Ján Rusnák

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Archivum Mathematicum, Vol. 26 (1990), No. 4, 201--205

Persistent URL: http://dml.cz/dmlcz/107389

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 4 (1990), 201–206

A REMARK ON A NONLINEAR BOUNDARY VALUE PROBLEM OF THE THIRD ORDER

J. RUSNÁK

(Received December 4, 1986)

Abstract. Existence theorem for a nonlinear boundary value problem of the third order is proved without requiring the existence of lower and upper solutions. The proof is based on the construction of a lower and upper solutions by using signs of Green's functions.

Key words. Three point boundary value problem, lower and upper solutions, Green's function.

MS Classification. 34 B 10.

1. INTRODUCTION

In this paper we investigate a boundary value problem

$$x''' = f(t, x, x', x''), \qquad (t, x, x', x'') \in [a_1, a_3] \times R^3, \tag{1}$$

$$\alpha_2 x'(a_1) - \alpha_3 x''(a_1) = A_1, \qquad x(a_2) = A_2, \qquad \gamma_2 x'(a_3) + \gamma_3 x''(a_3) = A_3,$$

$$\alpha_2, \alpha_3, \gamma_2, \gamma_3 \geqq 0, \qquad \alpha_2 + \alpha_3 > 0, \qquad \gamma_2 + \gamma_3 > 0, \tag{2}$$
$$\alpha_2 + \gamma_2 > 0, \qquad a_1 < a_2 < a_3.$$

Let $I = [a_1, a_3]$, $I_1 = [a_1, a_2]$, $I_2 = [a_2, a_3]$.

An existence theorem for (1) and (2) in [5] is proved under the assumption that lower and upper solutions of (1) and (2) exist. We prove an existence theorem without requiring the existence of lower and upper solutions. This theorem is an application of the theorem from [5]. The proof will be based on the construction of a lower and an upper solution by using signs of Green's functions. Thus the theorem is applicable to a fairly large class of functions.

Let $G_k(t, s)$, $k = 1, 2$ be Green's functions corresponding to (1) and (2). G_k are uniquely determined by the following three properties (see [2], [3], [4]):

for any point $s \in I_k$ there holds

1. $G_k, \dfrac{\partial G_k}{\partial t} = G_{kt}$ are continuous in t on I.

2. $\dfrac{\partial^2 G_k}{\partial t^2} = G_{ktt}$ is continuous everywhere on I except at the point s, where it has a discontinuity of the first kind and $G_{ktt}(s, +0, s) - G_{ktt}(s - 0, s) = 1$.

3. G_k as a function of t is a solution of $x'' = 0$ on intervals $[a_1, s)$, $(s, a_3]$ and satisfies the homogeneous boundary conditions (2) with $A_1 = A_2 = A_3 = 0$.

Let $\varphi(t)$ be a solution of the boundary value problem $x''' = 0$ and (2) and let $r(t) \in C_0(I)$. Then the solution x of boundary value problem $x''' = r(t)$ and (2) can be expressed in the form

$$x(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t, s)\, r(s)\, \mathrm{d}s, \qquad t \in I. \tag{3}$$

From explicit expression of Green's functions G_1, G_2 there follows immediately the following lemma on the signs of Green's functions and their derivatives.

Lemma 1. *For the Green's functions G_1, G_2 there holds:*

$$G_k \geqq 0 \quad \text{on} \quad I_1 \times I_k, \qquad G_k \leqq 0 \quad \text{on} \quad I_2 \times I_k,$$
$$G_{kt} \leqq 0 \quad \text{on} \quad I \times I_k, \qquad k = 1, 2.$$

