Archivum Mathematicum

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Archivum Mathematicum, Vol. 25 (1989), No. 4, 207--222

Persistent URL: http://dml.cz/dmlcz/107358

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 25, No. 4 (1989), 207–222

ON SOME NON-LINEAR BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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(Received October 5, 1988)

In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. Existence and uniqueness of the solution to some boundary value problems for the second-order differential equation in a critical case is proved by using the method of upper and lower solutions. Further boundary value problems with a parameter are investigated.

Key words. Neumann's conditions, periodic conditions, three and four point conditions. Peano's phenomen, Bernstein–Nagumo condition, boundary value problem with a parameter, isotone and antitone operator.

MS Classification. 34 B 15, 34 B 27.

The method of upper and lower solutions has been firstly used to solve the nonlinear boundary value problems (for short BVP-s) in a noncritical case (see e.g. [7]). In the last time some papers have appeared they use this method, sometimes with other arguments, in a critical case (e.g. [5], [2], [6]).

Here on the basis of this method combined with apriori estimates the solution of the differential equation

$$x'' = f(t, x, x') \tag{1}$$

is searched for which satisfies one of the following boundary conditions

$$(2_1) \qquad x'(a) = 0, \qquad x'(b) = 0, \qquad a < b \qquad \text{(Neumann's conditions)}$$

$$(2_2) \qquad x(a) - x(b) = 0, \qquad x'(a) - x'(b) = 0, \qquad a < b,$$
(periodic conditions),

$$(2_3) \qquad x'(a) = 0, \qquad x(b) - x(c) = 0, \qquad a < c < b,$$
(three point conditions),

$$(2_4) \qquad x(c) - x(a) = 0, \qquad x(b) - x(d) = 0, \qquad a < c < \leqq d < b,$$
(three or four points conditions).

We shall assume that $f \in C([a, b] \times R^2, R)$ and we shall show that all BVP-s (1), (2_j) $j = 1, 2, 3, 4$, have similar properties. Besides the existence, the problem of uniqueness of a solution to the BVP (1), (2_j) is studied together with the case that the set of all solutions to that problem is connected in the space $C([a, b], R)$ provided with the sup-norm (Peano's phenomenon). Further a BVP with a parameter is investigated and finally the theory of isotone and antitone operators (see [1], [8]) is applied to the investigation of a special case of the BVP (1), (2_j), $j = 1, 2, 3, 4$.

In what follows j will be an arbitrary, but fixed number, from the set $\{1, 2, 3, 4\}$.

LINEAR PROBLEM

The eigenvalue problem $x'' = \lambda x$, (2_j), has an eigenvalue $\lambda = 0$ and the corresponding eigenfunction $x_0(t) = c \neq 0$. This problem has no positive eigenvalue as the following lemma indicates.

Lemma 1. *Let $K < 0$. Then the problem (2_j),*

$$(3) \qquad x'' + Kx = 0,$$

has only the trivial solution.

Proof. Here and in the sequel only the case (3), (2_4) will be proved. In the other cases the proof is similar. By (3), each nontrivial solution $x(t)$ of (3) has neither a positive local maximum nor a negative local minimum.

Let $x(a) > 0$. Then $x(t)$ possesses a nonnegative local minimum in $[a, c]$ and hence $x'(c) \geqq 0$, $x''(c) > 0$. This implies that $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ in $(c, b]$ and hence the second of conditions (2_4) is not fulfilled. In case $x(a) < 0$ we come to contradiction, too. If $x(a) = 0$, then $x(t) = 0$ in $[a, c]$ and by the considerations as above we get that $x(t) = 0$ in $[c, b]$, too.

