity and some other good properties for the Laplace operators. We recall some facts from the theory of semigroups. If X is a Banach space, $x \in X$, $x \neq 0$, then there exists by the Hahn–Banach theorem an element $x^* \in X^*$ such that $\| x^* \| = \| x \|$ and $\langle x^*, x\rangle = \| x \|^2$. We call such an element a normalized tangent functional. Taking $X = {}^p\Omega^0(T^r_q \otimes E)$, $X^* = {}^{p'}\Omega^0(T^r_q \otimes E)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$, $0 \neq u \in X$, such an tangent functional can easily explicitly be written down by $c\,|u|^{p-2}u$ with $c = {}^p\| u \|^{-p/p'}$. Suppose $A\colon D_A \to X$, $D_A \subset X$ densely defined. A is said to be dissipative if for every $x \in D_A$ there exists a normalized tangent functional such that $\langle x^*, Ax\rangle \leqq 0$. The closure of a dissipative operator is dissipative. If X is a Hilbert space, $A\colon D_A \to X$ is symmetric and $\langle Ax, x\rangle \leqq 0$ for all $x \in D_A$ then surely A is dissipative.

A C^0-semigroup $\{T_t\}_{t\geqq 0}$ of bounded linear operators $T_t \in L(X, X)$, X a Banach space, is called a contraction semigroup if $\| T_t \| \leqq 1$, $0 \leqq t < \infty$. The infinitesimal generator A of a semigroup $\{T_t\}_{t\geqq 0}$ is defined by

$$A := s - \lim_{t\to 0+} (T_t - I)/t.$$

Lemma 3.1. *A closed, densely defined operator $A\colon D_A \to X$ is the infinitesimal generator of a contraction semigroup if and only if A is dissipative and* $\operatorname{im}(\mu - A) = X$ *for some $\mu > 0$* ([4]). □

The key for the existence of the heat semigroup $\{e^{-t\Delta}\rangle_{t\geqq 0}$ is to establish the conditions of the preceding lemma. We start with the condition $\operatorname{im}(\mu - (-\Delta)) = X$ *for some $\mu > 0$, or, equivalently,* $\operatorname{im}(\mu - \Delta) = X$ *for some $\mu < 0$.*

Lemma 3.2. *Suppose $1 < p \leqq q < 3$, $u \in {}^p\Omega^0(T^r_s \otimes E) + {}^q\Omega^0(T^r_s \otimes E)$ and $\Delta u = \mu u$ for some $\mu < 0$. Then u is identically zero.*

Proof. The following fact for complete open manifolds is standard. Suppose $x_0 \in M$, $B_r = B_r(x_0)$ the metric ball centered at x_0. Then there exists for every $s > r > 0$ a Lipschitz continuous and hence almost everywhere differentiable function $\Phi_{r,s}$ on M such that

a. $0 \leqq \Phi_{r,s}(x) \leqq 1$,
b. $\operatorname{supp} \Phi_{r,s} \subseteq B_s(x_0)$,
c. $\Phi_{r,s} = 1$ on $B_r(x_0)$,
d. $\lim\limits_{r,s\to\infty} \Phi_{r,s} = 1$,
e. $|d\Phi_{r,s}(x)| = |\nabla\Phi_{r,s}(x)| \leqq \dfrac{c}{s-r}$ almost everywhere.

Let Φ be one of the above functions $\Phi_{r,s}$ and $h(t)$ a smooth function $\geqq 0$ chosen as follows: $h(t) = t^{p-2}$ for $t \geqq 1$, $h(t) = (\varepsilon + t^2)^{(q-2)/2}$ for $t \leqq 1 - \varepsilon$, ε a small positive parameter. Then the equations $t \,.\, h'(t) = (p-2) \,.\, t^{p-2}$ for $t \geqq 1$ and $t \,.\, (\varepsilon + t^2)^{(q-2)/2-1} (q-2) \,.\, t = t \,.\, h'(t)$ for $t \leqq 1 - \varepsilon$ and $p, q < 3$ show that $|t \,.\, h'(t)| \leqq \eta \,.\, h(t)$ with some η, $0 < \eta < 1$. Between $1 - \varepsilon$ and 1 $h(t)$ can be arranged in such a manner that this latter inequality remains valid. For Δ acting on tensor fields with values in E we have $\Delta = \nabla^*\nabla$ which implies

$$\begin{aligned} \mu < \Phi^2 h(|u|)\, u, u\rangle &= \langle \Phi^2 h(|u|)\, u, \Delta u\rangle = \langle \Phi^2 h(|u|)\, u, \nabla^*\nabla u\rangle = \\ &= \langle \Phi^2 h(|u|)\, \nabla u, \nabla u + \langle \Phi^2 h'(|u|)\, (\nabla |u|) \otimes u, \nabla u\rangle + \\ &\quad + 2\langle \Phi h(|u|)\, \nabla\Phi \otimes u, \nabla u\rangle. \end{aligned} \tag{3.1}$$

