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REMARKS ON THE NIJENHUIS TENSOR AND ALMOST COMPLEX CONNECTIONS

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Abstract. Let M be a differentiable manifold and S a (1, 1)-tensor field on M. All natural (1, 2)-tensor fields which are of the same type as the Nijenhuis tensor N_S are found. It is proved that an affine connection polynomially naturally induced from S does not exist. All connections naturally induced from a given symmetric affine connection and from the tensor field S such that their torsion tensors are multiples of the Nijenhuis tensor are found. The conditions under which these connections are almost complex connections are deduced.

Key words. Affine connections, natural differential operator, differential invariant, Nijenhuis tensor, almost complex structure, almost complex connection.

MS Classification. 53 C 05, 54 C 15.

1. INTRODUCTION

This paper has been motivated by the theorem of Kobayashi and Nomizu [2], see also Yano [9].

Theorem of Kobayashi and Nomizu: Every almost complex manifold M with an almost complex structure J admits an almost complex affine connection $\tilde{\nabla}$ such that the torsion of $\tilde{\nabla}$ is $1/8N_J$, where N_J is the Nijenhuis tensor (or the torsion) of the almost complex structure J.

Kobayashi and Nomizu have given an example of such connection which is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - K(X, Y), \tag{1.1}$$

where ∇ is an arbitrary symmetric affine connection on M and K is a (1,2)-tensor field defined by

$$4K(X, Y) = (\nabla_{JY}J)\, X + J(\nabla_X)\, X + 2J(\nabla_X J)\, Y \tag{1.2}$$

and X, Y are some vector fields on M.

In this paper we try to explain why an auxiliary symmetric connection ∇ is needed in the proof of the above Theorem of Kobayashi and Nomizu. We try to answer two questions: 1. Does there exist a connection on M naturally polynomially induced from a (1,1)-tensor field only (without using an auxiliary connection)? 2. Is the connection of Kobayashi and Nomizu a unique affine connection naturally induced from a given symmetric affine connection and the almost complex structure J satisfying $8 \operatorname{Tor} \tilde{\nabla} = N_J$ and $\tilde{\nabla} J = 0$?

We shall use the methods of the general theory of natural bundles and natural differential operators, Janyška [1], Nijenhuis [7], Terng [8]. For the convenience of the actual coordinate calculations we shall use the methods of Krupka and Janyška [5], Krupka and Mikolášová [6].

All our considerations are in the category C^∞.

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2. NATURAL TENSORS OF THE NIJENHUIS TYPE

Let S be a (1,1)-tensor field on M, $\dim M = n$, (not necessarily an almost complex structure). The Nijenhuis tensor of S (or the torsion of S) is the (1,2)-tensor field N_S defined by

$$N_S(X, Y) = 2\{[SX, SY] + S^2[X, Y] - S[X, SY] - S[SX, Y]\}, \tag{2.1}$$

where X, Y are arbitrary vector fields on M and $[,]$ denotes the Lie bracket. It is obvious that $N_S(X, Y) = -N_S(Y, X)$ and from (2.1) the coordinate expression of N_S is given by

$$N^i_{jk}(x) = 2\left\{\frac{\partial S^i_k(x)}{\partial x^m} S^m_j(x) - \frac{\partial S^i_j(x)}{\partial x^m} S^m_k(x) - S^i_m(x)\frac{\partial S^m_k(x)}{\partial x^j} + S^i_m(x)\frac{\partial S^m_j(x)}{\partial x^k}\right\}, \tag{2.2}$$

where (x^i), $1 \leqq i \leqq n$, are some local coordinates on M and $S^i_j(x)$ is the coordinate expression of the tensor field S.

In the sense of the general geometrical theory of natural differential operators, [1], [8], the Nijenhuis tensor is a natural differential operator of order one from $T^{(1,1)}M$ (the space of (1,1)-tensors on M) to $TM \otimes \wedge^2 T^*M$. Under natural tensor field of the Nijenhuis type we shall understand a natural differential operator (of finite order) from $T^{(1,1)}M$ to $T^{(1,2)}M$ which is polynomial of degree two. By [5], [8] such an operator of finite order r is uniquely determined by an L^{r+1}_n-equivariant mapping (differential invariant of L^{r+1}_n) from $T^r_n P = J^r_0(R^n, P)$ to V, where P is the type fibre of the functor $T^{(1,1)}$, i.e. $P = R^n \otimes R^{n*}$ with the tensor action of the group L^1_n; and V is the type fibre of the functor $T^{(1,2)}$, i.e. $V = R^n \otimes \otimes^2 R^{n*}$ with the tensor action of L^1_n. The group L^r_n is the r-th differential group formed by all invertible r-jets from R^n into R^n with source and target in the origin of R^n.