Lemma 2. *Let $K < 0$. Then there exists the Green function $G(t, s)$ of the problem (3), (2_j). This function is continuous in $[a, b] \times [a, b]$ and $\frac{\partial G}{\partial t}$ is continuous in the triangles $a \leqq t \leqq s \leqq b$, $a \leqq s \leqq t \leqq b$.*

Proof. Let $g(t) \in C([a, b], R)$ and let $C(t, s) = [e^{\sqrt{-K}(t-s)} - e^{-\sqrt{-K}(t-s)}]/(2\sqrt{-K})$ be the Cauchy function for (3). Then the general solution of the equation $x'' + Kx = g(t)$ is of the form

(4) $$x(t) = c_1 e^{\sqrt{-K}t} + c_2 e^{-\sqrt{-K}t} + \int_a^t C(t, s) g(s) ds$$

and

$$x'(t) = \sqrt{-K}(c_1 e^{\sqrt{-K}t} - c_2 e^{-\sqrt{-K}t}) + \int_a^t \frac{\partial C(t, s)}{\partial t} g(s) ds, \qquad a \leqq t \leqq b.$$

By substituting $x(t)$ into (2_4) for c_1, c_2 we get the system of two conditions

$$c_1(e^{\sqrt{-K}c} - e^{\sqrt{-K}a}) + c_2(e^{-\sqrt{-K}c} - e^{-\sqrt{-K}a}) = -\int_a^c C(c, s) g(s) ds,$$

$$c_1(e^{\sqrt{-K}b} - e^{\sqrt{-K}d}) + c_2(e^{-\sqrt{-K}b} - e^{-\sqrt{-K}d}) = -\int_a^b C(b, s) g(s) ds + \\ + \int_a^d C(d, s) g(s) ds.$$

With respect to Lemma 1 this system has a unique solution (c_1, c_2). Putting this solution into (4) we get that

$$x(t) = \int_a^b G(t, s) g(s) ds, \qquad a \leqq t \leqq b,$$

with a uniquely determined function $G(t, s)$ and this function has all required properties.

Lemma 3. *Let $K < 0$. Then the Green function $G(t, s)$ for the problem* (3), (2_j), *satisfies the inequality*

(5) $$G(t, s) \leqq 0, \qquad a \leqq t, s \leqq b.$$

Proof. If suffices to show that for each function $x(t) \in C^2([a, b], R)$ satisfying the boundary conditions (2_j) the following implication holds:
If

(6) $$x''(t) + Kx(t) \geqq 0 \qquad \text{in } [a, b],$$

then

(7) $$x(t) \leqq 0 \qquad \text{for each } t \in [a, b].$$

Again only the problem (3), (2_4) will be considered. The solution $x(t)$ of (6) has the following property: If $x(t_0) > 0$, $x'(t_0) \geqq 0$ for a $t_0 \in (a, b)$, then $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ in $[t_0, b]$, while in the case $x(t_0) > 0$, $x'(t_0) < 0$ we have that $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$ in $[a, t_0]$.

If $x(a) > 0$, then $x'(a) \geqq 0$ leads to the inequalities $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ in $(a, b]$ which contradicts the second condition in (2_4). If $x(a) > 0$, $x'(a) < 0$, then $x(c) > 0$, $x'(c) \geqq 0$ and again we get contradiction with the second condition in (2_4). Hence $x(a) \leqq 0$, $x(c) \leqq 0$ and clearly $x(t) \leqq 0$ in $[a, c]$. This implies that there is no point $t_0 \in (a, b)$ with the property $x(t_0) > 0$, $x'(t_0) < 0$. Hence if $x(t_0) > 0$ in (c, b), then $x'(t) > 0$ in $(t_0, b]$ and again we come to contradiction with the second condition in (2_4). Therefore (7) is true.

PEANO'S PHENOMENON

Lemma 4. *Assume that*

(i) $f(t, ., y)$ *is nondecreasing in* R *for each* $(t, y) \in [a, b] \times R$,

(ii) *for each* $r > 0$ *there is an* $L_r > 0$ *such that*

$$| f(t, x, y) - f(t, x, z) | \leqq L_r | y - z |,$$

for each pair of points (t, x, y), $(t, x, z) \in [a, b] \times [-r, r] \times [-r, r]$.