According to the inequality $|\nabla |u|| \leqq |\nabla u|$ which implies

$$|\nabla |u| \otimes u| \leqq |u| \,.\, |\nabla u| \tag{3.2}$$

and to

$$|t \,.\, h'(t)| \leqq \eta \,.\, h(t),$$

there holds

$$-\eta\langle\Phi^2 h(|u|)\nabla u, \nabla u\rangle \leqq -\langle\Phi^2 |u| \, |h'(|u|)| \nabla u, \nabla u\rangle =$$
$$= -\int \Phi^2 \, |h'(|u|)| \, |u| \, |\nabla u| \, |\nabla u| \, \mathrm{dvol} \leqq$$
$$\leqq -\int \Phi^2 h'(|u|) \, |\nabla |u| \otimes u| \, |\nabla u| \, \mathrm{dvol} \leqq$$
$$\leqq \langle\Phi^2 h'(|u|) \nabla |u| \otimes u, \nabla u\rangle,$$

$$(1-\eta)\langle\Phi^2 h(|u|)\nabla u, \nabla u\rangle \leqq \langle\Phi^2 h(|u|)\nabla u, \nabla u\rangle +$$
$$+ \langle\Phi^2 h'(|u|)(\nabla |u|) \otimes u, \nabla u\rangle, \quad \text{i.e.}$$

together with (3.1)

$$(3.3) \qquad (1-\eta)\langle\Phi^2 h(|u|)\nabla u, \nabla u\rangle \leqq 2 \, |\langle\Phi h(|u|)(\nabla\Phi) \otimes u, \nabla u\rangle| .$$

The left hand side of (3.3) equals to

$$(1-\eta)\int \Phi^2 h(|u|) \, |\nabla u|^2 \, \mathrm{dvol}.$$

The right hand side we estimate according to Schwartz inequality from above by

$$2^{\infty} \, \|\nabla\Phi\| \left(\int \Phi^2 h(|u|) \, |\nabla u|^2 \, \mathrm{dvol}\right)^{1/2} \left(\int h(|u|) \, |u|^2 \, \mathrm{dvol}\right)^{1/2}.$$

Squaring both sides and division by $(1-\eta)^2 \int \Phi^2 h(|u|) \, |\nabla u|^2 \, \mathrm{dvol}$ now gives

$$(3.4) \qquad \int \Phi^2 h(|u|) \, |\nabla u|^2 \, \mathrm{dvol} = (1-\eta)^{-2} \, . \, {}^{\infty}\|\nabla\Phi\|^2 \, . \int_{\mathrm{supp}\,\Phi} h(|u|) \, |u|^2 \, \mathrm{dvol}.$$

Performing $\lim\limits_{\varepsilon\to 0}$, we obtain

$$h(|u|) \, |u|^2 = \begin{cases} |u|^p & \text{if} \quad |u| \geqq 1, \\ |u|^q & \text{if} \quad |u| \leqq 1, \end{cases}$$

i.e. $h(|u|) \, |u|^2$ is globally integrable since by assumption $u \in {}^p\Omega^0(T^r_s \otimes E) + {}^q\Omega^0(T^r_s \otimes E)$. Let $s \to \infty$. Then according to property *e.* of Φ the right hand side of (3.4) tends to zero, hence $\int_B h(|u|) \, |\nabla u|^2 \, \mathrm{dvol} = 0$, $\nabla u = 0$, $\Delta u = 0$, $u = \mu^{-1.r}\Delta u = 0$ as asserted. □

Lemma 3.3. *$-\Delta$ with domain $\Omega^0_0(T^r_s \otimes E)$ is dissipative on ${}^p\Omega^0(T^r_s \otimes E)$ for* $1 < p < 3$.

