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ODD FORMS ON A NON-ORIENTABLE MANIFOLD

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Abstract. In this paper we give a description the space of odd forms on a non-orientable manifold by means of ordinary forms on its double cover. In its second part we prove a relation between the integral of an odd form on the manifold and the integral of its canonical lift to the double cover.

Key words. Double cover of a manifold, odd differential form, integral of an odd differential form.

MS Classification. 58 C 35.

1. INTRODUCTION

This paper deals with some questions arising from extending the theory of integral of differential forms to non-orientable manifolds. The theory of odd differential forms (see e.g. [3], [4]) enables us to define an integral on a non-orientable manifold. It seems moreover that the integration theory of odd differential forms contains, as a special case, the integration theory of ordinary forms.

The fundamental concept is the double cover of a manifold (see e.g. [2]). This is a manifold which is always orientable and is topologically simpler than the initial manifold. In Section 3 we describe, using the double cover, the relation between the space of usual forms and the space of odd forms on a manifold X, and between the cohomology groups of these spaces.

We can imagine the double cover σ: Or $X \to X$ of manifold X as a "area" of X (for more detail see [2]). Then a question arises what is a relation between the integrals $\int_X \omega$, $\int_{\mathrm{Or}X} \sigma^*\omega$, that is a relation between the "volume" and "area" of X. We prove in Section 4 that $2\int_X \omega = \int_{\mathrm{Or}X} \sigma^*\omega$.

There is moreover an analogous relation for the boundaries $\partial\, \mathrm{Or}X$, ∂X, namely $2\int_{\partial X} \eta = \int_{\partial \mathrm{Or}X} \sigma^*\eta$.

2. NOTATION

The notation concerning odd forms is taken from [3] with the difference that $\Omega(X) = \Sigma_i \Omega^i(X)$ (resp. $\hat{\Omega}(X) = \Sigma_i \hat{\Omega}^i(X)$) denotes the graded real vector space of smooth differential forms (resp. odd smooth differential forms) on X. We note that most concepts and propositions adopted for manifolds in [3] can be formulated in the same way for manifolds with boundary. The graded vector cohomology spaces of differential spaces $(\Omega(X), \mathrm{d})$, $(\hat{\Omega}(X), \mathrm{d})$ are denoted by $H(X) = \Sigma_i H^i(X)$, $\hat{H}(X) = \Sigma_i \hat{H}^i(X)$.

The definition and the basic properties of the double cover of a manifold X, which will be denoted by $\sigma\colon \mathrm{Or}X \to X$, can be found in [2]. The canonical involution $\mathrm{Or}X \to \mathrm{Or}X$ will be denoted by ι, and its value at the point $x \in \mathrm{Or}X$ will be sometimes denoted by $-x$. (The double cover of a manifold with boundary can be introduced in the same manner as for a manifold without boundary.) If X is a manifold, we define

$$
\begin{aligned}
\Omega^i_+(\mathrm{Or}X) &= \{\eta \in \Omega^i(\mathrm{Or}X) \mid \iota^*\eta = \eta\},\\
\Omega^i_-(\mathrm{Or}X) &= \{\eta \in \Omega^i(\mathrm{Or}X) \mid \iota^*\eta = -\eta\},\\
\Omega_+(\mathrm{Or}X) &= \Sigma_i \Omega^i_+(\mathrm{Or}X),\\
\Omega_-(\mathrm{Or}X) &= \Sigma_i \Omega^i_-(\mathrm{Or}X).
\end{aligned}
$$

The graded cohomology space of the vector differential space $(\Omega_+(\mathrm{Or}X), \mathrm{d})$ (resp. $(\Omega_-(\mathrm{Or}X), \mathrm{d})$) will be denoted by $H_+(\mathrm{Or}X) = \Sigma_i H^i_+(\mathrm{Or}X)$ (resp. $H_-(\mathrm{Or}X) = \Sigma_i H^i_-(\mathrm{Or}X)$).

Finally, $\mathbf{N}$ denotes the set $\{1, 2, \dots\}$ of positive integers.

3. STRUCTURE OF SPACE OF ODD FORMS

In this section we will describe the structure of the space of odd forms on a manifold X with the help of an ordinary forms on X.