Let (S^i_j), $1 \leqq i, j \leqq n$, be the canonical global coordinates on P, then on $T^r_n P$ there are the induced coordinates $(S^i_j, S^i_{j,m}, \ldots, S^i_{j,m_1\ldots m_r})$, $1 \leqq i, j, m, m_1, \ldots, m_r \leqq$ $< n$, defined by $S^i_{j,m_1\ldots m_s}(j^r_0\sigma) = D_{m_1\ldots m_s}(\sigma^i_j)(0)$, $s = 0,1, \ldots, r$, $j^r_0\sigma \in T^r_n P$. Let $(a^i_j, \ldots, a^i_{j_1\ldots j_r})$, $1 \leqq i, j_1, \ldots, j_r \leqq n$, be the canonical coordinates on L^r_n, i.e. $a^i_{j_1\ldots j_s}(j^r_0\alpha) = D_{j_1\ldots j_s}(\alpha^i)(0)$, $s = 1, \ldots, r$, $j^r_0\alpha \in L^r_n$, $\alpha\colon R^n \to R^n$, $\alpha(0) = 0$. Then the coordinate expression of the action L^1_n on P can be written as

$$\bar{S}^i_j = a^i_p S^p_q b^q_j, \tag{2.3}$$

where $(b^i_j, \ldots, b^i_{j_1\ldots j_r})$ denote the coordinates of the inverse element of $j^r_0\alpha \in L^r_n$, i.e. $b^i_{j_1\ldots j_s}(j^r_0\alpha) = a^i_{j_1\ldots j_s}(j^r_0\alpha^{-1})$, $s = 1, \ldots, r$. The action (2.3) of L^1_n on P induces the action of L^{r+1}_n on $T^r_n P$ which is given in the canonical coordinates by (2.3) and by the formal differentiation of (2.3) up to the order r. For instance, if $r = 1$, the action of L^2_n on $T^1_n P$ is given by (2.3) and

$$\bar{S}^i_{j,m} = a^i_{pr} b^r_m S^p_q b^q_j + a^i_p(S^p_{q,r} b^q_j b^r_m + S^p_q b^q_{jm}). \tag{2.4}$$

The action of L^3_n on $T^2_n P$ is given by (2.3), (2.4) and the formal differentiation of (2.4), etc.

Let

$$\xi = \sum_{s=1}^{r+1} \xi^i_{j_1\ldots j_s} \frac{\partial}{\partial a^i_{j_1\ldots j_s}}, \tag{2.5}$$

be a member of the Lie algebra l^{r+1}_n of L^{r+1}_n. The fundamental vector field on $T^r_n P$ related to the action of L^{r+1}_n on $T^r_n P$ can be expressed by

$$\Xi(T^r_n P) = \sum_{s=1}^{r+1} \Xi^{j_1\ldots j_s}_i(T^r_n P)\, \xi^i_{j_1\ldots j_s}, \tag{2.6}$$

where $\Xi^{j_1\ldots j_s}_i(T^r_n P)$ are vector fields on $T^r_n P$. In the induced coordinates

$$\Xi^{j_1\ldots j_s}_i(T^r_n P) = \sum_{t=s-1}^{r} \left(\frac{\partial \bar{S}^p_{q,m_1\ldots m_t}}{\partial a^i_{j_1\ldots j_s}}\right) e \frac{\partial}{\partial S^p_{q,m_1\ldots m_t}}, \tag{2.7}$$

where $s = 1, \ldots, r + 1$ and $e \in L^{r+1}_n$ is the unity, i.e. $e = j^{r+1}_0 \mathrm{id}^n_R$.