Proof. If $u \in \Omega^0_0(T^r_s \otimes E)$ then

$$\langle |u|^{p-2} u, -\Delta u\rangle = -\langle\nabla(|u|^{p-2} u), \nabla u\rangle =$$
$$= -\langle |u|^{p-2} \nabla u, \nabla u\rangle - (p-2)\langle |u|^{p-3} \nabla |u| \otimes u, \nabla u\rangle. \quad \text{Using } (\varepsilon.2),$$

we obtain

$$|\langle |u|^{p-3} \nabla |u| \otimes u, \nabla u\rangle| \leqq \int |u|^{p-3} \, . \, |\nabla |u| \otimes u| \, . \, |\nabla u| \, \mathrm{dvol} \leqq$$
$$\leqq \int |u|^{p-3} \, . \, |u| \, |\nabla u| \, . \, |\nabla u| \, \mathrm{dvol} = \langle |u|^{p-2} \nabla u, \nabla u\rangle,$$

i.e. with $| p - 2 | < 1$

$$\langle | u |^{p-2} u, -\Delta u \rangle \leqq 0. \quad \square \tag{3.5}$$

Much more difficult becomes the situation if we consider the Laplace operator acting on q-forms with values in E. At the first glance the curvature endomorphism destroys the estimates of the preceding lemmas. But they remain still valid if we assume $\varrho \geqq 0$, i.e. $(\varrho u, u)_x \geqq 0$ for all x.

Lemma 3.4. *Suppose* $1 < p \leqq r < 3$, $\varrho \geqq 0$ *in* $\Delta = \nabla * \nabla + \varrho$, $u \in {}^p\Omega^q(E) + {}_r\Omega^q(E)$ *and* $\Delta u = \mu u$ *for* $\mu < 0$. *Then* u *is identically zero.*

Proof. On the right hand side of the equation corresponding to (3.1) appears additionally the term $\langle \varrho u, u \rangle$. Since $\langle \varrho u, u \rangle \geqq 0$ the analogue of (3.3) remains valid and we conclude in the same manner as in the proof of 3.2 $\nabla u = 0$, u has to be parallel. This implies $\varrho u = \mu u$, $\mu < 0$, which contradicts $\varrho \geqq 0$, i.e. $u = 0$. $\square$

In analogous manner 3.3 carries over to forms if $\varrho \geqq 0$.

Lemma 3.5. *Suppose in* $\Delta = \nabla * \nabla + \varrho$, $\varrho \geqq 0$, Δ *acting in* $\Omega^q(E)$. *Then* $-\Delta$ *with domain* $\Omega^q_0(E)$ *is dissipative on* ${}^p\Omega^q(E)$ *for* $1 < p < 3$. $\square$

We give some examples for $\varrho \geqq 0$.

1. If $q = 1$, the Ricci curvature of M^n nonnegative and $R^E = 0$, then according to (2.3) $\varrho \geqq 0$. For ordinary 1-forms the conditions Ricci curvature $\geqq 0$ and $\varrho \geqq 0$ are equivalent.

2. A sufficient condition for $q \geqq 1$ and ordinary forms (i.e. the trivial line bundle) is given by the nonnegativity of the curvature operator R^{op}. This we will shortly indicate. $R = R^M$ induces a symmetric linear operator $R^{op}: \Lambda^2 TM \to \Lambda^2 TM$ in the space of bivectors, called the curvature operator R^{op} and characterized by $(R^{op}(X \wedge Y), Z \wedge W)_x = (R(X, Y) W, Z)_x$. If $R^{op} \geqq \lambda$ then $(\varrho u, u)_x \geqq \lambda q(n-q) \cdot | u |^2_x$ ([3], p. 264), in particular $R^{op} \geqq 0$ implies $\varrho \geqq 0$.

3. Of particular interest are those cases where sectional curvature $K \geqq 0$ implies $R^{op} \geqq 0$ and hence $\varrho \geqq 0$.

If $f: M^n \to R^{n+2}$ is an isometric immersion, $n = 2k$, M^n open, complete, oriented, sectional curvature $K \geqq 0$ and at some point $x \in M$, $K > 0$ then $\varrho \geqq 0$ ([1]). A second class is given by manifolds with pure curvature operator. M^n has pure curvature operator if for each $x \in M$ there exists an orthonormal frame $(e_1, \dots, e_n)$ in $T_x M$ such that $R^{op}(e_i \wedge e_j) = K_{ij}(e_i \wedge e_j)$, K_{ij} = sectional curvature of the plane spanned by e_i, e_j. For a manifold with sectional curvature $\geqq 0$ and pure curvature operator there holds $\varrho \geqq 0$ ([1]).

An open manifold which belongs to all three classes is the rotating parabola in R^3.