If $x(t)$, $y(t)$ *are two solutions of* (1) *on* $[a, b]$ *and* $x(t) - y(t) \geqq 0$ in $[t_1, t_2] \subset [a, b]$, $x'(t_1) - y'(t_1) > 0 (x'(t_1) - y'(t_1) = 0)$, *then*

$$x(t) - y(t) > 0, x'(t) - y'(t) > 0 \text{ in } (t_1, b] \; (x'(t) - y'(t) \geqq 0 \text{ in } [t_1, t_2]).$$

Proof. Denote $v(t) = x(t) - y(t)$ in $[a, b]$. Then

$$v''(t) = [f(t, x(t), x'(t)) - f(t, y(t), x'(t))] + \\ + [f(t, y(t), x'(t)) - f(t, y(t), y'(t))] \text{ in } [a, b]. \tag{8}$$

Consider the case $v'(t_1) > 0$ and $v(t) \geqq 0$ in $[t_1, t_2]$. Then there is a maximal t_3, $t_1 < t_3 \leqq b$ such that $v'(t) > 0$, $v(t) > 0$ in (t_1, t_3). If $v'(t_3) = 0$, then from (8) we would have

$$v''(t) \geqq - | f(t, y(t), x'(t)) - f(t, y(t), y'(t)) | \geqq - L_r v'(t) \tag{9}$$

in $[t_1, t_3]$ with a suitable $r > 0$ and hence,

$$v'(t_3) \geqq v'(t_1) \exp [-L_r(t_3 - t_1)] > 0,$$

which gives that $v'(t) > 0$ must hold in $[t_1, b]$ and thus, $v(t) > 0$ in $(t_1, b]$. If $v'(t_1) = 0$, then from (9) we only get that $v'(t) \geqq 0$ in $[t_1, t_2]$.

Remark 1. By this lemma, there are no two solutions $x(t)$, $y(t)$ of (1) on $[a, b]$ such that $x(t_i) = y(t_i)$, $i = 1, 2$, and $x(t) > y(t)$ in (t_1, t_2). Hence, if $x(t_1) = y(t_1)$, $x'(t_1) = y'(t_1)$ and there are points $t_n \to t_1 +$ as $n \to \infty$ such that $x(t_n) > y(t_n)$, then $x(t) > y(t)$, $x'(t) > y'(t)$ in $(t_1, b]$.

Theorem 1 (Peano's phenomenon). *If the conditions of Lemma 4 are satisfied, and* $x(t)$, $y(t)$ *are two solutions of* (1), (2_j), *then*

(a) $x(t) - y(t) = c = \text{const}$ *in* $[a, b]$;

(b) *if* $c > 0 (c < 0)$, *then for each* c_1, $0 \leqq c_1 \leqq c$ $(0 \geqq c_1 \geqq c)$ *the function* $y(t) + c_1$ *is a solution of the problem* (1), (2_j).

Proof. Only the case (1), (2_4) will be considered. Denote $v(t) = x(t) - y(t)$ in $[a, b]$. By properly denoting the solutions $x(t)$, $y(t)$ we may assume that $v(a) \geqq 0$. By Lemma 4 the case $v(a) \geqq 0$, $v'(a) > 0$ would lead to contradiction with (2_4). If $v(a) > 0$, $v'(a) = 0$, then by this lemma $v(t)$ is a nondecreasing function in a maximal interval where $v(t) \geqq 0$, hence in $[a, b]$. If $v'(t_0) > 0$ for a $t_0 \in (a, b)$, then $v(t)$ would be increasing in $[t_0, b]$ which contradicts the second condition in (2_4). Thus $v(t) \equiv v(a) > 0$. Since $v(c) = v(a)$, the case $v(a) > 0$, $v'(a) < 0$ would imply that there is a point t_0, $a < t_0 < c$, such that $v(t_0) > 0$, $v'(t_0) > 0$ and, in view of Lemma 4, we again come to contradiction with (2_4). The case $v(a) = 0$, $v'(a) < 0$ can be inverted to the case $v(a) = 0$, $v'(a) > 0$ by relabelling the solutions $x(t)$, $y(t)$. If $v(a) = v'(a) = 0$, then either $v(t) \equiv 0$ in $[a, b]$, or by Remark 1, there is a point t_0, $a \leqq t_0 < b$ such that $v(t) \equiv 0$ in $[a, t_0]$ and either $v(t) > 0$, $v'(t) > 0$ in $(t_0, b]$ or $v(t) < 0$, $v'(t) < 0$ in $(t_0, b]$. In the last two cases we come to contradiction with (2_4). The statement (a) is completely proved.