Two functions $\alpha, \beta \in C_3(I)$ will be called a lower and upper solution of (1) and (2) respectively, if the following hold:

$$\alpha'(t) \leqq \beta'(t) \qquad \text{for any } t \in I,$$
$$\alpha_2\alpha'(a_1) - \alpha_3\alpha''(a_1) \leqq A_1, \qquad \alpha(a_2) = A_2, \qquad \gamma_2\alpha'(a_3) + \gamma_3\alpha''(a_3) \leqq A_3,$$
$$\alpha_2\beta'(a_1) - \alpha_3\beta''(a_1) \geqq A_1, \qquad \beta(a_2) = A_2, \qquad \gamma_2\beta'(a_3) + \gamma_3\beta''(a_3) \geqq A_3,$$
$$\alpha''' \geqq f(t, x, \alpha', \alpha''), \qquad \beta''' \leqq f(t, x, \beta', \beta'')$$

for all $t \in I$ and those x for which $\alpha(t) \leqq x \leqq \beta(t)$ when $\alpha(t) \leqq \beta(t)$ or for all $t \in I$ and those x for which $\beta(t) \leqq x \leqq \alpha(t)$ when $\beta(t) \geqq \alpha(t)$.

In the following lemma we give a version of [5, Theorem 2] modified in the view of [5, Remark 1]:

Lemma 2. *Let there exist lower and upper solutions $\alpha, \beta \in C_3(I)$ of* (1) *and* (2), *respectively. Further suppose that, for some positive constant L,*

$$| f(t, x, x', x'') - f(t, x, x', y'') | \leqq L \,| x'' - y'' |,$$

for all (t, x, x'): $\beta(t) \leqq x \leqq \alpha(t)$, $\alpha'(t) \leqq x' \leqq \beta'(t)$, $t \in I_1$ and $\alpha(t) \leqq x \leqq \beta(t)$, $\alpha'(t) \leqq x' \leqq \beta'(t)$, $t \in I_2$, $x'', y'' \in R$.

Then there exist at least one solution x of (1) *and* (2) *such that*

$$\beta(t) \leqq x(t) \leqq \alpha(t), \qquad t \in I_1, \qquad \alpha(t) \leqq x(t) \leqq \beta(t), \qquad t \in I_2,$$
$$\alpha'(t) \leqq x'(t) \leqq \beta'(t), \qquad t \in I.$$

2. EXISTENCE THEOREM

Theorem. *Let the function f satisfy:*

(i) *for any $(t, x, x', x'') \in I \times R^3$ either $f(t, x, x', x'') \geqq M$ or $f(t, x, x', x'') \leqq M$, where M* is *a real number.*

(ii) *f is non-increasing in x on R for $t \in I_1$ and f is nondecreasing in x' on R for $t \in I_2$.*

(iii) *f is non-decreasing in x' on R.*

(iv) *there exists a positive constant L such that*

$$|f(t, x, x', x'') - f(t, x, x', y'')| \leqq L\,|x'' - y''|$$
$$\text{for all } (t, x, x') \in I \times R^2, \qquad x'', y'' \in R.$$

Then there exists at least one solution x of (1) *and* (2).

Proof. First we shall prove the theorem under the assumption $f \geqq M$. Without loss of generality we suppose that $M < 0$. We shall prove the existence of a lower solution α and an upper solution β of (1) and (2) which satisfies the hypotheses of Lemma 2. The proof will then be complete because the Lipschitz condition of Lemma 2 follows from (iv).

Let $\beta(t)$ be a solution of $x''' = M$ and (2). From (3) $\beta(t)$ can be expressed in the form

$$\beta(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t, s)\, M \,\mathrm{d}s. \tag{4}$$

$\beta(t)$ is an upper solution of (1) and (2) and in view of Lemma 1 and (4) we have

$$\begin{aligned} &\beta(t) \leqq \varphi(t) \quad \text{for } t \in I_1, \qquad \varphi(t) \leqq \beta(t) \quad \text{for } t \in I_2,\\ &\qquad\qquad \varphi'(t) \leqq \beta'(t) \quad \text{for } t \in I. \end{aligned} \tag{5}$$