If X is orientable, the situation is simple. In this case there exists a smooth field of unit odd scalars δ on X (see [3], Theorem 1.2.), and the mapping $\Omega(X) \ni \varrho \mapsto \delta \otimes \varrho \in \hat{\Omega}(X)$ is an isomorphism of vector spaces. From the definition of an exterior derivative it follows that this mapping is an isomorphism of differential spaces $(\Omega(X), \mathrm{d})$, $(\hat{\Omega}(X), \mathrm{d})$ as well. Thus we have:

Proposition 3.1. *Let X be an orientable manifold. Then*

$$\Omega(X) \cong \hat{\Omega}(X), \qquad H(X) \cong \hat{H}(X).$$

If the manifold X is non-orientable, the situation is more complicated. Let X be a connected non-orientable manifold, let δ be a field of unit odd scalars on OrX. (A field like this exists because the manifold OrX is orientable.) Let $\omega \in \hat{\Omega}^p(X)$ be any odd p-form on X. For the odd form $\sigma^*\omega \in \hat{\Omega}^p(\mathrm{Or}X)$ there exists a unique element, let us denote it by $\chi(\omega)$, belonging to space $\Omega^p(\mathrm{Or}X)$ such that $\sigma^*\omega = \delta \otimes \chi(\omega)$. In this way we get a correctly defined mapping χ: $\hat{\Omega}(X) \to \Omega^p(\mathrm{Or}X)$. This mapping is obviously linear.

Lemma 3.1. *Let X be a connected non-orientable manifold. Then* $\mathrm{Im}\,\chi \subseteq \Omega^p_-(\mathrm{Or}\,X)$.

Proof. Let (V, ψ), $\psi = (y^i)$ be a chart on X, let (U, φ), $\varphi = (x^i)$ be a chart on OrX such that $\sigma|_U : U \to V$ is a diffeomorphism and that $\varphi = \psi \circ \sigma|_U$. Let us denote $-U = \iota(U)$, where ι: Or$X \to$ OrX is the canonical involution. Set $\varphi^- = \varphi \circ \iota|_{-U} = (\bar{x}^i)$.

Let $\omega \in \hat{\Omega}^p (p \geq 0)$, let $\omega = \omega_{i_1,\dots,i_p}\hat{\psi} \otimes \mathrm{d}y^{i_1} \wedge \dots \wedge \mathrm{d}y^{i_p}$ be the chart expression for ω with respect to the chart (V, ψ). Then the coordinate expression for $\sigma^*\omega$ with respect to (U, φ) is $\sigma^*\omega = (\omega_{i_1,\dots,i_p} \circ \sigma|_U)\,\hat{\varphi} \otimes \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}$, and the coordinate expression for $\sigma^*\omega$ with respect to $(-U, \varphi^-)$ is $\sigma^*\omega = (\omega_{i_1,\dots,i_p} \circ \sigma|_{-U})\,\hat{\varphi}^- \otimes \mathrm{d}\bar{x}^{i_1} \wedge \dots \wedge \mathrm{d}\bar{x}^{i_p}$ (see [3], p. 166).

Let δ be a field of unit odd scalars on OrX, let $\xi_\varphi, \xi_{\varphi^-} \in \{-1, 1\}$ be such integers that on U (resp. on $-U$), $\xi_\varphi\hat{\varphi} = \delta$ (resp. $\xi_{\varphi^-}\hat{\varphi}^- = \delta$). Then the coordinate expression of $\sigma^*\omega$ with respect to (U, φ) (resp. to $(-U, \varphi^-)$) is given by $\sigma^*\omega = \delta \otimes [\xi_\varphi \cdot (\omega_{i_1,\dots,i_p} \circ \sigma|_U)\,\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}]$ (resp. $\sigma^*\omega = \delta \otimes [\xi_{\varphi^-}(\omega_{i_1,\dots,i_p} \circ \sigma|_{-U})\,\mathrm{d}\bar{x}^{i_1} \wedge \dots \wedge \mathrm{d}\bar{x}^{i_p}]$). Thus the coordinate expression of $\chi(\omega)$ with respect to (U, φ) (resp. to $(-U, \varphi^-)$) is given by $\chi(\omega) = \xi_\varphi(\omega_{i_1,\dots,i_p} \circ \sigma|_U)\,\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}$ (resp. $\chi(\omega) = \xi_{\varphi^-}(\omega_{i_1,\dots,i_p} \circ \sigma|_{-U})\,\mathrm{d}\bar{x}^{i_1} \wedge \dots \wedge \mathrm{d}\bar{x}^{i_p}$).

