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SINGULAR QUADRATIC FUNCTIONALS AND TRANSFORMATION OF LINEAR HAMILTONIAN SYSTEMS

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. Singular quadratic functionals with a single singular end-point are investigated using the transformation theory of linear Hamiltonian systems. In particular, there are established results for self-adjoint 2n-order functionals.

Key words. Transformation of linear Hamiltonian systems and functionals, singularity condion, self-adjoint functionals of higher order.

MS Classification. Primary 34 C 10, 34 A 30, 49 B 10.

1. INTRODUCTION

The theory of singular quadratic functionals as introduced by Morse and Leighton [11] and followed by [12, 13] involves the study of functional

$$J[y; s_1, s_2] = \int_{s_1}^{s_2} [p(t)\, y'^2(t) - q(t)\, y^2(t)]\, dt, \tag{1}$$

$a < s_1 < s_2 < b$ as $s_1 \to a$, $s_2 \to b$ and y belongs to the prescribed class of "admissible arcs" defined on (a, b). Morse and Leighton [11] discovered a condition termed the „singularity condition" which with the classical condition (disconjugacy of the corresponding Euler equation) yields necessary and sufficient condition for singular functional to be nonnegative, i.e. $\liminf_{s_1 \to a \; s_2 \to b} J[y; s_1, s_2] \geqq 0$. Comprehensive bibliography concerning the problem may be found in [16].

In this paper we solve the problem of minimizing of the singular quadratic functionals corresponding to linear Hamiltonian systems. The principal idea we use is the application of the transformation theory of linear Hamiltonian systems and corresponding quadratic functionals. In particular, we establish results for singular functionals in terms of a singular condition similar to that of [11, 17] and for regular functionals in terms of a phase matrix. In the case of the second order linear differential equation this approach was originally proposed by J. Krbiľa [10] for associated regular functionals (on the compact interval) and later in [8, 9] for singular functionals (1). However, the result in [8, 9] is incorrect as provides the counter example in [5].

Statement of the problem. We suppose the second order variational problem corresponding with the linear Hamiltonian system

$$
\begin{aligned} y' &= B(t)\,y + C(t)\,z, \\ z' &= -A(t)\,y - B^T(t)\,z, \end{aligned} \tag{2}
$$

where $A(t)$, $B(t)$, $C(t)$ are $n \times n$ matrices of real-valued functions continuous on the interval $I = [a, \infty)$, the matrices $A(t)$, $C(t)$ are symmetric.

We suppose (1) to be *identically normal* on I, i.e. the trivial solution $(y, z) \equiv (0, 0)$ is the only one solution of (2) for which $y(t) = 0$ on a nondegenerate subinterval of I.

We consider the functional

$$
J[y, z; a, b] = \int_a^b [z^T(t)\,C(t)\,z(t) - y^T(t)\,A(t)\,y(t)]\,\mathrm{d}t, \tag{3}
$$

$a < b < \infty$. Integrals employed throughout are Lebesgue integrals and their extensions.

We say that vector functions $y(t)$, $z(t)$ are *admissible* curves on $I = [a, \infty)$ with respect to (2) if

i) $z(t)$ is (Lebesgue) measurable on I and $y(t)$ is a solution of $y' = B(t)\,y + C(t)\,z(t)$ a.e., satisfying boundary conditions $y(a) = 0$, $\lim_{t\to\infty} y(t) = 0$;

ii) $\int_a^b z^T(t)\,z(t)\,\mathrm{d}t < \infty$ for every b, $a < b < \infty$.

We seek conditions under which

$$
\liminf_{t\to\infty} J[y, z; a, t] \geq 0 \tag{4}
$$

for all admissible functions $y(t)$, $z(t)$ on $[a, \infty)$ with respect to (2). Whenever (4) holds for the admissible class of curves we say that $[a, \infty)$ affords a *minimum limit* to J.