Consider a differential equation

$$x''' - L\,|x''| - K = 0, \tag{6}$$

where

$$K = \max_I \,(L, |\varphi(t)| + f(t, \beta(t), \beta'(t), 0)).$$

We shall show that there exists a solution Φ of (6) which fulfils conditions

$$\begin{aligned} &\alpha_2\Phi'(a_1) - \alpha_3\Phi''(a_1) \leqq 0, \qquad \Phi(a_2) = 0, \qquad \gamma_2\Phi'(a_3) + \gamma_3\Phi''(a_3) \leqq 0,\\ &\Phi(t) \geqq 0 \quad \text{for any } t \in I_1, \qquad \Phi(t) \leqq 0 \quad \text{for any } t \in I_2,\\ &\qquad\qquad \Phi'(t) \leqq 0 \quad \text{for any } t \in I. \end{aligned} \tag{7}$$

For the solution x of (6) we have

$$(8)\qquad x(t) = \begin{cases} c_1 + \left(c_2 - \frac{2K}{L} t_0\right) t + \frac{K}{2L} t^2 - \frac{K}{L^3} e^{L(t_0 - t)} & t \leqq t_0, \\ c_1 - \frac{2K}{L^3} - \frac{K}{L} t_0^2 + c_2 t - \frac{K}{2L} t^2 + \frac{K}{L^3} e^{L(t - t_0)} & t \geqq t_0, \end{cases}$$

where t_0, c_1, c_2 are real constants. Then for any t_0 there are constants c_1 and c_2 such that $x(t)$ satisfies (7).

Further choose α in the form $\alpha(t) = \varphi(t) + \Phi(t)$, for every $t \in I$. From (5) and (7) it follows that

$$\beta(t) \leqq \alpha(t) \quad \text{for } t \in I_1, \qquad \alpha(t) \leqq \beta(t) \quad \text{for } t \in I_2,$$
$$\alpha'(t) \leqq \beta'(t) \quad \text{for } t \in I.$$

Let $t \in I_1$ and $x \geqq \beta(t)$ or $t \in I_2$ and $x \leqq \beta(t)$. Then from (ii), (iii) and (iv) we get

$$f(t, x, \alpha', \alpha'') \leqq f(t, \beta, \beta', \alpha'') \leqq L \mid \alpha'' \mid + f(t, \beta, \beta', 0) \leqq$$
$$\leqq L \mid \Phi'' \mid + L \mid \varphi'' \mid + f(t, \beta, \beta', 0) \leqq L \mid \Phi'' \mid + K = \Phi''' = \alpha'''.$$

Hence α and β are lower and upper solutions of (1) and (2) which satisfy the hypotheses of Lemma 2.

The proof of the Theorem under the assumption $f \leqq M$ is similar. In this case we can assume that $M > 0$. We show only the construction of $\alpha(t)$ and $\beta(t)$ in this case.

$\alpha(t)$ will be a solution of $x''' = M$ and (2). For $\beta(t)$ there will hold: $\beta(t) = \varphi(t) + \psi(t)$, where $\psi(t)$ is a solution of

$$x''' + L \mid x'' \mid - K = 0,$$
$$K = \min_I \left(-L \mid \varphi''(t) \mid + f(t, \alpha(t), \alpha'(t), 0)\right),$$

satisfying

$$\alpha_2 \psi'(a_1) - \alpha_3 \psi''(a_1) \geqq 0, \qquad \psi(a_2) = 0, \qquad \gamma_2 \psi'(a_3) + \gamma_3 \psi''(a_3) \geqq 0$$

and $\quad \psi(t) \leqq 0 \quad$ on I_1, $\quad \psi(t) \geqq 0 \quad$ on $I_2 \quad$ and $\quad \psi'(t) \geqq 0 \quad$ on I.

Remark. The Theorem and the method of proof are generalizations of [1, Theorem 4] and [6, Corollary 5] for two point nonlinear boundary value problem of the second order.

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J. Rusnák
Department of Mathematics VŠDS
Marxa–Engelsa 15
010 88 Žilina
Czechoslovakia