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ON CENTERS OF TYPE B OF POLYNOMIAL SYSTEMS

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Dedicated to Academician Otakar Borůvka on his 90th birthday

Abstract. The continuous band of cycles surrounding a center of type B of a polynomial system of degree n in $\mathbf{R}^2$ is bounded by a number of orbits $\leqq n + 1$. Examples 2.1, 2, 3, 4 show that such number can be $= n - 1$. It is conjectured that it cannot be greater than $n - 1$. The same examples show that a system of degree n can have up to n centers of type B. It is conjectured that the number of such centers cannot be greater than n.

Key words. Ordinary differential equations. Polynomial systems. Centers.

MS Classification. 34 C 05, 34 C 25.

I.

A polynomial planar system is a pair of ordinary differential equations

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y), \tag{1.1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $\dot{y} = \mathrm{d}y/\mathrm{d}t$, as usual, $t \in \mathbf{R}$, and X, Y are polynomials of $(x, y) \in \mathbf{R}^2$ with real coefficients, relatively prime. By definition, the degree of (1.1) is the maximum degree of X, Y.

A singular point of (1.1) is a center if there exists a neighborhood entirely covered by cycles surrounding the point itself.

Let S be a center, let G_S be the family of cycles γ surrounding S and no other singular point and let int γ denote the region interior to γ. We denote by N_S the region

$$N_S = \bigcup_{\gamma \in G_S} \operatorname{int} \gamma.$$

It is easy to show that the boundary ∂N_S of N_S is either empty or the finite union of singular points and open orbits of (1.1).

A center will be said to be of type B if ∂N_S is the union of open orbits only. If the degree n of (1.1) is $= 1$ then ∂N_S is empty, so S cannot be of type B.

On the contrary, for an arbitrary integer $n > 1$ there are polynomial systems of degree n with centers of type B. This will appear from examples in Sec. 2.

The same examples will also suggest two conjectures about polynomial systems with centers of type B.

II.

Example 2.1.

The quadratic system

$$\dot{x} = -2y^2 + 1, \qquad \dot{y} = 2xy$$

has two singular points, $S' = (0, \ -1/\sqrt{2})$, $S'' = (0, \ 1/\sqrt{2})$ and the orbits are represented by $[\exp(-x^2 - y^2)]\, y = c$. Therefore S' and S'' are both centers of type B and the straight line $y = 0$ represents $\partial N_{S'} = \partial N_{S''}$.

Example 2.2.

Let ν be a positive integer, let

$$P(x) = \prod_1^{\nu}{}_s\, (x^2 - s^2) \tag{2.1}$$

and let $q > 0$.
The function V defined by

$$V(x, y) = \exp(-x^2 - y^2)\, [P(x)\, y - q] \tag{2.2}$$

is an integral of the polynomial system (1.1) of degree $n = 2\nu + 2$ with

$$\begin{cases} X(x, y) = -2P(x)\, y^2 + 2qy + P(x), \\ Y(x, y) = [2xP(x) - P'(x)]\, y - 2qx, \end{cases} \tag{2.3}$$

where

$$P'(x) = 2xP(x) \sum_1^{\nu}{}_s\, (x^2 - s^2)^{-1}. \tag{2.4}$$

Therefore the level lines of $V(x, y) = c$ of the surface $z = V(x, y)$ represent the orbits of the system (1.1) defined by (2.3).

Since $V(x, y) \to 0$ as $x^2 + y^2 \to +\infty$, V must have a minimum point S, at least, necessarily lying in the region

$$E = \{(x, y) : P(x)\, y - q < 0\}$$

and one maximum point, at least, inside each of the $n - 1 = 2\nu + 1$ regions whose union is the set $\mathbf{R}^2 \setminus E$. Therefore (Cf. J. K. Hale [3], pp. 172–173) the system has $n = 2\nu + 2$ centers at least.

It q is close to zero E may contain other singular points than S. This does not happen if q is sufficiently large, so that S is a center of type B and $\partial N_S = \partial E$ is the union of $n - 1 = 2\nu + 1$ orbits.

To show this notice that the vertical isocline $X(x, y) = 0$ has one branch contained into E, namely

$$2P(x)\,y = q - [q^2 + 2P^2(x)]^{1/2}. \tag{2.5}$$

It has an intersection with the horizontal isocline $Y(x, y) = 0$, i.e.,

$$x\{P(x)\,[1 - \sum_{1}^{\nu}{}_s\,(x^2 - s^2)^{-1}]\,y - q\} = 0 \tag{2.6}$$

at the point $(0, \{q - [q^2 + 2P^2(0)]^{1/2}\}/2P(0))$ and no other intersection if q is large enough.