Let (v^i_{jk}), $1 \leqq i, j, k \leqq n$, be the canonical global coordinates on V. The action of L^1_n on V is given in the coordinates by

$$\bar{v}^i_{jk} = a^i_p v^p_{qr} b^q_j b^r_k. \tag{2.8}$$

Then the corresponding fundamental vector field on V induced by $\xi \in l^1_n$ is expressed by

$$\Xi(V) = \Xi^j_i(V)\xi^i_j, \tag{2.9}$$

where

$$(2.10)\qquad \Xi_i^j(V) = \left(\frac{\partial \bar{v}^p_{qr}}{\partial a^i_j}\right)_e \frac{\partial}{\partial v^p_{qr}} = = (\delta^p_i v^j_{qr} - v^p_{ir}\delta^j_q - v^p_{qi}\delta^j_r)\frac{\partial}{\partial v^p_{qr}}\,.$$

To calculate differential invariants we shall use the following

Lemma 1. *Let U and W be two L^r_n-manifolds, $f: U \to W$ a mapping. The following conditions are equivalent:*

(i) *f is a differential invariant.*

(ii) *For each element $\xi \in l^r_n$*

$$(2.11)\qquad \partial_\xi f = 0,$$

where ∂_ξ denotes the Lie derivative with respect to ξ, and there exists an element $A_0 \in L^{r(-)}_n$ such that $f(A_0 u) = A_0 f(u)$ for all $u \in U$.

For the proof see [3], [5].

As a corollary of Lemma 1 we obtain that f: $T^r_n P \to V$ is a differential invariant of the group L^{r+1}_n if and only if the vector fields $\Xi_i^{j_1 \dots j_s}(T^r_n P)$ and $\Xi_i^{j_1 \dots j_s}(V)$, $s = 1, \dots, r+1$, are f-related and there exists an element $A_0 \in L^{r+1(-)}_n$, $\det(a^i_j(A_0)) < 0$, such that $f(A_0 p) = A_0 f(p)$, $p \in T^r_n P$. Let us remark that $\Xi_i^{j_1 \dots j_s}(V)$ is the zero vector field if $s \geqq 2$.

Lemma 2. *Polynomial differential invariants from $T^r_n P$ to V depend on variables from $T^1_n P$ only.*

Proof. If $f: T^r_n P \to V$ is a differential invariant, then the vector fields $\Xi_i^j(T^r_n P)$ and $\Xi_i^j(V)$ are f-related. From (2.7)

$$(2.12)\qquad \Xi_i^j(T^r_n P) = \sum_{s=0}^{r} \left(\frac{\partial \bar{S}^r_{q,m_1 \dots m_s}}{\partial a^i_j}\right)_e \frac{\partial}{\partial S^p_{q,m_1 \dots m_s}}\,.$$

If $v^a_{bc} = f^a_{bc}(S^p_q, S^p_{q,m_1}, \dots, S^p_{q,m_1 \dots m_r})$ is the coordinate expression of f, then by Lemma 1 f^a_{bc} have to satisfy the following system of partial differential equations

$$(2.13)\qquad \sum_{s=0}^{r} \left(\frac{\partial \bar{S}^p_{q,m_1 \dots m_s}}{\partial a^i_j}\right)_e \frac{\partial f^a_{bc}}{\partial S^p_{q,m_1 \dots m_s}} = = \delta^a_i f^j_{bc} - f^a_{ic}\delta^j_b - f^a_{bi}\delta^j_c\,.$$

From the coordinate expression of the action of L^{r+1}_n on $T^r_n P$ we obtain for $i = j$ (no summation over i) the system

$$(2.14)\qquad \sum_{s=0}^{r} s \cdot \frac{\partial f^a_{bc}}{\partial S^p_{q,m_1 \dots m_s}} S^p_{q,m_1 \dots m_s} = f^a_{bc}\,.$$

By [5] it follows that polynomial solutions of (2.14) have to be polynomials of degrees a_s in variables $S^p_{q,m_1\dots m_s}$, $s = 0, \dots, r$, such that

$$\sum_{s=0}^{r} s \,.\, a_s = 1. \tag{2.15}$$

(2.15) is satisfied for arbitrary a_0, $a_1 = 1$ and $a_i = 0$, $1 < i \leqq r$. Thus polynomial differential invariants from $T^r_n P$ to V depend on S^p_q (in any degree) and on $S^p_{q,m}$ (linearly) only which proves our Lemma 2.