Now it is enough to compare the coordinate representation of the forms $\iota^*\chi(\omega)$ and $\chi(\omega)$ on (U, φ). For any point $x \in U$ and any vectors $\xi_1, \dots, \xi_p \in T_x\,\mathrm{Or}X$ it holds

$$\iota^*\chi(\omega)(x)(\xi_1, \dots, \xi_p) = \chi(\omega)(\iota(x))(T\iota\xi_1, \dots, T\iota\xi_p);$$

an elementary calculation shows that

$$\iota^*\chi(\omega)(x)(\xi_1, \dots, \xi_p) = \xi_{\varphi^-}(\omega_{i_1,\dots,i_p} \circ \sigma|_{-U})(\iota(x)) \,.\, \mathrm{d}\bar{x}^{i_1} \wedge \dots$$
$$\dots \wedge \mathrm{d}\bar{x}^{i_p}(T\iota\xi_1, \dots, T\iota\xi_p) = \xi_{\varphi^-}(\omega_{i_1,\dots,i_p} \circ \sigma|_U)(x)\,\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}(\xi_1, \dots, \xi_p).$$

If X is connected and non-orientable, then the canonical involution ι: Or$X \to$ OrX changes the orientation (see [2], Proposition 5.14.). Then it holds $\xi_{\varphi^-} = -\xi_\varphi$. From this it follows that $\iota^*\chi(\omega) = -\chi(\omega)$ on (U, φ). Since the charts are arbitrary, we have $i^*\chi(\omega) = -\chi(\omega)$ so that $\chi(\omega) \in \Omega^p_-(\mathrm{Or}X)$. □

Proposition 3.2. *Let X be a connected non-orientable manifold. Then the mapping χ: $\hat{\Omega}(X) \to \Omega_-(\mathrm{Or}X)$ is an isomorphism of graded differential spaces.*

Proof. Linearity of the mapping χ is evident. Thus it is enough to prove that χ is a bijection and that $\chi \circ \mathrm{d} = \mathrm{d} \circ \chi$. First let us prove a bijectivity of χ. The mapping σ is a surjective local diffeomorphism and so σ^* as well as χ is an injection.

Let $\eta \in \Omega^p_-(\mathrm{Or}X)$ be any form. Let (V, ψ), (U, φ), $(-U, \varphi^-)$ be the charts from the previous lemma. Let the expression of η relative to (U, φ) (resp. to $(-U, \varphi^-)$) be given by $\eta = \eta_{i_1,\dots,i_p} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}$ (resp. $\eta = \bar{\eta}_{i_1,\dots,i_p} \mathrm{d}\bar{x}^{i_1} \wedge \dots \wedge \mathrm{d}\bar{x}^{i_p}$). From our assumption $\iota^*\eta = -\eta$ it follows that $\bar{\eta}_{i_1,\dots,i_p} = -\eta_{i_1,\dots,i_p} \circ \iota|_{-U}$. Let ξ_φ, $\xi_{-\varphi}$ be integers from the proof of the previous lemma, let us set $\omega_{i_1,\dots,i_p} = \xi_\varphi \eta_{i_1,\dots,i_p} \circ (\sigma|_U)^{-1}$.

Define an odd form ω on V by $\omega = \omega_{i_1,\dots,i_p} \tilde{\psi} \otimes \mathrm{d}y^{i_1} \wedge \dots \wedge \mathrm{d}y^{i_p}$. Then

$$\begin{aligned}\xi_\varphi(\eta_{i_1,\dots,i_p} \circ (\sigma|_U)^{-1}) &= -\xi_{\varphi^-}(-\bar{\eta}_{i_1,\dots,i_p} \circ \iota|_U \circ (\sigma|_U)^{-1}) = \\ &= \xi_{\varphi^-}\bar{\eta}_{i_1,\dots,i_p} \circ (\sigma|_{-U})^{-1}.\end{aligned}$$

Consequently, for the given choice of (V, ψ), ω is defined correctly.

It remains to show that in this way it is possible to define an odd form ω on the whole manifold X, i.e. that this definition does not depend on the choice of the chart (V, ψ). Then let (A, γ), $\gamma = (g^i)$ (resp. (B, τ), $\tau = (t^i)$) be other charts on X (resp. $\mathrm{Or}X$) such that $\sigma|_B \colon B \to A$ is a diffeomorphism and that $\tau = \gamma \circ \sigma|_B$. Let $A \cap V = C \neq 0$. Express η with respect to (B, τ) by $\eta = \bar{\bar{\eta}}_{i_1,\dots,i_p} \mathrm{d}t^{i_1} \wedge \dots \wedge \mathrm{d}t^{i_p}$. The construction described above defines the odd form $\bar{\omega} = \bar{\omega}_{i_1,\dots,i_p} \tilde{\gamma} \otimes \mathrm{d}g^{i_1} \wedge \dots \wedge \mathrm{d}g^{i_p}$, on A, where $\bar{\omega}_{i_1,\dots,i_p} = \xi_\tau \bar{\bar{\eta}}_{i_1,\dots,i_p} \circ (\sigma|_B)^{-1}$. We must prove that on C, $\omega = \bar{\omega}$. Let us denote the set $B \cap U$ by D. On D it holds $\eta_{j_1,\dots,j_p} = \dfrac{\partial x^{i_1}}{\partial t^{j_1}} \cdots \dfrac{\partial x^{i_p}}{\partial t^{j_p}} \bar{\bar{\eta}}_{i_1,\dots,i_p}$ and then on C it holds