In fact

$$-P^2(x)\,[1 - \sum_{1}^{\nu}{}_s\,(x^2 - s^2)^{-1}] = -x^{4\nu} + ax^{4\nu-2} + \ldots + b,$$

so that

$$\max\{-P^2(x)\,[1 - \sum_{1}^{\nu}{}_s\,(x^2 - s^2)^{-1}], \qquad x \in \mathbf{R}\} = \mu < +\infty.$$

Since, obviously,

$$q^2 + q[q^2 + 2P^2(x)]^{1/2} \geqq 2q^2, \qquad x \in \mathbf{R},$$

if we take

$$2q^2 > \mu, \tag{2.7}$$

we have

$$q^2 + q[q^2 + 2P^2(x)]^{1/2} > -P^2(x)\,[1 - \sum_{1}^{\nu}{}_s\,(x^2 - s^2)^{-1}], \qquad x \in R,$$

which means that the branch (2.5) of $X(x, y) = 0$ cannot intersect the horizontal isocline $Y(x, y) = 0$ at any point of

$$P(x)\,[1 - \sum_{1}^{\nu}{}_s\,(x^2 - s^2)^{-1}]\,y - q = 0.$$

On the other hand the rest of the vertical isocline $X(x, y) = 0$ is represented by

$$2P(x)\,y = q + [q^2 + 2P^2(x)]^{1/2}, \tag{2.8}$$

so that it is contained into $\mathbf{R}^2 \backslash E$.

Therefore, if (2.7) holds, the only singular point in E is the center $S = (0, \{q - [q^2 + 2P^2(0)]^{1/2}\}/2P(0)) = (0, (-1)^\nu \{q - [q^2 + 2(\nu!)^4]^{1/2}\}/2(\nu!)^2)$, of type B with $\partial N_S = \partial E$.

We want to complete our analysis by showing that, independently of (2.7), the singular points in $\mathbf{R}^2 \backslash E$ are exactly $2\nu + 1 = n - 1$.

The singular points of (1.1) with X, Y defined by (2.3) are also the solutions (x, y), $y \neq 0$ of $X(x, y) = 0$, $xX(x, y) + yY(x, y) = 0$ and viceversa. Therefore it remains to look for the solutions (x, y), $y \neq 0$, of (2.8) and

$$P'(x)\, y^2 - xP(x) = 0. \tag{2.9}$$

Since P has 2ν simple zeros at $x = \pm s$, $s = 1, 2, \ldots, \nu$, so P' has $2\nu - 1$ simple zeros, namely $x = 0$ and $x = \pm\alpha_1$, $1 < \alpha_1 < 2 < \alpha_2 < \ldots < \alpha_\nu < \nu$. Therefore (2.9) consists of the straight line $x = 0$ plus ν branches through the points $(s, 0)$, $s = 1, 2, \ldots, \nu$ and their symmetricals with respect to the y-axis. Each branch is symmetrical with respect to the x-axis. In the half plane $y \geqq 0$ the branch through $(\nu, 0)$ is the graph of an analytical function $x \mapsto y(x)$ defined for $x \geqq \nu$, strictly increasing from 0 to $+\infty$. If $\nu > 1$ the branch through $(s, 0)$, $s = 1, 2, \ldots, \nu - 1$, in the half plane $y \geqq 0$ is the graph of an analytical function $x \mapsto y(x)$ defined for $s \leqq x < \alpha_s$, strictly increasing from 0 to $+\infty$.

On the other hand, according to (2.8), the part of the vertical isocline $X(x, y) = 0$ lying within $\mathbf{R}^2 \backslash E$ is the graph of a function $x \mapsto y_v(x)$ defined by

$$y_v(x) = \frac{q + [q^2 + 2P^2(x)]^{1/2}}{2P(x)}, \tag{2.10}$$

for $x \neq \pm s$, $s = 1, 2, \ldots, \nu$, with vertical asymptotes at $x = \pm s$, $s = 1, 2, \ldots, \nu$, and a horizontal asymptote $y = 1/\sqrt{2}$. Since

$$y_v'(x) = -\frac{1}{2}\, \frac{q[q^2 + 2P^2(x)]^{1/2} + q^2}{P^2(x)\,[q^2 + 2P^2(x)]^{1/2}}\, P'(x),$$

y_v' has an extremum at each one of the zeros of P' and $y_v'(x)\, P'(x) < 0$ otherwise.

It follows that each branch of (2.10) meets just one of the branches of (2.9) and just once, so that the total number of singular points in $\mathbf{R}^2 \backslash E$ is $n - 1 = 2\nu + 1$ and so they are all centers of type B.

Example 2.3.

Let V be defined by $V(x, y) = \exp(-x^2 - y^2)\,[xy - q]$. Then it is easily seen that if $4q^2 > 1$ the cubic system

$$\dot{x} = x + 2qy - 2xy^2, \qquad \dot{y} = -2qx - y + 2x^2y$$

has a center of type B at 0, and $\partial N_0 = \{(x, y) : xy - q = 0\}$, so that ∂N_0 consists of two orbits. The other singular points are $(-(1/2 + q)^{1/2}, -(1/2 + q)^{1/2})$, $((1/2 + q)^{1/2}, (1/2 + q)^{1/2})$, which are both centers of type B.

Example 2.4.

