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SEMIREGULAR FRAMES

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Abstract. The properties of semiregular frames are studied. Any dense homomorphic image of a semiregular frame is semiregular. A sum of semiregular frames is semiregular. If L is a semiregular frame then there exists a compact spatial semiregular frame $R(L)$ and a surjective dense frame homomorphism $\sigma : R(L) \to L$. There exists a compact normal Hausdorff frame which is not semiregular, i.e., is not a topology.

Key words. Semiregular element of a frame, (hereditary) semiregular frame, sums and homomorphisms, almost compact frame.

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In connections with investigations of separation axioms for frames (see for example [2], [9]) it is natural to introduce separation axioms for frames such that the subcategory F of the category **Frm** of frames, corresponding to the given separation axiom, is closed with respect to homomorphic images and sums. Further, we want that this subcategory F is determined by the corresponding subcategory T of the category **Top** of topological spaces given by the same separation axiom in the sense that $F \cap \mathbf{Top} = T$. This problem was solved for completely regular frames by B. Banaschewski and C. J. Mulvey [1] and for T_1-frames by J. Rosický and B. Šmarda [8].

In the case of a T_2-axiom (see for example [5] or [7]) several subcategories of **Frm** are described, closed under sums and homomorphic images.

In this paper we shall investigate similar questions for semiregular frames. Any dense homomorphic image of a semiregular frame is semiregular. A sum of semiregular frames is semiregular. If L is a semiregular frame then there exist a compact spatial semiregular frame $R(L)$ and a surjective dense frame homomorphism $\sigma : R(L) \to L$. There exists a compact normal Hausdorff frame which is not semiregular, i.e., it is not a topology.

All unexplained facts concerning frames can be found in P. T. Johnstone [4]. Recall that a *frame* is a complete lattice L in which the infinite distributive law

$a \wedge \bigvee S = \bigvee\{a \wedge s: s \in S\}$ holds for all $a \in L$, $S \subseteqq L$. The set of all open sets of a topological space forms a frame. These frames and frames isomorphic with them are called *spatial* or *topologies*.

Regular and normal frames are defined in [2]. *Hausdorff frame* is a frame L with the property: $a, b \in L, 1 \neq a \nleqq b \Rightarrow \exists c \in L : c^* \nleqq a, c \nleqq b$. L is a Hausdorff frame iff $a = \bigvee(x \in L : x \leqq a, x^* \nleqq a)$ for any $1 \neq a \in L$ (see [5]).

We say that an element $a \in L$, $1 \neq a$ of a frame is *prime*, or *semiprime* resp., if

$$x \wedge y \leqq a \Rightarrow x \leqq a \quad \text{or} \quad y \leqq a, \qquad \text{or}$$
$$x \wedge y = 0 \Rightarrow x \leqq a \quad \text{or} \quad y \leqq a \qquad \text{resp.},$$

for any $x, y \in L$. If we denote $D(L)$, $P(L)$ resp., $S(L)$ resp., the set of all dual atoms, prime elements resp., semiprime elements resp., in L then $D(L) \subseteqq P(L) \subseteqq S(L)$. We say that L is an S-frame if $S(L) = D(L)$. Spatial Hausdorff frames or S-frames correspond to topologies of usual Hausdorff topological spaces.

§ 1. SEMIREGULARITY IN FRAMES

Definition. Let L be a frame. We say that an element $a \in L$ is *semiregular* if $a = \bigvee(x \in L : x^{**} \leqq a)$. *Let* $Sreg(L)$ be the set of all semiregular elements of L. We say that L is *semiregular* if $L = Sreg(L)$.

Remark. Any regular frame is semiregular ([4], 1.8, p. 89). Semiregular spatial frames are topologies of usual semiregular topological spaces (for example see [10]). Adding a new top element to the four element Boolean algebra, we get a semiregular spatial frame which is not a T_1-frame and its homomorphic images are semiregular.

We denote by $L_r = \{a \in L: a^{**} = a\}$.