$$\begin{aligned}\omega_{j_1,\dots,j_p} &= \xi_\varphi[\eta_{j_1,\dots,j_p} \circ (\sigma|_D)^{-1}] = \\ &= \xi_\varphi\left[\frac{\partial x^{i_1}}{\partial t^{j_1}} \circ (\sigma|_D)^{-1} \cdots \frac{\partial x^{i_p}}{\partial t^{j_p}} \circ (\sigma|_D)^{-1}\right](\bar{\bar{\eta}}_{i_1,\dots,i_p} \circ (\sigma|_D)^{-1}).\end{aligned}$$

Further,

$$(D_{\varphi \circ \tau^{-1}}) \circ \tau \circ (\sigma|_D)^{-1} = (D_{\psi \circ (\sigma|_D)^{-1} \circ (\sigma|_D) \circ \gamma^{-1}}) \circ \tau \circ (\sigma|_D)^{-1} = (D_{\psi \circ \gamma^{-1}}) \circ \gamma.$$

Thus for any indices i, j,

$$\frac{\partial x^i}{\partial \tau^j} \circ (\sigma|_D)^{-1} = \frac{\partial y^i}{\partial g^j}.$$

From that it follows

$$\omega_{j_1,\dots,j_p} = \xi_\varphi\left(\frac{\partial y^{i_1}}{\partial g^{j_1}} \cdots \frac{\partial y^{i_p}}{\partial g^{j_p}}\right)\bar{\bar{\eta}}_{i_1,\dots,i_p} \circ (\sigma|_D)^{-1}.$$

Moreover $\xi_\varphi = (\operatorname{sign}\det D_{\varphi\circ\tau^{-1}})\,\xi_\tau = (\operatorname{sign}\det D_{\psi\circ\gamma^{-1}})\,\xi_\tau$ so that

$$\omega_{j_1,\dots,j_p} = \xi_\tau(\operatorname{sign}\det D_{\psi\circ\gamma^{-1}})\left(\frac{\partial y^{i_1}}{\partial g^{j_1}}\cdots\frac{\partial y^{i_p}}{\partial g^{j_p}}\right)\bar{\eta}_{i_1,\dots,i_p}\circ(\sigma|_D)^{-1} =$$

$$= (\operatorname{sign}\det D_{\psi\circ\gamma^{-1}})\left(\frac{\partial y^{i_1}}{\partial g^{j_1}}\cdots\frac{\partial y^{i_p}}{\partial g^{j_p}}\right)\bar{\omega}_{i_1,\dots,i_p}.$$

We see that the components of ω transform like the components of an odd differential form; therefore, on C, $\omega = \bar{\omega}$. Then we can construct the form ω globally and the form constructed in this way fulfils $\sigma^*\omega = \delta\otimes\eta$, i.e. $\chi(\omega) = \eta$. Thus a mapping χ is surjective. This completes the proof of bijectivity of χ.

It remains to prove that $\mathrm{d}\chi = \chi\mathrm{d}$. Let $\omega\in\hat{\Omega}^p(X)$, let (V,ψ), (U,φ) be charts on X, $\mathrm{Or}X$ such that $\sigma|_U : U\to V$ is a diffeomorphism and that $\varphi = \psi\circ\sigma|_U$. Let $\omega = \omega_{i_1,\dots,i_p}\hat{\psi}\otimes\mathrm{d}y^{i_1}\wedge\ldots\wedge\mathrm{d}y^{i_p}$ be the chart expression for ω relative to (V,ψ). Then the coordinate expression for $\mathrm{d}\omega$ is

$$\mathrm{d}\omega = \frac{\partial\omega_{i_1,\dots,i_p}}{\partial y^i}\hat{\psi}\otimes\mathrm{d}y^i\wedge\mathrm{d}y^{i_1}\wedge\ldots\wedge\mathrm{d}y^{i_p}.$$

On (U,φ) it holds

$$\chi(\mathrm{d}\omega) = \left(\frac{\partial\omega_{i_1,\dots,i_p}}{\partial y^i}\circ\sigma|_U\right)\xi_\varphi\,\mathrm{d}x^i\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p},$$

$$\mathrm{d}\chi(\omega) = \xi_\varphi\frac{\partial}{\partial x^i}(\omega_{i_1,\dots,i_p}\circ\sigma|_U)\,\mathrm{d}x^i\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p};$$

since

$$\frac{\partial}{\partial x^i}(\omega_{i_1,\dots,i_p}\circ\sigma|_U) = \left(\frac{\partial\omega_{i_1,\dots,i_p}}{\partial y^k}\circ\sigma|_U\right)\left(\frac{\partial}{\partial x^i}(y^k\circ\sigma|_U)\right) =$$

$$= \left(\frac{\partial\omega_{i_1,\dots,i_p}}{\partial y^k}\circ\sigma|_U\right)\frac{\partial x^k}{\partial x^i} = \frac{\partial\omega_{i_1,\dots,i_p}}{\partial y^i}\circ\sigma|_U,$$

we have $\mathrm{d}\chi(\omega) = \chi(\mathrm{d}\omega)$. Since ω and the charts we use are arbitrary, χ is an isomorphism of differential spaces. □