Let ν be a positive integer, let $P(x)$ be the polynomial of degree 2ν defined by (2.1) and let $q > 0$. Then the function V defined by

(2.11) $$V(x, y) = \exp(-x^2 - y^2)[xP(x)y - q]$$

is an integral of the polynomial system (1.1) of degree $n = 2\nu + 3$ with

(2.12) $$\begin{cases} X(x, y) = -2xP(x)y^2 + 2qy + xP(x), \\ Y(x, y) = [2x^2P(x) - P(x) - xP'(x)]y - 2qx. \end{cases}$$

This time the region E is defined by

$$E = \{(x, y) : xP(x)y - q < 0\}$$

and $\mathbf{R}^2 \backslash E$ is the union of $n - 1 = 2\nu + 2$ unbounded regions.

By the same argument used for the case (2.3) we see that the system (1.1) defined by (2.12) has at least $n = 2\nu + 3$ centers, one in E and one in each region of $\mathbf{R}^2 \backslash E$.

Also, this time 0 is a singular point and $0 \in E$.

We want to show that there are exactly $2\nu + 3$ singular points so that they all are centers of type B and, in particular, $\partial N_0 = \partial E$.

The vertical isocline $X(x, y) = 0$ has one branch

(2.13) $$2xP(x)y = q - [q^2 + 2x^2P^2(x)]^{1/2}$$

contained into E. It has an intersection with $Y(x, y) = 0$ at 0 and no other intersection if q is large enough. This can be seen by an argument similar to the one used for (2.3). Therefore 0 is a center of type B and $\partial N_0 = \partial E$ is the union of $n - 1 = 2\nu + 2$ orbits.

To look for singular points in $\mathbf{R}^2 \backslash E$ we can replace (2.13) by

(2.14) $$2xP(x)y = q + [q^2 + 2x^2P^2(x)]^{1/2}$$

and $Y(x, y) = 0$ by $xX(x, y) + yY(x, y) = 0$, i.e., by

(2.15) $$[xP'(x) + P(x)]y^2 - x^2P(x) = 0.$$

By means of arguments similar to those used for (2.3) we see that both (2.14) and (2.15) are composed by $2\nu + 2$ branches each. Each branch of (2.14) meets only one branch of (2.15) and only once, so the total number of intersections is $n - 1 = 2\nu + 2$ and they all must be centers of type B.

III.

The number of orbits contained into ∂N_S for a center S of type B of a polynomia system of degree n is $\leqq n + 1$. To show this recall that given an algebraic curve C of order k in $\mathbf{R}^2$ represented by $f(x, y) = 0$, a point (x, y) is said to be a contact point on C with the vector field (X, Y) if it is a solution of the system of algebraic equations

$$(3.1) \qquad f(x, y) = 0, \qquad f_x(x, y)\, X(x, y) + f_y(x, y)\; Y(x, y) = 0,$$

i.e., if either (x, y) is a singular point of (X, Y) on C or the vector (X, Y) is tangent to C at (x, y).

Now let S be a center of type B for a polynomial system. Assume that ∂N_S contains $k \geqq n + 2$ orbits. Then we could take k points, one on each orbit, and a circle C large enough so as to contain all such points. Then C would be divided by the orbits of N_S into $2k$ arcs at least, each containing a contact point, so that there would exist $2(n + 2)$ contact points at least on C. This contradicts the fact that, by Bézout's theorem applied to (3.1) if $f(x, y) = 0$ represents C the number of solutions of (3.1) cannot be greater than $2(n + 1)$ unless C is an orbit, which is not the case.

Let us denote by B_n the class of all the polynomial systems (1.1) of a given degree $n > 1$ having a canter of type B, and let $k(n)$ be the maximum number in B_n of orbits $\subset \partial N_S$.

From what precedes and from the examples of Sec. 2 it follows

$$(3.2) \qquad n - 1 \leqq k(n) \leqq n + 1, \qquad n = 2, 3, \ldots$$

For $n = 2$ we have

$$k(2) = 1.$$

In fact, when $n = 2$, N_S is a convex region (Cf. W. A. Coppel [2]), so if ∂N_S contained two orbits they ought to be two parallel straight lines, so that their union would be an isocline of the system and consequently it ought to contain S.

On the other hand I was unable to find examples of polynomial systems of degree $n > 2$ with a center S of type B and more than $n - 1$ orbits $\subset \partial N_S$.

All these facts suggest the conjecture that (3.2) can be replaced by $k(n) = n - 1$, $n = 2, 3 \ldots$

IV.

The number of centers of type B for systems in the class B_n has a maximum $b(n)$, obviously $\leqq n^2$.

It can be shown (Cf. R. Conti [1]) that

$$b(2) = 2.$$

The examples of Sec. 2 show that

$$n \leqq b(n), \qquad n = 2, 3, \ldots$$

but, again. I was unable to find examples showing that $b(n)$ can be greater than n, so it seems reasonable to conjecture that $b(n) = n$, $n = 2, 3, \ldots$

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