1.1. Lemma. *Let L be a frame. Then* $Sreg(L)$ *is a semiregular subframe of L.*

Proof. If $a, b \in Sreg(L)$ then $a \wedge b = \bigvee(x \wedge y : x^{**} \leqq a, y^{**} \leqq b) = \bigvee(x \wedge y: (x \wedge y)^{**} = x^{**} \wedge y^{**} \leqq a \wedge b) = \bigvee(z: z^{**} \leqq a \wedge b)$. If $a_i \in Sreg(L)$ then $\bigvee a_i = \bigvee(x_{ij}: x_{ij}^{**} \leqq a_i) = \bigvee(z: z^{**} \leqq \bigvee a_i)$. Since $L_r \subseteqq Sreg(L)$, we have that $Sreg(L)$ is semiregular.

Recall that any frame homomorphism $f: K \to L$ determines a mapping $f_0 : L \to K$ such that $f_0(a) = \bigvee\{x \in K: f(x) \leqq a\}$. It is easy to see that f_0 preserves prime and semiprime elements. Consequently, if $p \in P(L)$ then $\bigvee\{x \leqq p: x \in Sreg(L)\}$ is a prime in $Sreg(L)$. The fact that $x \wedge y = 0$ implies $x^{**} \wedge y^{**} = 0$, for $x, y \in L$, follows that semiprime elements in $Sreg(L)$ are semiprime in L. Consequently, if p is semiprime element in L then $\bigvee\{x \leqq p: x \in Sreg(L)\}$ is semiprime in L.

1.2. Proposition. *If L is an S-frame then $Sreg(L)$ is an S-frame.*

Proof. Any semiprime element p in $Sreg(L)$ is semiprime in L, i.e., p is a dual atom in L.

1.3. Proposition. *If L is a Hausdorff frame then $Sreg(L)$ is a Hausdorff frame.*

Proof. If $a, b \in Sreg(L)$, $1 \neq a \leqq b$ then there exists $k, l \in L$ such that $k \nleqq a$, $l \nleqq b$, $k \wedge l = 0$. Clearly, $k^{**} \nleqq a$, $l^{**} \nleqq b$, $k^{**} \wedge l^{**} = 0$, $k^{**}, l^{**} \in Sreg(L)$.

1.4. Proposition. *If L is a normal frame then $Sreg(L)$ is normal.*

Proof. If $a, b \in Sreg(L)$, $a \vee b = 1$ then there exist $c, d \in L$ such that $c \vee b = 1 = a \vee d, \vee d$, $c \wedge d = 0$. Clearly, $c^{**} \vee b = 1 = a \vee d^{**}$, $c^{**} \wedge d^{**} = 0$, $c^{**}, d^{**} \in Sreg(L)$, i.e., $Sreg(L)$ is normal.

Definition. Let j be a nucleus on a frame L. We say that an element $a \in L$ is *j-regular* if $a = \bigvee(x \in L: j(x) \leqq a) = \bigvee(j(x): j(x) \leqq a)$.

Let $L(j)$ be the set of all j-regular elements of L. Clearly, $L(j)$ is a subframe of L. We say that j is *regular* if j is dense (i.e., $j(a) = 0 \Rightarrow a = 0$) and $L(j) = L$.

1.5. Lemma. *If L is a frame, $j : L \to L$ is a dense nucleus on L then $L_r \subseteqq Sreg(L) \subseteqq L(j)$.*

Proof. If $x \in L_r$ then $0 = j(x \wedge x^*) = j(x) \wedge j(x^*) \leqq j(x) \wedge x^*$, i.e., $j(x) \leqq x^{**} = x$. Now, we have $j(x) = x$ for any $x \in L_r$. The rest is obvious.

1.6. Theorem. *If L is a frame then the following conditions are equivalent:*

(i) *L is semiregular.*

(ii) *Any dense nucleus on L is regular.*

Proof. (i) $\Rightarrow$ (ii): Clearly, $L = Sreg(L) \subseteqq L(j) \subseteqq L$ for any dense nucleus j on L.