Remark 1. Some special cases of the problem have been investigated in the past. If $n = 1$, $C(t) \neq 0$ then (2) and (3) corresponds to the second order equation $(p(t)\,y')' + q(t)\,y = 0$ and to (1), respectively (Case I). If $B(t) = 0$, $C(t)$ being invertible then we have quadratic functionals of n dependent variables corresponding to the second order linear system

$$\int_a^b (y^{T\prime}C^{-1}y' - y^TAy)\,\mathrm{d}t \to (C^{-1}Y')' + AY = 0,$$

investigeted by Tomastik [17, 18] (Case II).

The condition of the identical normality of (2) eliminates pathologies in the investigation of conjugate points present in an abnormal differential system (2), see [16]; in the terminology of [4] this condition is called "controllability condition".

The introduced definition of admissible functions agrees with that of [4, 16] for the compact interval and with that of [17, 18] for Case II.

2. PRELIMINERIES

Corresponding to (2), we have the matrix equation

$$(2)^* \qquad \begin{aligned} Y' &= B(t)\,Y + C(t)\,Z,\\ Z' &= -A(t)\,Y - B^T(t)\,Z. \end{aligned}$$

In accordance with [4, 16] we use the following notation. We say that $(Y(t), Z(t))$ is a solution of (2)* if $Y(t), Z(t) \in \mathcal{AC}(I)$ (absolutely continuous) and (2)* satisfy a.e. on I. If $(Y(t), Z(t))$ is a solution of (2)* then $Y^T(t)\,Z(t) - Z^T(t)\,Y(t) = K$, where K is a constant $n \times n$ matrix. If $K = 0$ then $(Y(t), Z(t))$ is called *conjoined* (an alternate terminology for this concept is isotropic; see [4]). Two points $a, b \in \mathbf{R}$ are *conjugate* with respect to (2) if there exists a non-trivial solution $(y(t), z(t))$ of (2) such that $y(a) = 0$, $y(b) = 0$. (2) is disconjugate on I if there exist no two distinct points from I that are conjugate with respect to (2).

Let be (2) disconjugate on $[a, \infty)$. Then there exists a conjoined solution $(Y_0(t), Z_0(t))$ of (2)* such that the matrix $Y_0(t)$ is nonsingular on (a, ∞) and

$$\lim_{t\to\infty} [\int^t Y_0^{-1}(t)\,C(t)\,(Y_0^T)^{-1}(t)]^{-1} = 0.$$

The solution $(Y_0(t), Z_0(t))$ with these properties is called *principal* at infinity. A principal solution (Y_a, Z_a) at a is defined similarly; one can verify that this solution satisfies the initial condition $Y_a(a) = 0$, $Z_a(a) = N$ where N is a non-singular matrix. A solution $(Y(t), Z(t))$ of (2)* is called *antiprincipal* at infinity if it is conjoined, $Y(t)$ is non-singular for large t and

$$\lim_{t\to\infty} [\int^{t} Y^{-1}(t)\, C(t)\, Y^{T-1}(t)]^{-1} = M,$$

where M is a non-singular matrix.

If $(Y(t), Z(t))$ is a solution of (2)* such that $Y(t)$ is invertible for all t then $W(t) = Z(t)\, Y^{-1}(t)$ is a solution of the Riccati equation

$$W' + A(t) + WB(t) + B^T(t)\, W + WC(t)\, W = 0. \tag{5}$$

The solution $(Y(t), Z(t))$ is conjoined if and only if the corresponding solution $W(t)$ of (5) is symmetric. If $(Y_a(t), Z_a(t))$ is the principal solution at a then the solution $W_a(t) = Z_a(t)\, Y_a^{-1}(t)$ of (5) is called the *distinguished* solution at a.

Our method will be based on the transformation of linear Hamiltonian system given in the following two theorems.