Corollary 3.1. *Let X be a connected non-orientable manifold. Then $\hat{\Omega}(X)\cong\Omega_-(\mathrm{Or}X)$, $\hat{H}(X)\cong H_-(\mathrm{Or}X)$. Moreover $\Omega(\mathrm{Or}X)\cong\Omega(X)\oplus\hat{\Omega}(X)$, $H(\mathrm{Or}X)\cong H(X)\oplus\hat{H}(X)$.*

Proof. It is a direct consequence of Proposition 3.2. and the relations

$$\Omega(\mathrm{Or}X)\cong\Omega_+(\mathrm{Or}X)\oplus\Omega_-(\mathrm{Or}X),$$
$$H(\mathrm{Or}X)\cong H_+(\mathrm{Or}X)\oplus H_-(\mathrm{Or}X),$$
$$\Omega(X)\cong\Omega_+(\mathrm{Or}X),\qquad H(X)\cong H_+(\mathrm{Or}X),$$

which are proved in [2], part 5.7. □

Let us denote by $H_c(X)$ the space of smooth differential forms with compact support on a manifold X, let $H_c(X)^*$ be the dual vector space of $H_c(X)$.

Corollary 3.2. *Let X be a connected non-orientable manifold. Then $\hat{H}(X) \cong$ $\cong H_c(X)^*$.*

Proof. It is known that if X is connected, non-orientable manifold, then $H_-(\mathrm{Or}X) \cong H_c(X)^*$ (see [2], part 5.14, Corollary 2). On the other hand, in Corollary 3.1., we have proved that $\hat{H}(X) \cong H_-(\mathrm{Or}X)$. Consequently, $\hat{H}(X) \cong$ $\cong H_c(X)^*$. □

4. INTEGRATION OF ODD FORMS ON THE DOUBLE COVER

The aim of this chapter is to compare the integrals $\int\limits_X \omega$, $\int\limits_{\mathrm{Or}X} \sigma^*\omega$ for an odd n-form ω on an n-manifold X with boundary. Similarly as in the proof of Lemma 3.1, let us introduce the following notation. For the chart (U, φ), $\varphi = (x^i)$ on $\mathrm{Or}X$, denote $-U = \iota(U)$, $\varphi^- = \varphi \circ \iota|_{-U} = (\bar{x}^i)$.

For the given smooth structure of the manifold with boundary on X (resp. $\mathrm{Or}X$) it is obviously possible to choose an atlas $\mathscr{A}_X$ (resp. $\mathscr{A}_{\mathrm{Or}X}$) such that the following conditions are satisfied:

(C_1) Atlases $\mathscr{A}_{\mathrm{Or}X}$ and $\mathscr{A}_X$ have the form

$$\mathscr{A}_{\mathrm{Or}X} = \{(U_i, \varphi_i) \mid i \in \mathrm{N}\} \cup \{(-U_i, \varphi_i^-) \mid i \in \mathrm{N}\},$$
$$\mathscr{A}_X \quad = \{(V_i, \psi_i) \mid i \in \mathrm{N}\}.$$

(C_2) $\sigma(U_i) = V_i$ and $\sigma|_{U_i} : U_i \to V_i$ is a diffeomorphism for each $i \in \mathrm{N}$. Moreover for each $i \in \mathrm{N}$, it holds $\psi_i = \varphi_i \circ (\sigma|_{U_i})^{-1}$.

(C_3) Atlases $\mathscr{A}_{\mathrm{Or}X}$, $\mathscr{A}_X$ are locally finite and all sets V_i are connected.

In order to guarentee the existence of integrals, we will work only with continuous odd forms and compact manifolds with boundary. Let us remark that the double cover of a compact manifold with boundary is again a compact manifold with boundary.