Corollary 1. *Every non-zero polynomial natural differential operator of finite order from $T^{(1,1)}M$ to $T^{(1,2)}M$ is of order one.*

Now we are able to prove the following

Proposition 1. *If $n \geqq 2$, there exists a 9-parameter family of natural tensor fields of the Nijenhuis type. This family is a vector space over R generated by the tensor fields N_S, $S \otimes \mathrm{d}(\mathrm{tr}\, S)$, $\mathrm{d}(\mathrm{tr}\, S) \otimes S$, $I \otimes C^1_2(S \otimes \mathrm{d}(\mathrm{tr}\, S))$, $C^1_2(S \otimes \mathrm{d}(\mathrm{tr}\, S)) \otimes I$, $(I \otimes \mathrm{d}(\mathrm{tr}\, S))\,\mathrm{tr}\, S$, $(\mathrm{d}(\mathrm{tr}\, S) \otimes I)\,\mathrm{tr}\, S$, $I \otimes \mathrm{d}(\mathrm{tr}\, S^2)$, $\mathrm{d}(\mathrm{tr}\, S^2) \otimes I$, where* tr *denotes the trace of S, C^i_j denotes the contraction with respect to the i-th superscript and j-th subscript, d is the exterior derivative and $I = (\delta^i_j)$ is the absolute invariant tensor field.*

Proof. By Lemma 2 every tensor field of the Nijenhuis type is given by a quadratic mapping from $T^1_n P$ to V which can be expressed in the coordinates by

$$v^i_{jk} = A^{iqs_1s_2}_{jkpr} S^p_q S^r_{s_1s_2}, \tag{2.16}$$

where $A^{iqs_1s_2}_{jkpr}$ are real coefficients. From L^2_n-equivariancy of (2.16) we obtain that $A^{iqs_1s_2}_{jkpr}$ is an absolute invariant tensor, [5], i.e.

$$A^{iqs_1s_2}_{jkpr} = \sum_{\sigma} c_\sigma \delta^i_{\sigma(j)} \delta^q_{\sigma(k)} \delta^{s_1}_{\sigma(p)} \delta^{s_2}_{\sigma(r)}, \tag{2.17}$$

where σ runs all permutations of four indices (j, k, p, r) and $c_\sigma \in R$. Further $\Xi^{j_1j_2}_i(T^1_n P)$ and $\Xi^{j_1j_2}_i(V)$ have to be related with respect to (2.16), i.e.

$$\left(\frac{\partial S^p_{q,m}}{\partial a^i_{j_1j_2}}\right) e \frac{\partial A^{aluv}_{bckt} S^k_l S^t_{u,v}}{\partial S^p_{q,m}} = 0. \tag{2.18}$$

Using (2.4) and (2.17) we obtain for 24 coefficients c_σ a system of linear homogeneous equations which has 9 independent variables and (2.16) has the expression

$$\begin{aligned} v^i_{jk} = {} & A_1(S^m_j S^i_{k,m} - S^m_k S^i_{j,m} - S^i_m S^m_{k,j} + S^i_m S^m_{j,k}) + \\ & + A_2 S^i_j S^m_{m,k} + A_3 S^i_k S^m_{m,j} + A_4 \delta^i_j S^p_k S^m_{m,p} + A_5 \delta^i_k S^p_j S^m_{m,p} + \\ & + A_6 \delta^i_j S^p_{p,k} S^m_m + A_7 \delta^i_k S^p_{p,j} S^m_m + A_8 \delta^i_j S^p_m S^m_{p,k} + \\ & + A_9 \delta^i_k S^p_m S^m_{p,j}, \end{aligned} \tag{2.19}$$

where $A_i \in R$, $i = 1, \dots, 9$.

It is easy to see that (2.19) is the differential invariant which corresponds to the family of natural tensor fields of the Nijenhuis type given by Proposition 1.

The Nijenhuis tensor N_S is the unique operator from the list of natural operators of Proposition 1 the values of which are in $TM \otimes \wedge^2 T^*M$. Using the antisymmetrisation with respect to subscripts we immediately obtain

Corollary 2. *If $n \geqq 2$, there exists a 5-parameter family of natural differential operators of the Nijenhuis type from $T^{(1,1)}M$ to $TM \otimes \wedge^2 T^*M$. This family is a linear combination (with real coefficients) of operators N_S, $S \wedge \mathrm{d}(\mathrm{tr}\, S)$, $I \wedge \wedge C_2^1(S \otimes \mathrm{d}(\mathrm{tr}\, S))$, $(I \wedge \mathrm{d}(\mathrm{tr}\, S))\, \mathrm{tr}\, S$, $I \wedge \mathrm{d}(\mathrm{tr}\, S^2)$, where $\wedge$ denotes the tensor product combined with the antisymmetrisation with respect to subscripts.*

Corollary 3. *The Nijenhuis tensor N_J is the unique non-trivial operator of the Nijenhuis type which transforms a given almost complex structure J to a (1,2)-tensor field.*

Proof. If J is an almost complex structure, then $\mathrm{tr}\, J = 0$ and $\mathrm{tr}\, J^2 = -n$. Our Corollary 3 now immediately follows from Proposition 1 putting $S = J$.