Theorem A. [1, Theorem 6.3]. *Let $D(t)$, $E(t) \in \mathscr{AC}(I)$ be $n \times n$ matrices $D(t)$ being non-singular, for which $D^T(t)\, E(t) = E^T(t)\, D(t)$.*

Then the transformation

$$\begin{aligned} y &= D(t)\, u, \\ z &= E(t)\, u + D^{T-1}(t)\, v \end{aligned} \tag{6}$$

transforms (2) into the system

$$\begin{aligned} u' &= B_0(t)\, u + C_0(t)\, v, \\ v' &= -A_0(t)\, u - B_0^T(t)\, v, \end{aligned} \tag{7}$$

where $B_0(t) = D^{-1}(-D' + BD + CE)$, $C_0(t) = D^{-1}CDT^{-1}$, $A_0(t) = D^T(E' + AD + B^TE) + (-D' + BD + CE)$.

Remark 2. The transformation (6) keeps the identical normality, disconjugacy on the given interval I, and a principal (antiprincipal, conjoined) solution is transformed into that of the same type.

Obviously, the transformation (6) with $E(t) = 0$, $D' = B(t)\, D$ transforms (2)* into the "off-diagonal" system

$$\begin{aligned} U' &= \bar{C}(t)\, V, \\ V' &= -\bar{A}(t)\, U, \end{aligned} \tag{8}$$

where $\bar{C}(t) = D^{-1}CD^{T-1}$, $\bar{A}(t) = D^TAD$.

Theorem B. [5, Theorem 1]. *There exist $n \times n$ matrices $D(t)$, $E(t) \in \mathscr{AC}$, $D(t)$ being nonsingular, such that the transformation $U = D(t)\, Y, V = E(t)\, Y + D^{T-1}(t)\, Z$ transforms the system (8) into the system*

$$(9)\qquad \begin{aligned} Y' &= Q(t)\,Z,\\ Z' &= -Q(t)\,Y, \end{aligned}$$

where $Q(t) = D^{-1}CD^{T-1}$. *The matrix* $A(t) = \int_a^t Q(s)\,\mathrm{d}s$ *is called a phase matrix of the system* (2)*.

3. TRANSFORMATION OF FUNCTIONALS AND SINGULARITY CONDITION

The symbol $y \in \mathcal{D}[a, b] : z$ will denote those functions $y \in \mathcal{AC}[a, b]$ for which there exists a $z(t)$ measurable, satisfying condition ii) from the definition of admissible functions and such that $y' = B(t)\,y + C(t)\,z(t)$ a.e. on $[a, b]$.

Theorem 1. Let $y \in \mathcal{D}[a, b] : z$. Then functions u, v given by the transformation (6) satisfy

$$\int_a^b (z^TCz - y^TAy)\,\mathrm{d}t = \int_a^b (v^TC_0v - u^TA_0u)\,\mathrm{d}t + [y^TED^{-1}y]_a^b.$$

Proof. According to Theorem A it holds $u' = B_0(t)\,u + C_0(t)\,v$ and $u = D^{-1}y$. Using the transformation (6) we get

$$(10)\quad \int_a^b (z^TCz - y^TAy)\,\mathrm{d}t = \int_a^b [(u^TE^T + v^TD^{-1})\,C(Eu + D^{T-1}v) - u^TD^TADu]\,\mathrm{d}t =$$

$$= \int_a^b (v^TD^{-1}CD^{T-1}v + u^TE^TCEu + v^TD^{-1}CEu + u^TE^TCD^{T-1}v - u^TD^TADu)\,\mathrm{d}t.$$

Further it holds $(u^T)' = u^TB_0^T + v^TC_0 = u^T(-D^{T\prime} + D^TB^T + E^TC)\,D^{T-1} + v^TD^{-1}CD^{T-1}$, thus

$$\begin{gathered}(u^TD^TEu)' = u^T(-D^{T\prime} + D^TB^T + E^TC)\,D^{T-1}D^TEu + v^TD^{-1}CD^{T-1}D^TEu +\\ + u^TD^{T\prime}Eu + u^TD^TE'u + u^TD^TED^{-1}(-D' + BD + CE)\,u + u^TD^TED^{-1}CD^{T-1}v =\\ = v^TD^{-1}CEu + u^TE^TCD^{T-1}v +\\ + u^T(-D^{T\prime}E + D^TB^TE + E^TCE + D^{T\prime}E + D^TE' - E^TD' + E^TBD + E^TCE)\,u =\\ = v^TD^{-1}CEu + u^TE^TCD^{T-1}v +\\ + u^T(D^TE' - E^TD' + D^TB^TE + 2E^TCE + E^TBD)\,u.\end{gathered}$$