4. PROPERTIES OF THE HEAT SEMIGROUP

We now give some answers to the questions of section 2.

Theorem 4.1. *Suppose (M^n, g) open, complete, $(E, h) \to M$ a Riemannian vector bundle. Denote by $\{e^{-t\Delta}\} = \{\int e^{-t\lambda}\, dE_\lambda\}$ the heat semigroup acting on ${}^2\Omega^0(T^r_s \otimes E)$. Then ${}^p\| e^{-t\Delta}u \| \leqq {}^p\| u \|$ for all $u \in {}^p\Omega^0(T^r_s \otimes E) \cap {}^2\Omega^0(T^r_s \otimes E)$ and $\frac{3}{2} \leqq p \leqq 3$. Therefore $\{e^{-t\Delta}\}$ extends to a contration semigroup on ${}^p\Omega^0(T^r_s \otimes E)$ for these p. $e^{-t\Delta}u$ satisfies the heat equation $\frac{\partial}{\partial t} e^{-t\Delta}u = -\Delta e^{-t\Delta}u$ for $u \in {}^p\Omega^0(T^r_s \otimes E)$, and $\{e^{-t\Delta}\}^n$ is the unique semigroup with these properties for $\frac{3}{2} \leqq p \leqq 3$.*

Proof. The closure L of $-\Delta\, |_{\Omega^0_0(T^r_s \otimes E)}$ in ${}^p\Omega^0(T^r_s \otimes E)$ is dissipative for $1 < p < 3$. Furthermore, $\mu - L$ is surjective for $\mu > 0$ and the above p: If this would not be the case there would not be the case there would exist an $u \in {}^{p'}\Omega^0(T^r_s \otimes E)$ such that $\langle u, (\mu - L)\, v\rangle = 0$ for all $v \in \Omega^0_0(T^r_s \otimes E)$. This would imply $\Delta u = -\mu u$, $\mu > 0$, and contradict 3.2. From $p' < 3$ we get the restriction $p > \frac{3}{2}$. Altogether, L generates according to 3.1. a contraction semigroup $\{Q_t\}_{t \geqq 0}$ for $\frac{3}{2} < p < 3$. Next we show that the semigroups Q_t and $e^{-t\Delta}$ agree on $L_2 \cap L_p = {}^2\Omega^0(T^r_s \otimes E) \cap {}^p\Omega^0(T^r_s \otimes E)$. For this it suffices to show that the two resolvents $(\mu - (-\Delta))^{-1}$ and $(\mu - L)^{-1}$ coincide on $L_2 \cap L_p$. Suppose $u \in L_2 \cap L_p$, $(\mu - (-\Delta))\, u^{-1} = v$, $(\mu - L)^{-1}\, u = w$. Then $v \in L_2$, $w \in L_p$, $v - w \in L_2 + L_p$ and $\Delta(v - w) = -\mu(v - w)$, $\mu > 0$. According to lemma 3.2 we have $v = w$, $\{Q_t\}_t = \{e^{-t\Delta}\}_t$ on $L_p \cap L_2$. This proves the estimate ${}^p\| e^{-t\Delta}u \| \leqq {}^p\| u \|$ for $\frac{3}{2} < p < 3$. By a limiting argument we conclude this for the endpoints of the interval too. Since $e^{-t\Delta}u$ satisfies the heat equation for $u \in D_\Delta$ and since this domain is dense in ${}^p\Omega^0(T^r_s \otimes E)$, $e^{-t\Delta}u$ also satisfies the heat equation for $u \in {}^p\Omega^0(T^r_s \otimes E)$, but at the first instance only in the distributional sense. The hypoellipticity of the heat operator implies this in the pointwise sense. We conclude with the proof of the uniqueness statement. If L' is the infinitesimal generator of another contraction semigroup $\{P_t\}_t$ such that $P_t u$ satisfies the heat equation, then we have to show $(\mu - L')^{-1} = (\mu - (-\Delta))^{-1}$. $(\mu - L')^{-1}u = v$ means $v \in D_L$, and $(\mu - L')\, v = u$. If $v \in D_L$, then $t^{-1}(P_t v - v) \to L'v$ in ${}^p\Omega^0(T^r_s \otimes E)$, $t^{-1}(P_{s+t}v - P_s v) \to P_s L'v \in {}^p\Omega^0(T^r_s \otimes E)$ for any fixed $s > 0$ (since P_s is a bounded operator). $P_t u$ satisfies the heat equation. Therefore $t^{-1}(P_{s+t}v - P_s v) \to \frac{\partial}{\partial s} P_s v = -\Delta P_s v$, i.e. altogether $P_s L'v = -\Delta P_s v$. Performing $\lim\limits_{s \to 0}$ we obtain $L'v = \lim\limits_{s \to 0} -\Delta P_s v = -\Delta v$ in the distributional sense. It follows $v \in {}^p\Omega^0(T^r_s \otimes E)$ satisfies $(\mu - (-\Delta))\, v = u$. On the other hand, if $(\mu - (-\Delta))^{-1} u = w$, then $w \in {}^p\Omega^0(T^r_s \otimes E)$, $(\mu - (-\Delta))\, w = u$, $\Delta(v - w) = -\mu(v - w)$, $\mu > 0$. According to 3.2 $v = w$, the resolvents and the semigroups coincide. □