3. CONNECTIONS NATURALLY INDUCED FROM A (1,1)-TENSOR FIELD

Let $CM \to M$ be the fibred space of elements of affine connections on M, i.e. C is a lifting functor of order 2 with the type fibre $Q = R^n \otimes \otimes^2 R^{n*}$, where the action of the group L_n^2 on Q is given, in the canonical coordinates (Γ_{jk}^i), $1 \leqq i, j, k \leqq n$, by

$$\bar{\Gamma}_{jk}^i = a_p^i(\Gamma_{qr}^p b_j^q b_k^r + b_{jk}^p). \tag{3.1}$$

Under an affine connection of order r naturally induced from a (1,1)-tensor field we understand a natural differential operator of order r from $T^{(1,1)}M$ to CM. Every such operator is uniquely given by an L_n^{r+1}-equivariant mapping from $T_n^r P$ to Q. By Lemma 1 if $f : T_n^r P \to Q$ is a differential invariant, then the corresponding fundamental vector fields on $T_n^r P$ and Q are f-related. The fundamental vector fields on $T_n^r P$ are given by (2.7) and from (3.1)

$$\Xi_i^j(Q) = \left(\frac{\partial \bar{\Gamma}_{qr}^p}{\partial a_j^i}\right) e \frac{\partial}{\partial \Gamma_{qr}^p} = (\delta_i^p \Gamma_{qr}^{\ j} - \Gamma_{ir}^p \delta_q^j - \Gamma_{qi}^p \delta_r^j) \frac{\partial}{\partial \Gamma_{qr}^p}, \tag{3.2}$$

$$\Xi_i^{j_1 j_2}(Q) = \left(\frac{\partial \bar{\Gamma}_{qr}^p}{\partial a_{j_1 j_2}^i}\right) e \frac{\partial}{\partial \Gamma_{qr}^p} = -\frac{1}{2} \delta_i^p (\delta_q^{j_1} \delta_r^{j_2} + \delta_r^{j_1} \delta_q^{j_2}) \frac{\partial}{\partial \Gamma_{qr}^p}. \tag{3.3}$$

Using the same methods as in Lemma 2 we deduce that all polynomial L_n^{r+1}-equivariant mappings from $T_n^r P$ to Q depend on variables from $T_n^1 P$ only and so all polynomial natural affine connections naturally depending on a (1,1)-tensor field and its finite order derivatives are of order one.

Lemma 3. *There is no non-zero polynomial L^2_n-equivariant mapping from T^1_nP to Q.*

Proof. From Lemma 1, (2.7), (3.2) and (3.3) we obtain that every differential invariant from T^1_nP to Q has to satisfy the following system of partial differential equations

$$(3.4)\qquad (\delta^p_i S^j_q - S^p_i \delta^j_q)\frac{\partial f^a_{bc}}{\partial S^p_q} + (\delta^p_i S^j_{q,r} - S^p_{i,r}\delta^j_q - S^p_{q,i}\delta^j_r)\frac{\partial f^a_{bc}}{\partial S^p_{q,r}} = \\ = \delta^a_i f^j_{bc} - f^a_{ic}\delta^j_b - f^a_{bi}\delta^j_c,$$

$$(3.5)\qquad (\delta^p_i S^{j_1}_q \delta^{j_2}_r + \delta^p_i S^{j_2}_q \delta^{j_1}_r - S^p_i \delta^{j_1}_q \delta^{j_2}_r - S^p_i \delta^{j_1}_r \delta^{j_2}_q)\frac{\partial f^a_{bc}}{\partial S^p_{q,r}} = \\ = \delta^a_i(\delta^{j_1}_b \delta^{j_2}_c + \delta^{j_1}_c \delta^{j_2}_b),$$

where $\Gamma_{bc}^{\ a} = f^a_{bc}(S^p_q, S^p_{q,r})$ is the coordinate expression of $f: T^1_nP \to Q$.

By [5] a polynomial solution of (3.4) has to be a sum of polynomials which are linear in $S^p_{q,r}$ and of arbitrary degrees in S^p_q, i.e.

$$(3.6)\qquad f^a_{bc} = \sum_m A^{aqrl_1\dots l_m}_{bcpk_1\dots k_m} S^p_{q,r} S^{k_1}_{l_1} \dots S^{k_m}_{l_m},$$

where $m = 0, 1, \dots$ and $A^{aqrl_1\dots l_m}_{bcpk_1\dots k_m}$ are absolute invariant tensors. (3.6) has to satisfy the system (3.5). Substituting (3.6) into (3.5) we obtain for every m a contradiction which proves our Lemma 3.