Proposition 4.1. *Let X be a compact n-manifold with boundary, let ω be a continuous odd n-form on X. Then*

$$\int\limits_{\mathrm{Or}X} \sigma^*\omega = 2 \int\limits_X \omega.$$

Proof. Let $\mathscr{A}_{\mathrm{Or}X}$, $\mathscr{A}_X$ be atlases on $\mathrm{Or}X$, X fulfilling conditions (C_1), (C_2), (C_3). Let (U, φ), $\varphi = (x^i)$ be a chart from $\mathscr{A}_{\mathrm{Or}X}$, let (V, ψ), $\psi = (y^i)$ be a corresponding chart from $\mathscr{A}_X$. Consider a continuous odd n-form ω on X. Let $\omega = \omega_\psi \hat{\psi} \otimes \otimes \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^n$ be the chart expression for ω relative to the chart (V, ψ). Then the coordinate expression of the odd form $\sigma^*\omega$ relative to (U, φ) (resp. to $(-U, \varphi^-)$) is given by $\sigma^*\omega = (\omega_\psi \circ \sigma|_U) \hat{\varphi} \otimes \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ (resp. $\sigma^*\omega = (\omega_\psi \circ \sigma|_{-U})\hat{\varphi}^- \otimes \otimes \mathrm{d}\bar{x}^1 \wedge \ldots \wedge \mathrm{d}\bar{x}^n$) (see [3], relation (1.3.15)). Then according to the definition,

$$\int_V \omega = \int_{\psi(V)} \omega_\psi \circ \psi^{-1} = \int_{\varphi \circ (\sigma|_U)^{-1}(V)} \omega_\psi \circ (\sigma|_U) \circ \varphi^{-1} = \int_{\varphi(U)} \omega_\psi \circ \sigma \circ \varphi^{-1},$$

$$\int_U \sigma^*\omega = \int_{\varphi(U)} \omega_\psi \circ \sigma \circ \varphi^{-1},$$

$$\int_{-U} \sigma^*\omega = \int_{\varphi(U)} \omega_\psi \circ (\sigma|_{-U}) \circ (\varphi^-)^{-1} = \int_{\varphi(U)} \omega_\psi \circ \sigma \circ l \circ l^{-1} \circ \varphi^{-1} = \int_{\varphi(U)} \omega_\psi \circ \sigma \circ \varphi^{-1}.$$

Then on the whole $\int_V \omega = \int_U \sigma^*\omega = \int_{-U} \sigma^*\omega = \int_{\varphi(U)} \omega_\psi \circ \sigma \circ \varphi^{-1}$.

Now we can proceed to the comparison of integrals on the whole manifold X and on $\mathrm{Or}X$. Let $\{\chi_i \mid i \in \mathbf{N}\} \cup \{\chi_i^- \mid i \in \mathbf{N}\}$ be a partition of unity on $\mathrm{Or}X$ subordinate with the atlas $\mathscr{A}_{\mathrm{Or}X}$ such that $\operatorname{supp} \chi_i \subseteq U_i$, $\operatorname{supp} \chi_i^- \subseteq -U_i$. For each $i \in \mathbf{N}$ define a function $\lambda_i : X \to \mathbf{R}$ by the formula

$$\lambda_i(x) = \begin{cases} \frac{1}{2}[\chi_i \circ (\sigma|_{U_i})^{-1} + \chi_i^- \circ (\sigma|_{-U_i})^{-1}](x), & x \in V_i, \\ 0, & x \notin V_i. \end{cases}$$

We will show that the system $\{\lambda_i \mid i \in \mathbf{N}\}$ is a partition of unity on X subordinate with the atlas $\mathscr{A}_X$. $\lambda_i(x) \neq 0$ holds for $x \in V_i$ iff $\chi_i \circ (\sigma|_{U_i})^{-1}(x) \neq 0$ or $\chi_i^- \circ \circ (\sigma|_{-U_i})^{-1}(x) \neq 0$. Then $\operatorname{supp} \lambda_i = \sigma(\operatorname{supp} \chi_i) \cup \sigma(\operatorname{supp} \chi_i^-) \subseteq \sigma(U_i) \cup \sigma(-U_i) = V_i$. But on V_i, the function λ_i is smooth, i.e. $\lambda_i \in C^\infty(X)$. Evidently, $0 \leqq \lambda_i \leqq 1$. It remains to show that $\Sigma\lambda_i = 1$. Let $x \in X$ be any element. Let $i_j, j \in \{1, \ldots, k\}$ be those indexes from $\mathbf{N}$, for which $x \in V_{i_j}$. Denote $\{\xi_x, -\xi_x\} = \sigma^{-1}(x)$. Then only for the indexes i_j, $j = 1, \ldots, k$ it holds $\xi_x \in U_{i_j}$ or $\xi_x \in -U_{i_j}$. Then

$$\sum_{i \in \mathbf{N}} \lambda_i(x) = \sum_{j=1}^{k} \lambda_{i_j}(x) = \frac{1}{2} \sum_j \chi_{i_j}((\sigma|_{U_{i_j}})^{-1}(x)) + \frac{1}{2} \sum_j \chi_{i_j}^-((\sigma|_{-U_{i_j}})^{-1}(x)).$$