Integrating the last equality we get

$$(11)\qquad \int_a^b (u^TE^TCEu + v^TD^{-1}CEu + u^TE^TCD^{T-1}v)\,\mathrm{d}t = [u^TD^TEu]_a^b +$$

$$+ \int_a^b [-u^T(D^TE' - E^TD' + D^TB^TE + E^TCE + E^TBD)\,u]\,\mathrm{d}t.$$

Finally, by substitution (11) into (10) we have

$$\int_a^b (z^T Cz - y^T Ay)\,dt = \int_a^b [v^T C_0 v - u^T(D^T AD + D^T E' - E^T D' + D^T B^T E + E^T CE + E^T BD)\,u]\,dt = \int_a^b (v^T C_0 v - u^T A_0 u)\,dt + [u^T D^T E u]_a^b = \int_a^b (v^T C_0 v - u^T A_0 u)\,dt + [y^T E D^{-1} y]_a^b. \blacksquare$$

We can use Theorem 1 to have a non-negativity of functionals. In the following if C is symmetric $n \times n$ matrix (i.e. $C^T = C$), $C \geqq 0$ means that C is non-negative definite.

Theorem 2. *Let* $C(t) \geqq 0$ on $[a, \infty)$. *In order that* (4) *holds for all admissible functions* $y(t)$, $z(t)$ *on* $[a, \infty)$ *with respect to* (2) *it is necessary and sufficient*

i) (2) *is disconjugate on* $[a, \infty)$,

ii) *singularity condition is satisfied*, i.e. *for all* $y(t)$, $z(t)$ *admissible on* $[a, \infty)$ *with respect to* (2) *such that*

$$\liminf_{t\to\infty} \int_a^t (z^T Cz - y^T Ay)\,dt < \infty.$$

it holds

$$\liminf_{t\to\infty} y^T(t)\, W_a(t)\, y(t) \geqq 0,$$

where $W_a(t)$ *is the distinguished solution of* (5).

Proof. I. Note that if $y(t), z(t)$ are admissible functions with respect to (2) then $y \in \mathscr{D}[a, b] : z$ and by virtue of the boundary condition at a it holds $[y^T E D^{-1} y]_{t=a} = 0$.

Let (2)* be disconjugate on $[a, \infty)$ and (Y, Z) be a principal solution of (2)* at a. Then $W_a(t) = Z(t)\, Y^{-1}(t)$ is the distinguish solution of (5) at a and the transformation (6) with

$$D(t) = Y(t), \qquad E(t) = Z(t)$$

yields

$$B_0 = Y^{-1}(-Y' + BY + CZ) = Y^{-1}(-BY - CZ + BY + CZ) = 0,$$
$$C_0 = Y^{-1} C Y^{T-1},$$
$$A_0 = Y^T(Z' + AY + B^T Z) = Y^T(-AY - B^T Z + AY + B^T Z) = 0.$$

By Theorem 1

$$\int_a^b (z^T Cz - y^T Ay)\,dt = \int_a^b (v^T C_0 v)\,dt + [y^T W_a y]_{t=b}$$

holds for all corresponding couples of functions $y(t)$, $z(t)$ and $u(t)$, $v(t)$. From the inequality

$$\liminf_{t\to\infty} \int_a^b (z^T Cz - y^T Ay)\,ds \geqq \liminf_{t\to\infty} \int_a^t (v^T C_0 v)\,dt + \liminf_{t\to\infty} y^T W_a y,$$

it follows the sufficiency of the singular condition.