From Lemma 3 it immediately follows

Proposition 2. *There is no non-zero polynomial affine connection of finite order naturally induced from a* (1,1)*-tensor field.*

Remark 1. From Proposition 2 it follows that a connection, depending naturally on an almost complex structure only such that its torsion tensor is a multiple of the Nijenhuis tensor, does not exist. This is the reason why Kobayashi and Nomizu had to use an auxiliary symmetric connection.

4. ON NON-UNIQUENESS OF THE ALMOST COMPLEX CONNECTION

Now let an affine connection $\tilde{\nabla}$ be naturally induced from a given symmetric affine connection ∇ and a (1,1)-tensor field S, i.e. $\tilde{\nabla}(\nabla, S)$ is a natural differential operator from $C_SM \oplus T^{(1,1)}M$ to CM, where C_SM is the subbundle in CM formed by elements of all symmetric affine connections on M and $\oplus$ denotes the Whitney's sum. It is obvious that such a connection $\tilde{\nabla}$ can be expressed by

$$\tilde{\nabla}_X Y = \nabla_X Y - L(X, Y), \tag{4.1}$$

where X, Y are arbitrary vector fields on M and L is a (1,2)-tensor field naturally induced from ∇ and S, i.e. L is a natural differential operator from $C_S M \otimes T^{(1,1)}M$ to $T^{(1,2)}M$.

We shall need the following

Lemma 4. *Let W be a left L^r_n-manifold and U a left L^1_n-manifold. Let K^r_n be the kernel of the canonical group homomorphism $\pi^{r,1} : L^r_n \to L^1_n$, $\pi^{r,1}(j^r_0\alpha) = j^1_0\alpha$, and $\pi : W \to W/K^r_n$ the canonical projection onto the orbit space. Each L^r_n-equivariant mapping $f : W \to U$ is of the form*

$$f = F \circ \pi, \tag{4.2}$$

where $F : W/K^r_n \to U$ is a uniquely determined L^1_n-equivariant mapping.

For the proof see [4], [5].

Proposition 3. *All affine connections $\tilde{\nabla}$ on M naturally induced from a symmetric affine connection ∇ and a (1,1)-tensor field S such that*

$$\operatorname{Tor} \tilde{\nabla} = \lambda N_S, \tag{4.3}$$

$\lambda \in R$, are of the form

$$\begin{aligned} \tilde{\nabla}_X Y = \nabla_X Y - \{&\alpha S(\nabla_Y S)\, X + (\alpha + 2\lambda)\, S(\nabla_X S)\, Y + \\ &+ \beta(\nabla_{SX} S)\, Y + (\beta + 2\lambda)\,(\nabla_{SY} S)\, X\} - P(X, Y), \end{aligned} \tag{4.4}$$

where $\alpha, \beta \in R$ and P is a natural differential operator from $C_S M \oplus T^{(1,1)}M$ to $TM \otimes \odot\, T^*M$.

Proof. Every desired connection is of the form (4.1) and the assumption (4.3) implies

$$\lambda N_S(X, Y) = L(Y, X) - L(X, Y). \tag{4.5}$$

Hence we have to find natural differential operators L from $C_S M \oplus T^{(1,1)}M$ to $T^{(1,2)}M$ satisfying (4.5). Let L be an r-th order operator, then the corresponding differential invariant is an L^{r+2}_n-equivariant mapping from $T^r_n Q_S \times T^{r+1}_n P$ to V, $Q_S = R^n \otimes \odot^2 R^{n*} \subset Q$. The canonical coordinates (Γ^i_{jk}), $j \leq k$, on Q_S and (S^i_j) on P induce the coordinates $(\Gamma^i_{jk}, \Gamma^i_{jk,m_1}, \ldots, \Gamma^i_{jk,m_1\ldots m_r}, S^i_j, S^i_{j,m_1}, \ldots, S^i_{j,m_1\ldots m_{r+1}})$, $1 \leqq m_1 \leqq m_2 \leqq \ldots \leqq n$, on $T^r_n Q_S \times T^{r+1}_n P$. Let us consider on $T^r_n Q_S \times T^{r+1}_n P$, $r \geqq 1$, the system of functions

$$\Gamma^i_{jkm_1\ldots m_s} = \Gamma^i_{(jk,m_1\ldots m_s)}, \qquad s = 0, \ldots, r, \tag{4.6}$$