Let us divide the indexes $i_1, \ldots, i_k$ into two disjoint sets $\{\alpha_1, \ldots, \alpha_p\}$, $\{\alpha_{p+1}, \ldots, \alpha_k\}$ so that $(\sigma|_{U_{\alpha_l}})^{-1}(x) = \xi_x$ for $l = 1, \ldots, p$ and $(\sigma|_{U_{\alpha_l}})^{-1}(x) = -\xi_x$ for $l = p + 1, \ldots, k$. Then

$$\sum_{i \in \mathbf{N}} \lambda_i(x) = \frac{1}{2}[\sum_{l=1}^{p} \chi_{\alpha_l}(\xi_x) + \sum_{l=p+1}^{k} \chi_{\alpha_l}(-\xi_x) + \sum_{l=1}^{p} \chi_{\alpha_l}^-(-\xi_x) + \sum_{l=p+1}^{k} \chi_{\alpha_l}^-(\xi_x)].$$

It holds $\xi_x \in U_{\alpha_l}$ for $l = 1, \ldots, p$, and $-\xi_x \in U_{\alpha_l}$ for $l = p + 1, \ldots, k$. Then, by condition (C_2), $\xi_x \notin U_{\alpha_l}$ $(l = p + 1, \ldots, k)$, $-\xi_x \notin U_{\alpha_l}$ $(l = 1, \ldots, p)$. Thus we get

$$\sum_{i\in\mathbf{N}}\lambda_i(x)=\frac{1}{2}[\sum_{l=1}^{k}\chi_{\alpha_l}(\xi_x)+\sum_{l=1}^{k}\chi_{\alpha_l}(-\xi_x)+\sum_{l=1}^{k}\chi_{\alpha_l}^{-}(-\xi_x)+\sum_{l=1}^{k}\chi_{\alpha_l}^{-}(\xi_x)]=$$

$$=\frac{1}{2}[(\sum_{i\in\mathbf{N}}\chi_i(\xi_x)+\sum_{i\in\mathbf{N}}\chi_i^{-}(\xi_x))+(\sum_{i\in\mathbf{N}}\chi_i(-\xi_x)+\sum_{i\in\mathbf{N}}\chi_i^{-}(-\xi_x))]=\frac{1}{2}(1+1)=1.$$

Therefore the system $\{\lambda_i \mid i\in\mathbf{N}\}$ is a partition of unity on X subordinate with the atlas $\mathscr{A}_X$.

Then, of course it holds

$$\int_{\mathrm{Or}X}\sigma^*\omega=\sum_i\int_{U_i}\chi_i\sigma^*\omega+\sum_i\int_{-U_i}\chi_i^{-}\sigma^*\omega=$$

$$=\sum_i\int_{U_i}\sigma^*[(\chi_i\circ(\sigma|_{U_i})^{-1})\,\omega]+\sum_i\int_{-U_i}\sigma^*[(\chi_i^{-}\circ(\sigma|_{-U_i})^{-1})\,\omega]=$$

$$=\sum_i\int_{V_i}(\chi_i\circ(\sigma|_{U_i})^{-1})\,\omega+\sum_i\int_{V_i}(\chi_i^{-}\circ(\sigma|_{-U_i})^{-1})\,\omega=$$

$$=\sum_i\int_{V_i}2\lambda_i\omega=2\sum_i\int_{V_i}\lambda_i\omega=2\int_X\omega.$$

□

Lemma 4.1. *Let X be a manifold with boundary. Then the manifolds $\partial\,\mathrm{Or}X$ and $\mathrm{Or}\,\partial X$ are diffeomorphic.*

Proof. Let $\mathscr{A}_{\mathrm{Or}X}$, $\mathscr{A}_X$ be atlases on n-manifolds $\mathrm{Or}X$, X from Proposition 4.1., let $\|\ \|$ (resp. $|\ |$) be a norm on $\wedge^n TX$ (resp. on $\wedge^{n-1}T\,\partial X$) induced by the Riemannian metric used for the construction of the double cover $\mathrm{Or}X$ (resp. $\mathrm{Or}\,\partial X$). Let ν be nowhere zero field of the exterior normal on the boundary ∂X of the manifold X. Let $(U,\varphi)\in\mathscr{A}_{\mathrm{Or}X}$, $(V,\psi)\in\mathscr{A}_X$ be the corresponding charts on $\mathrm{Or}X$ and on X, such that $\partial V=V\cap\partial X\neq\emptyset$. (Let us remark that $\xi_x\in\sigma^{-1}(x)$ falls into $\partial\,\mathrm{Or}X$ iff $x\in\partial X$). Then $(\partial U,\varphi|_{\partial U})$, $(\partial V,\psi|_{\partial V})$ are charts on $\partial\,\mathrm{Or}X$ and ∂X such that $\sigma|_{\partial U}:\partial U\to\partial V$ is a diffeomorphism.