II. We now follow a method which was used in the scalar case by Morse and Leighton [11]. Suppose there exists a couple of admissible functions y, z such that $\liminf_{t\to\infty} J[y, z; a, t] < \infty$ and the singularity condition is not satisfied for this couple i.e., $\liminf_{t\to\infty} y^T(t)\, W_a(t)\, y(t) = -k^2$, where $W_a(t)$ is the distinguish solution of (5) at a and k is a real constant. Let $e \in (a, \infty)$. We construct a couple of vector functions

$$(y_e(t), z_e(t)) = \begin{cases} (y(t), z(t)) & \text{for } t \in (e, \infty), \\ (Y_a(t)\, c, Z_a(t)\, c) & \text{for } t \in (a, e], \end{cases}$$

where (Y_a, Z_a) is the principal solution of (2)* at a, c is a constant vector such that $(y(e), z(e)) = (Y_a(e)\, c, Z_a(e)\, c)$. It holds

$$\int_a^t (z_e^T Cz_e - y_e^T Ay_e)\,ds = \int_a^e c^T(Z_a CZ_a - Y_a^T AY_a)\, c\,dt + \int_e^t (z^T Cz - y^T Ay)\,ds =$$

$$= -c^T Y_a^T(e)\, W_a(e)\, Y_a(e)\, c + \int_e^t (z^T Cz - y^T Ay)\,ds =$$

$$= -y^T(e)\, W_a(e)\, y(e) + \int_e^t (z^T Cz - y^T Ay)\,ds.$$

Since $\liminf_{t\to\infty} (-y^T(t)\, W_a(t)\, y(t)) = -k^2$ and $\liminf_{t\to\infty} \int_a^t (z^T Cz - y^T Ay)\,ds < \infty$ choosing e sufficiently large, we have $-y^T(e)\, W_a(e)\, y(e) < -2k^2/3$ and $\liminf_{t\to\infty} \int_e^t . (z^T Cz - y^T Ay)\,ds < k^2/3$.

Consequently, we have $\liminf_{t\to\infty} \int_a^t (z_e^T Cz_e - y_e Ay_e)\,dt < -k^2/3$ which is a contradiction. ■

Remark that in special Cases I and II (see Remark 1) the singularity condition complies with that one introduced in [11] and [17], respectively.

The following theorem gives sufficient condi ions for singularity condition to be satisfied. Since every system can be transformed to "off-diagonal" form (see Remark 2) we suppose $B(t) = 0$ in (2)* without loss of generality.

Theorem 3. *Let* $B(t) = 0$, $C(t) \geqq 0$ on $[a, \infty)$. *If the system* (2)* *is disconjugate on* $(a - \varepsilon, \infty)$ *for some* $\varepsilon > 0$, $\int_a^\infty C(s)\,ds < \infty$ *and* $\int_a^\infty \max|a_{ij}(s)|\,ds < \infty$ *then* $[a, \infty)$ *affords a minimum limit to J.*

Proof. Let (Y_a, Z_a) be a principal solution of (2)* at a. In the light of the fact that $W = W_a = Z_a Y_a^{-1}$ is a solution of the Riccati equation

$$W' + A(t) + WC(t)W = 0,$$

it holds

$$W(t) = W(b) - \int_b^t W(s)\,C(s)\,W(s)\,ds - \int_b^t A(s)\,ds, \qquad a < b < t$$

and using the symmetry of $W(t)$ we get

$$W(t) = W(b) - \int_b^t Z_a Y_a^{-1} C Y_a^{T-1} Z_a^T\,ds - \int_b^t A(s)\,ds. \tag{12}$$

The fact that (Y_a, Z_a) is a principal solution and disconjugacy of (2) on $(a - \varepsilon, \infty)$ for some $\varepsilon > 0$ imply that (Y_a, Z_a) is a antiprincipal solution of (2)* at infinity. Thus $\int_b^t Y_a^{-1} C Y_a^{T-1}\,ds$ is bounded as well as $Z_a(t) = -\int_b^t AY_a\,ds$.

Now, we use the following lemma [17, Lemma 6.3].