(4.7) $$R^i_{jkl;m_1;\dots;m_s}, \qquad s = 0, \dots, r-1,$$

(4.8) $$S^i_{j;m_1;\dots;m_s}, \qquad s = 0, \dots, r+1,$$

where $(jk, m_1 \dots m_s)$ denotes the symmetrisation of the indices, R^i_{jkl} is the formal curvature tensor of Γ^i_{jk}, i.e.

(4.9) $$R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^i_{pl}\Gamma^p_{jk} - \Gamma^i_{pk}\Gamma^p_{jl},$$

and ";" denotes the formal covariant derivative with respect to Γ^i_{jk}, i.e. for instance

(4.10) $$S^i_{j;k} = S^i_{j,k} + \Gamma^i_{pk}S^p_j - \Gamma^p_{jk}S^i_p.$$

In general, the s-th formal covariant derivative is deduced formally in the same way as the s-th covariant derivative of a tensor field with respect to a symmetric affine connection. The systems of functions (4.6)–(4.8) contain a subsystem defining a global chart on $T^r_nQ_S \times T^{r+1}_nP$, [5]. Each such global chart on $T^r_nQ_S \times T^{r+1}_nP$ will be called an adapted chart and the functions belonging to an adapted chart will be called adapted coordinates.

If $r = 0$, then $(\Gamma^i_{jk}, S^i_j, S^i_{j;k})$ form a new global chart on $Q_S \times T^1_nP$ which will be also called an adapted chart.

Using Lemma 4 we obtain that every L^{r+2}_n-equivariant mapping f: $T^r_nQ_S \times T^{r+1}_nP \to V$ determines a unique L^1_n-equivariant mapping F: $(T^r_nQ_S \times \times T^{r+1}_nP)/K^{r+2}_n \to V$ such that $f = F \circ \pi$.

Calculating the L^1_n-equivariant mapping F in some adapted coordinates we deduce that F is a function of $R^i_{jkl;m_1;\dots;m_s}$, $s = 0, \dots, r-1$, and $S^i_{j;m_1;\dots;m_s}$, $s = 0, \dots, r+1$, only, i.e. F is the sum of

(4.11) $$v^i_{jk} = \bar{F}^i_{jk}(R^i_{jkl}, \dots, R^i_{jkl;m_1;\dots;m_{r-1}}, S^i_j, \dots, S^i_{j;m_1;\dots;m_{r+1}})$$

$r \geqslant 1$, and

(4.12) $$v^i_{jk} = F^i_{jk}(S^i_j, S^i_{j;k}).$$

From the assumption (4.5) it follows that we are interested in this part of the L^1_n-equivariant mapping (4.12) which is linear in S^i_j and $S^i_{j;k}$. By Lemma 1 such a polynomial mapping has to satisfy the following system of partial differential equations

(4.13) $$\left(\frac{\partial \bar{S}^p_q}{\partial a^i_j}\right)_e \frac{\partial F^a_{bc}}{\partial S^p_q} + \left(\frac{\partial \bar{S}^p_{q;r}}{\partial a^i_j}\right)_e \frac{\partial F^a_{bc}}{\partial S^p_{q;r}} = \\ = \delta^a_i F^j_{bc} - F^a_{ic}\delta^j_b - F^a_{bi}\delta^j_c.$$

By [5] the desired polynomial solution of (4.13) has the form

$$F^i_{jk} = a_1\delta^i_j S^p_k S^q_{q;p} + a_2\delta^i_k S^p_j S^q_{q;p} + a_3\delta^i_j S^p_p S^q_{k;q} + \\ + a_4\delta^i_k S^p_p S^q_{j;q} + a_5\delta^i_j S^p_p S^q_{q;k} + a_6\delta^i_k S^p_p S^q_{q;j} + a_7\delta^i_j S^p_q S^q_{p;k} + \\ + a_8\delta^i_k S^p_q S^q_{p;j} + a_9\delta^i_j S^p_k S^q_{p;q} + a_{10}\delta^i_k S^p_j S^q_{p;q} + \\ + a_{11}\delta^i_j S^p_q S^q_{k;p} + a_{12}\delta^i_k S^p_q S^q_{j;p} + a_{13} S^i_j S^p_{k;p} + a_{14} S^i_k S^p_{j;p} + \\ + a_{15} S^i_j S^p_{p;k} + a_{16} S^i_k S^p_{p;j} + a_{17} S^p_j S^i_{p;k} + a_{18} S^p_k S^i_{p;j} + \\ + a_{19} S^p_p S^i_{j;k} + a_{20} S^p_p S^i_{k;j} + a_{21} S^i_p S^p_{j;k} + a_{22} S^i_p S^p_{k;j} + \\ + a_{23} S^p_j S^i_{k;p} + a_{24} S^p_k S^i_{j;p}, \tag{4.14}$$