(ii) $\Rightarrow$ (i): Let $r: L \to L_r$ be a nucleus defined by $r(a) = a^{**}$ for any $a \in L$. Then r is dense and for $z \in L$ we have $z = \bigvee(r(x): r(x) \leqq z) = \bigvee(x^{**}: x^{**} \leqq z)$, i.e., $z \in Sreg(L)$.

1.7. Corollary. *Any dense homomorphic image of a semiregular frame is semiregular.*

1.8. Proposition. *A sum of semiregular frames is semiregular.*

Proof. Let L_γ, $\gamma \in \Gamma$ be semiregular frames, $i_\gamma: L_\gamma \to \Sigma L_\gamma$ canonical injections. Then $i_\gamma(x_\gamma)$ is a semiregular element of ΣL_γ for any $x_\gamma \in L_\gamma$, $\gamma \in \Gamma$. Namely, $i_\gamma(x_\gamma) = \bigvee(i_\gamma(y): y^{**} \leqq x_\gamma) = \bigvee(i_\gamma(y): i_\gamma(y)^{**} \leqq i_\gamma(x_\gamma))$. Since elements of this form generate ΣL_γ, we have that ΣL_γ is semiregular.

§ 2. HEREDITARY SEMIREGULAR FRAMES

Definition. We say that a frame L is *hereditary semiregular* if any its homomorphic image is semiregular.

Clearly, any regular frame is hereditary semiregular. We remark that hereditary semiregular topological spaces were introduced by M. Katětov [6] as spaces such that all subspaces are semiregular. Any topological space can be embedded in a semiregular space (see [10]). Consequently, a subspace of a semiregular space is not semiregular, in general.

2.1. Proposition. *L is a hereditary semiregular frame if and only if any closed homomorphic image of L is semiregular.*

Proof. From [4], Th. 1.2, p. 40 it follows that any surjective homomorphism f of frames we can factorize in the form $f = \bar{f} . c$, where $\bar{f}$ is dense and c is closed. The rest follows from 1.7.

We don't know when a sum of hereditary semiregular frames is hereditary semiregular.

2.2. Corollary. *If T is a hereditary semiregular topological space then the frame $O(T)$ of all open sets of T is hereditary semiregular.*

Proof. It follows from 2.1 and the fact that any closed homomorphic image of a topology is again a topology.

Let L be a frame, $Id(L)$ the frame of all ideals in L and $R(L) = Sreg(Id(L))$. Clearly, $R(L)$ is generated by the elements $\downarrow a$, $a \in L_r$, $R(L)$ is compact and spatial because $Id(L)$ is compact and spatial.

2.3. Theorem. *Let L be a semiregular frame. Then there exists a surjective dense frame homomorphism $\sigma: R(L) \to L$.*

Proof. Put $\varphi(A) = \bigvee A$ for any $A \in Id(L)$. It is well known that φ is a surjective dense frame homomorphism. If we define $\sigma = \varphi/_{R(L)}$ then σ is a dense frame homomorphism. If $l \in L$ then $\sigma(I_l) = l$, where I_l is the ideal in L generated by the elements $x \in L_r$, $x \leqq l$.

2.4. Proposition. *Let L be a semiregular frame. Then L is hereditary semiregular iff $R(L)$ is hereditary semiregular.*

Proof. $\Leftarrow$: It follows from 2.3.

$\Rightarrow$: If f: $R(L) \to H$ is a surjective frame homomorphism then the elements $f(\downarrow a)$ for $a \in L_r$ generate the whole H. Let us define a map g: $L \to H$ by the formula $g(a) = f(\downarrow a)$ for any $a \in L_r$. It is easy to verify that g is a surjective frame homomorphism, i.e., H is semiregular.