In similar manner we conclude for q-forms with values in E.

Theorem 4.2. *Suppose (M^n, g) open, complete, $(E, h) \to M$ a Riemannian vector bundle and $\varrho \geqq 0$ for the endomorphism ϱ in $\Delta = \nabla * \nabla + \varrho$, acting on q-forms with values in E. If $\{e^{-t\Delta}\}_{t \geqq 0}$ denotes the semigroup on ${}^2\Omega^q(E)$, then ${}^p\| e^{-t\Delta} u \| \leqq \leqq {}^p\| u \|$ for all $u \in {}^p\Omega^q(E) \cap {}^2\Omega^q(E)$ and $\frac{3}{2} \leqq p \leqq 3$. Therefore $\{e^{-t\Delta}\}_{t \geqq 0}$ extends to a contraction semigroup on these ${}^p\Omega^q(E)$. $e^{-t\Delta}u$ satisfies the heat equation $\frac{\partial}{\partial t} e^{-t\Delta} u = -\Delta\, e^{-t\Delta} u$ for all $u \in {}^p\Omega^q(E)$, and $\{e^{-t\Delta}\}_{t \geqq 0}$ is the unique semigroup with these properties for $\frac{3}{2} < p < 3$.*

Proof. The proof is performed quite analogous to that of 4.1 essentially using 3.4, 3.5. □

Corollary 4.3. *The assertions of 4.2 are valid for ordinary q-forms in each of the following cases.*

a. *$q = 1$ and Ricci curvature $\geqq 0$.*

b. *$q \geqq 1$ and $R^{op} \geqq 0$.*

c. *$q \geqq 1$, $M^n \hookrightarrow R^{n+2}$ as an isometric immersion, $n = 2k$, sectional curvature $K \geqq 0$ and at some point $x \in M$, $K > 0$.*

d. *$q \geqq 1$, sectional curvature $K \geqq 0$ and M has pure curvature operator.* □

Finally we turn to the initial value problem for the heat equation

$$(4.1) \qquad \frac{\partial}{\partial t} v = -\Delta v \text{ on } M \times R_+, \qquad v(x, 0) = u_0(x) \text{ on } M.$$

Theorems 4.1, 4.2 immediately imply

Theorem 4.4. *Suppose (M^n, g) open, complete, $(E, h) \to M$ a Riemannian vector bundle, $\frac{3}{2} < p < 3$. Then the initial value problem* (4.1) *is solvable in the following cases.*

a. *$v(\ , t) \in {}^p\Omega^0(T^r_s \otimes E)$ and $u_0 \in {}^p\Omega^0(T^r_s \otimes E)$.*

b. *$v(\ , t) \in {}^p\Omega^q(E)$ and $u_0 \in {}^p\Omega^q(E)$ and $\varrho \geqq 0$, in particular in all cases of corollary 4.3.* □

The remaining open question is the uniqueness which is partially answered by

Theorem 4.5. *Suppose (M^n, g) open, complete, $(E, h) \to M$ a Riemannian vector bundle,* $\frac{3}{2} < p < 3$, *$v(x, t)$ a solution of the heat equation with $v(\ , t) \in {}^p\Omega^0(T^r_s \otimes E)$ or $v(\ ,t) \in {}^p\Omega^q(E)$ and $\varrho \geqq 0$, respectively. Assume further ${}^p\| v(\ , t) \| \leqq a \cdot e^{bt}$. Then there exists a uniquely determined $u_0 \in {}^p\Omega^0(T^r_s \otimes E)$ or $u_0 \in {}^p\Omega^q(E)$, respectively, such $v = e^{-t\Delta} u_0$.*