Lemma. *If* $Q(t)$ *is a positive definite matrix on* $[a, \infty)$, $\int_a^t Q(s)\,ds$ *is bounded and* $A(t)$ *is bounded matrix then* $\int_a^t A^T(s)\,Q(s)\,A(s)\,ds$ *is bounded.*

According to this Lemma the first integral in (12) is bounded and thus $W(t)$ is bounded. Hence $\lim_{t\to\infty} y^T(t)\,W(t)\,y(t) = 0$ i.e., the singularity condition is satisfied. ∎

In the following, we denote $l_n(Q)$ the maximal eigenvalue of the matrix $Q(t)$. If $\int_a^t l_n(Q) < \pi$ then (9) is disconjugate on $[a, t]$ (see e.g. [16, p. 366]). This fact together with Theorem 3 is used in the following example.

Example 1. Let $Q(t) \geqq 0$ on $[a, \infty)$ and $\int_a^\infty l_n(Q) < \pi$. Then it holds

$$\liminf_{t\to\infty} \int_a^t (z^T(s)\,Q(s)\,z(s) - y^T(s)\,Q(s)\,y(s))\,ds \geqq 0,$$

for all $y(t)$, $z(t)$ admissible on $[a, \infty)$ with respect to (9) i.e. $y, z \in \mathscr{AC}$ such that $y' = Q(t)\,z$, $y(a) = 0 = \lim_{t\to\infty} y(t)$.

This example corresponds in the scalar case to the well-known fact that $\int_a^b q(t)(y'^2 - y^2)\,dt > 0$, $y(a) = 0 = y(b)$, $y \not\equiv 0$, whenever $\int_a^b q(t)\,dt < \pi$.

Till now we have used transformation of the functional (3) into the functional $\int_a^b (v^T C_0 v)\,dt$ which is allways non-negative (if $C \geqq 0$). Now we use another method consisting in the fact that every system (2) can be transformed into the system (9) whose solutions are the so called trigonometric matrices (see [2]). This method follows the idea of [8, 9, 10] consisting in the fact that the equation $(p(t)\,y')' + q(t)\,y = 0$ can be (globally) transformed into the equation $u'' + u = 0$ whose solutions are the sine and cosine functions.

The following statement sketches the application of this idea.

Corollary 1. *Let $A(t)$, $C(t) \in \mathscr{C}[a, b]$, $B(t) = 0$ and $Q(t)$ be a derivative of the phase matrix of (2)* satisfying $\int_a^b l_n(Q) < \pi$. Then*

$$\int_a^b (z^T(s)\,C(s)\,z(s) - y^T(s)\,A(s)\,y(s))\,ds \geqq 0,$$

for all $y(t)$, $z(t)$ admissible on $[a, \infty)$ with respect to (2).

Especially, if $n = 1$ we get results of [10].

Proof. It follows immediately from Remark 2 and Theorem B.

4. SELF-ADJOINT FUNCTIONALS OF HIGHER ORDER

Consider a self-adjoint linear differential equation of the 2n order

$$\sum_{k=0}^{n} (-1)^k \,[p_k(t)\,u^{(k)}]^{(k)} = 0, \tag{13}$$

where $p_k(t) \in \mathscr{C}^k[a, \infty)$, $k = 0, \ldots, n$ and $p_n(t) > 0$ for $t \in I = [a, \infty)$.

Putting

$$y = (u, u', \ldots, u^{(n-1)})^T, \qquad z = (z_1, \ldots, z_n)^T, \qquad z_k = \sum_{j=k}^{n} (-1)^{j-k}\,[p_j u^{(j)}]^{(j-k)}, \tag{14}$$

we can write the equation (13) as a linear Hamiltonian system (2) where

$$A = -\operatorname{diag}[p_0, p_1, \ldots, p_{n-1}], \qquad C = p_n^{-1} \operatorname{diag}[0, \ldots, 0, 1]$$
$$B = (b_{ij}), \qquad b_{ij} = \begin{cases} 1 & i = j + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

It is easy to verify that $C(t) \geqq 0$ and the system (2) with A, B, C given by (15) is identically normal. In accordance with [4] we call points a, b conjugate with respect to (13) if there exists a nontrivial solution of (13) having zeros of multiplicity n at a and b. We say that (13) is disconjugate on I if there exists no couple points from I conjugate with respect to (13).