Remarks. 1. We define a relation ϱ on a frame L such that $a\varrho b$ iff $a \leqq a_1 \vee \ldots \ldots \vee a_k \leqq b$, where $a_1, \ldots, a_k \in L_r$ and $a, b \in L$. Then it holds: L is a semiregular frame iff $a = \bigvee(x\colon x\varrho a)$ for any $a \in L$. $A \in Sreg(Id(L))$ iff for any $a \in A$ there exists $b \in A$ such that $a\varrho b$. These facts are similar as results of B. Banaschewski and C. J. Mulvey (see [1]) for completely regular frames.

2. Unfortunately, the homomorphism $\sigma\colon R(L) \to L$ has no universal property.

Let us recall that a frame L is *almost compact* if any covering of L has a finite dense subset. Some properties of almost compact frames are in [7].

2.5. Proposition. *If L is an almost compact frame then $Sreg(L)$ is almost compact.*

Proof. If $x_i \in Sreg(L)$, $\bigvee(x_i\colon i \in I) = 1$ then $0 = [\bigvee(x_i\colon i \in F)]^* \in Sreg(L)$ for some finite set $F \subseteqq I$.

2.6. Corollary. *If L is a semiregular Hausdorff frame then there exists an almost compact semiregular Hausdorff frame K such that L is a dense homomorphic image of K.*

Proof. If L_β is the H-closed extension of L defined in [7] then if we put $K = Sreg(L_\beta)$ is is easy to verify that K is an almost compact semiregular Hausdorff frame and L is a dense homomorphic image of K.

2.7. Proposition. *If L is a frame then $K(L) = \{(u, v)\colon u \in L,\ v \in L_r,\ u \leqq v\}$ is a frame with the following properties:*

1. *L is normal iff $K(L)$ is normal.*
2. *$K(L)$ is not semiregular.*

Proof. 1. $\Rightarrow$: If L is normal, $(a_1, a_2), (b_1, b_2) \in K(L)$, $(a_1, a_2) \vee (b_1, b_2) = (1,1)$ then $a_1 \vee b_1 = 1$, i.e., there exist $c_1, d_1 \in L$ such that $a_1 \vee d_1 = 1 = b_1 \vee c_1$, $c_1 \wedge d_1 = 0$. Clearly, $(a_1, a_2) \vee (d_1, d_1^{**}) = (1,1) = (c_1, c_1^{**}) \vee (b_1, b_2), (c_1, c_1^{**}) \wedge (d_1, d_1^{**}) = (0,0)$.

$\Rightarrow$: Conversely, if $K(L)$ is normal, $a \vee b = 1$, $a, b \in L$ then $(a, 1) \vee (b, 1) = (1,1)$, $(a, 1), (b, 1) \in K(L)$. Now, there exist $(c_1, c_2), (d_1, d_2) \in K(L)$ such that $(a, 1) \vee (d_1, d_2) = (1,1) = (b, 1) \vee (c_1, c_2), (c_1, c_2) \wedge (d_1, d_2) = (0,0)$. Clearly, $a \vee d_1 = b \vee c_1 = 1$, $c_1 \wedge d_1 = 0$.

2. If we consider an element $(0,1) \in K(L)$ then $\bigvee((x, y) \in K(L)\colon (x, y)^{**} = \bigvee((y^{**}, y^{**}) \leqq (0,1)) = (0,0)$, i.e., $K(L)$ is not semiregular.

2.8. Corollary. *There exists a compact normal Hausdorff frame which is not semiregular, i.e., is not spatial.*

Proof. Let I be the closed interval $[0,1]$ with the usual topology $0(I)$. From [7], Proposition 2.4 we know that $K(0(I))$ is a compact Hausdorff frame. Now we have that $K(0(I))$ is a normal frame, which is not semiregular.

M. Katětov [6] gives an example of a hereditary semiregular Hausdorff space which is not regular.

2.9. Proposition. *There exists a compact spatial hereditary semiregular frame which is not regular.*

Proof. If L is a regular frame which is not completely regular then $R(L)$ is a compact spatial hereditary semiregular frame. In the case that $R(L)$ is regular then $R(L)$ is completely regular what is in a contradiction with the fact that L is a homomorphic image of L.

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