Proof. We treat both cases simultaneously, denoting the corresponding solution space by L_p. If $u_0 = \lim\limits_{t_k \to 0} v(\ , t_k)$ in the weak star topology, $u = v - e^{-t\Delta} u_0$, then

$$
{}^p\| u(\ , t) \| \leqq a\, e^{bt} \tag{4.2}
$$

and

$$
u(\ , t_k) \underset{t_k \to 0}{\to} 0 \tag{4.3}
$$

in the distributional sense. Furthermore, u satisfies the heat equation since each term does. We have to show $u = 0$. To do this we consider the Laplace transform $w_\lambda(x) = \int\limits_0^\infty e^{-t\lambda} u(x, t)\, dt$ of u. According to (4.2) the integral converges absolutely for sufficiently large λ and almost every x. Moreover, $w_\lambda \in L_p$. Next we show $\Delta w_\lambda = -\lambda w_\lambda$ in the distributional sense. For any $\mathscr{S} \in \Omega^0_0(T^r_s \otimes E)$ or $\mathscr{S} \in \Omega^q_0(E)$, respectively,

$$
\langle \mathscr{S}, \Delta w_\lambda \rangle = \langle \Delta \mathscr{S}, w_\lambda \rangle = \int\limits_0^\infty e^{-t\lambda} \langle \Delta \mathscr{S}, u(\ , t) \rangle\, dt. \tag{4.4}
$$

According to (4.2) the double integral (4.4) converges absolutely for large λ. Using the heat equation

$$
\langle \Delta \mathscr{S}, u(\ , t) \rangle = -\frac{\partial}{\partial t} \langle \mathscr{S}, u(\ , t) \rangle,
$$

we obtain by integration of parts

$$
\begin{aligned}
\langle \mathscr{S}, \Delta w_\lambda \rangle &= -\int\limits_0^\infty e^{-t\lambda} \frac{\partial}{\partial t} \langle \mathscr{S}, u(\ , t) \rangle\, dt = \\
&= -\lim_{\substack{t_k \to 0 \\ N \to \infty}} \int\limits_{t_k}^{N} e^{-t\lambda} \frac{\partial}{\partial t} \langle \mathscr{S}, u(\ , t) \rangle\, dt = \\
&= -\lim_{\substack{t_k \to 0 \\ N \to \infty}} \Big[\lambda \int\limits_{t_k}^{N} e^{-t\lambda} \langle \mathscr{S}, u(\ , t) \rangle\, dt + e^{-N\lambda} \langle \mathscr{S}, u(\ , N) \rangle - e^{-t_k\lambda} \langle \mathscr{S}, u(\ , t_k) \rangle\Big] = \\
&= -\lambda \int\limits_0^\infty e^{-t\lambda} \langle \mathscr{S}, u(\ , t) \rangle\, dt
\end{aligned}
$$

since $e^{-N\lambda} \langle \mathscr{S}, u(\ , N) \rangle \to 0$ by (4.2) and $e^{-t_k\lambda} \langle \mathscr{S}, u(\ , t^k) \rangle \to 0$ by (4.3). Altogether $\Delta w_\lambda = -\lambda w_\lambda$ in the distributional sense. Now $w_\lambda = 0$ by 3.2 or 3.4 respectively. From the uniqueness of the Laplace transform we conclude $u(x, t) = 0$ for almost every x, i.e. $u = 0$. If $v = e^{-t\Delta} u_0'$, then

$$(4.5) \quad \| u_0 - u_0' \| \leqq \| e^{-t\Delta} u_0' - u_0' \| + \| u_0 - e^{-t\Delta} u_0 \| + \| e^{-t\Delta} u_0 - e^{-t\Delta} u_0' \| .$$

The first two terms of (4.5) tend to zero if $t \to 0$, the third term equals to zero by assumption. Therefore $u_0 = u_0'$. □

Concluding remarks. For q-forms in our approach the assumption $\varrho \geqq 0$ or the modified versions of this were very essential. In [2] Dodziuk proved a uniqueness theorem for the initial value problem (4.1) in ${}^\infty\Omega^q$ under the assumption that (M^n, g) has Ricci curvature bounded from below and $\varrho \geqq 0$. On the other hand, Strichartz has shown in [5] that on the hyperbolic plane H^2_{-1} for $q = 1\{e^{-t\Delta}\}_{t \geqq 0}$ is not a contraction semigroup on ${}^p\Omega^1$. This altogether supports the hypothesis that some kind of nonnegativity of the curvature should be connected with the contraction property.

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