The equation (13) is Euler–Lagrange equation for quadratic functional

$$J_s(u) = \int_a^b [p_n(u^{(n)})^2 + p_{n-1}(u^{(n-1)})^2 + \ldots + p_0 u^2] \, dt.$$

The functional $J_s(u)$ will be investigated on the class of admissible functions $u(t)$ on $[a, \infty)$ i.e. $u \in \mathscr{C}^{n-1}$, $u^{(n)} \in \mathscr{AC}$, $u^{(i)}(a) = 0 = \lim u^{(i)}(t)$, $i = 0, \ldots, n-1$, $(p_n u^{(n)})^{(k)}$ are measurable and $\int_a^b (p_n u^{(n)})^{(k)} (p_n u^{(n)})^{(j)} < \infty$ for every $b > a$; $k, j = 0, \ldots, n-1$.

Note that admissible functions defined in such a way are admissible functions of the corresponding system (2) with matrices (15) as well as the definition of conjugate points with respect to (13) corresponds to that one of (2). Hence, we can apply Theorem 3.

Corollary 2. *If the equation* (13) *is disconjugate on* $(a - \varepsilon, \infty)$ *for some* $\varepsilon > 0$ *and*

$$\int_a^\infty p_n^{-1}(t)\, t^{2n-2} \, dt < \infty, \qquad \int_a^\infty | p_i(t) | \, t^{2n-2(i+1)} \, dt < \infty, \qquad i = 0, \ldots, n-1,$$

then

$$\liminf_{t \to \infty} \int_a^t \sum_{k=0}^n p_k(s)[u^{(k)}(s)]^2 \, ds \geqq 0$$

for all admissible functions $u(t)$ *on* $[a, \infty)$.

Proof. Using Theorem A and Remark 2 we transform (2) with matrices (15) into "off-diagonal" system (8). The equation $D' = BD$ yields

$$D = (d_{ij}), \qquad d_{ij} = \begin{cases} 0 & i > j, \\ t^{j-i}/(j-i)! & i \leqq j. \end{cases}$$

Then

$$D^{-1} = (\bar{d}_{ij}), \qquad \bar{d}_{ij} = \begin{cases} 0 & i > j \\ (-1)^{i+j} t^{j-i}/(j-i)! & i \leqq j \end{cases}$$

and by a straightforward computation we get $\bar{C}(t) = D^{-1} C D^{T-1} = (\bar{c}_{ij})$ and $\bar{A}(t) = D^T A D = (\bar{a}_{ij})$

$$\bar{c}_{ij} = (-1)^{i+j} \frac{t^{2n-i-j}}{(n-i)!\,(n-j)!} \qquad \bar{a}_{ij} = \sum_{k=1}^{\min\{i,j\}} \frac{t^{2k-i-j}}{(i-k)!\,(j-k)!} p_{k-1}(t).$$

Now, Theorem 3 can be used to obtain the desired result.

Example 2. Consider the self-adjoint equation of the fourth order

$$(p(t)\, y'')'' + q(t)\, y = 0 \qquad t \in (0, \infty), \tag{16}$$

where $p(t) > 0$, $p \in \mathscr{C}^2$ and (i) $q(t) < 0$ for $t \in (0, \infty)$,

(ii) $$\int_1^\infty t^2 q > -\infty,$$

(iii) $$\int_1^\infty t^2 p^{-1} < \infty.$$

Assumptions (i), (ii) and $\int^\infty p^{-1} < \infty$ ensure disconjugacy of (16) on $[a, \infty)$ where a is sufficiently large (see [7]). Hence, according to Corollary 2 it holds

$$\liminf_{t \to \infty} \int_a^t (p(s)\, u''^2 + q(s)\, u^2)\, ds \geqq 0,$$

for all admissible functions $u(t)$ i.e. $u \in \mathscr{C}^3[a, \infty)$, $u^{(i)}(a) = 0$, $\lim_{t \to \infty} u^{(i)}(t) = 0$, $i = 0, 1$.

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