where $a_i \in R$, $i = 1, \ldots, 24$. From (4.5) we have $F^i_{kj} - F^i_{jk} = \lambda N^i_{jk}$ and by (2.2)

$$a_{22} - a_{21} = 2\lambda, \qquad a_{24} - a_{23} = 2\lambda, \\ a_{h+1} = a_h, \qquad h = 2g + 1, \qquad g = 0, \ldots, 9. \tag{4.15}$$

Let us put $a_{21} = \alpha$, $a_{23} = \beta$, then the differential invariant corresponding to L is of the form

$$L^i_{jk} = \alpha S^i_p S^p_{j;k} + (\alpha + 2\lambda) S^i_p S^p_{k;j} + \beta S^p_j S^i_{k;p} + \\ + (\beta + 2\lambda) S^p_k S^i_{j;p} + P^i_{jk}, \tag{4.16}$$

where $P^i_{jk} = P^i_{kj}$ includes the symmetric solutions of (4.13) and symmetric L^1_n-equivariant mappings given by (4.11). It is easy to see that the differential invariant (4.16) corresponds to the operator which can be expressed by

$$L(X, Y) = \alpha S(\nabla_Y S) X + (\alpha + 2\lambda) S(\nabla_X S) Y + \\ + \beta(\nabla_{SX} S) Y + (\beta + 2\lambda)(\nabla_{SY} S) X + P(X, Y), \tag{4.17}$$

which proves our Proposition 3.

From now let S be an almost complex structure on M, i.e. $S = J$, $J^2 = -I$. The connection (4.4) coincides with the example of Kobayashi and Nomizu if and only if $\lambda = 1/8$, $\alpha = 1/4$, $\beta = 0$ and P is the zero tensor field.

Proposition 4. *Let M be an almost complex manifold with an almost complex structure J. Then a natural connection $\tilde{\nabla}$ given by* (4.4) *is an almost complex connection if and only if*

$$\tilde{\nabla}_X Y = \nabla_X Y - K(X, Y) - P(X, Y), \tag{4.18}$$

where $K(X, Y)$ is given by (1.2) *and $P(X, Y)$ satisfies*

$$P(X, Y) = -JP(X, JY). \tag{4.19}$$

Proof. Let $\tilde{\nabla}_X Y = \nabla_X Y - L(X, Y)$, where $L(X, Y)$ is given by (4.17). From the assumption $\tilde{\nabla} J = 0$ we obtain $J(\tilde{\nabla}_X Y) = \tilde{\nabla}_X(JY)$ which implies

$$(\nabla_X J) Y = L(X, JY) - JL(X, Y). \tag{4.20}$$

Substituting (4.17) into (4.20) and using $0 = \nabla_X J^2 = (\nabla_X J) J + J(\nabla_X J)$ we obtain

$$(4.21) \qquad 0 = (-1 + 2\alpha + 4\lambda)(\nabla_X J) Y + 2\beta(\nabla_{JX} J) JY + \\ + (\beta + 2\lambda - \alpha)((\nabla_{JY} J) JX - (\nabla_Y J) X) + P(X, JY) - JP(X, Y),$$

which implies our Proposition 4.

Remark 2. The example of Kobayashi and Nomizu is not unique natural almost complex connection satisfying Tor $\tilde{\nabla} = \lambda N_J$. But other natural almost complex connections satisfying this condition differ from the example of Kobayashi and Nomizu only by a natural symmetric (with respect to subscripts) (1,2)-tensor field which satisfies (4.19). Such non-trivial tensor fields exist. For instance

$$(4.22) \qquad P = I \odot C^{12}_{23}(J \otimes \nabla J) + J \odot C^1_2 \nabla J,$$

where $\odot$ denotes the tensor product combined with the symmetrisation with respect to subscripts, is the example of natural operator which satisfies (4.19).

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