Let $e_1,\ldots,e_{n-1}$ be vector fields on $T\,\partial V$ such that

$$(\sigma|_{\partial U})^{-1}=\frac{e_1\wedge\ldots\wedge e_{n-1}\wedge\nu}{\|e_1\wedge\ldots\wedge e_{n-1}\wedge\nu\|}.$$

Then the mapping

$$s_U=\frac{e_1\wedge\ldots\wedge e_{n-1}}{|e_1\wedge\ldots\wedge e_{n-1}|}$$

is a continuous global section of the double cover $\varrho|_{\varrho^{-1}(\partial V)}:\varrho^{-1}(\partial V)\to\partial V$ of the manifold ∂V ($\varrho:\mathrm{Or}\,\partial X\to\partial X$ is a double cover of the manifold ∂X). The manifold ∂V is orientable, we can assume that ∂V is connected (see condition (C_3)). Thus its double cover is trivial and consists of two components, denoted by W_1, W_2, diffeomorphic to ∂V. Then $s_U=(\varrho|_W)^{-1}$, where W is one of the sets W_1, W_2, is a diffeomorphism.

To each chart (U,φ) from $\mathscr{A}_{\mathrm{Or}X}$ such that $U\cap\partial\,\mathrm{Or}X\neq\emptyset$ we assigned an open set $s_U(\partial V)$ and a diffeomorphism $T_U:\partial U\to s_U(\partial V)$ defined by $T_U=s_U\circ\sigma|_{\partial U}$.

Moreover, the system $\{s_U(\partial V) \mid U \in \mathscr{A}_{\mathrm{Or}\,X},\ U \cap \partial\, \mathrm{Or}X \neq \emptyset\}$ covers $\mathrm{Or}\,\partial X$. Let us now define the mapping $\Theta : \partial\,\mathrm{Or}X \to \mathrm{Or}\,\partial X$ as follows. If $z \in (U, \varphi) \in \mathscr{A}_{\mathrm{Or}\,X}$, $z \in \partial\,\mathrm{Or}X$, then we set $\Theta(z) = T_U(z)$. This definition is obviously correct and Θ is a bijection. The restriction of Θ on each chart from $\mathscr{A}_{\mathrm{Or}\,X}$ is a diffeomorphism; hence Θ is a local diffeomorphism. From the bijectivity it follows that Θ is a diffeomorphism; that means that the manifolds $\partial\,\mathrm{Or}X$ and $\mathrm{Or}\,\partial X$ are diffeomorphic. □

Proposition 4.2. *Let η be a continuous odd $(n - 1)$-form on a compact n-manifold X with boundary ∂X. Then*

$$\int_{\partial\,\mathrm{Or}\,X} \sigma^*\eta = 2 \int_{\partial X} \eta.$$

Proof. Denote by $j : \partial\,\mathrm{Or}X \to \mathrm{Or}X$, $k : \partial X \to X$ natural inclusions. Then

$$\int_{\partial X} \eta = \int_{\partial X} k^*\eta, \qquad \int_{\partial\,\mathrm{Or}\,X} \sigma^*\eta = \int_{\partial\,\mathrm{Or}\,X} j^*\sigma^*\eta.$$

Let $\varrho : \mathrm{Or}\,\partial X \to \partial X$ be the double cover of the manifold ∂X. Then, according to Proposition 4.1,

$$2 \int_{\partial X} \eta = 2 \int_{\partial X} k^*\eta = \int_{\mathrm{Or}\partial X} \varrho^*k^*\eta.$$

Moreover

$$\int_{\mathrm{Or}\partial X} \varrho^*k^*\eta = \int_{\partial\,\mathrm{Or}\,X} \Theta^*\varrho^*k^*\eta.$$

(see [3], Theorem 1.4). Therefore,

$$2 \int_{\partial X} \eta = \int_{\partial\,\mathrm{Or}\,X} \Theta^*\varrho^*k^*\eta = \int_{\partial\,\mathrm{Or}\,X} (k \circ \varrho \circ \Theta)^*\eta =$$
$$= \int_{\partial\,\mathrm{Or}\,X} (\sigma \circ j)^*\eta = \int_{\partial\,\mathrm{Or}\,X} j^*(\sigma^*\eta) = \int_{\partial\,\mathrm{Or}\,X} \sigma^*\eta.$$
□

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