To prove (b), suppose that $c > 0$ and $0 \leqq c_1 \leqq c$. Then $(y(t) + c_1)'' = y''(t) = f(t, y(t), y'(t)) = f(t, y(t) + c_1, (y(t) + c_1)')$ for each $t \in [a, b]$, since $x''(t) = f(t, y(t) + c, y'(t)) = f(t, y(t), y'(t)) = y''(t)$ in $[a, b]$ and $f(t, ., y)$ is nondecreasing in R.

Theorem 2. *If f satisfies the strenghthened condition* (i)

(i′) $f(t, ., y)$ *is increasing in R for each* $(t, y) \in [a, b] \times R$, *then there exists at most one solution of* (1), (2_j).

Proof. Only the case (1), (2_4) is proved. Suppose that there are two solutions $x(t)$, $y(t)$ of (1), (2_4) and that the function $v(t) = x(t) - y(t)$ has a positive local maximum at t_0. If $a < t_0 < b$, then $v(t_0) > 0$, $v'(t_0) = 0$, $v''(t_0) \leqq 0$. On the other hand, by (i′) $v''(t_0) = x''(t_0) - y''(t_0) = f(t_0, x(t_0), x'(t_0)) - f(t_0, y(t_0), x'(t_0)) > 0$ which gives a contradiction. If $t_0 = a$ or $t_0 = b$, then v attains a positive local maximum at c or at d, and hence the same conclusion follows.

METHOD OF LOWER AND UPPER SOLUTIONS

The notion of a lower and upper solution can be defined for the problem (1), (2_j).

Definition 1. We say that $\alpha(t) \in C^2([a, b], R)$ $(\beta(t) \in C^2([a, b], R))$ is a lower solution for (1), (2_j) (an upper solution for (1), (2_j)) if

$$\alpha''(t) \geqq f(t, \alpha(t), \alpha'(t)), \qquad (\beta''(t) \leqq f(t, \beta(t), \beta'(t)) \qquad \text{for every } t \in [a, b]$$

and in case (2_1)

(11) $$\alpha'(a) \geqq 0, \qquad \alpha'(b) \leqq 0 \qquad (\beta'(a) \leqq 0, \beta'(b) \geqq 0);$$

in case (2_2)

(12) $$\alpha(a) - \alpha(b) = 0, \quad \alpha'(a) - \alpha'(b) \geqq 0 \quad (\beta(a) - \beta(b) = 0, \beta'(a) - \beta'(b) \leqq 0);$$

in case (2_3)

(13) $$\alpha'(a) = 0, \qquad \alpha(b) - \alpha(c) \leqq 0 \qquad (\beta'(a) = 0, \beta(b) - \beta(c) \geqq 0);$$

in case (2_4)

(14) $$\alpha(c) - \alpha(a) = 0, \qquad \alpha(b) - \alpha(d) \leqq 0 \qquad (\beta(c) - \beta(a) = 0, \beta(b) - \beta(d) \geqq 0).$$

Remark 2. If we denote

$$g(t) = \alpha''(t) - f(t, \alpha(t), \alpha'(t)), \qquad h(t) = \beta''(t) - f(t, \beta(t), \beta'(t)), \qquad t \in [a, b],$$

and $v(t)$ ($w(t)$) is the solution of (3) for $K < 0$ which satisfies the same boundary conditions as $\alpha(t)$ ($\beta(t)$), e.g. in case (2_4)

$$v(c) - v(a) = \alpha(c) - \alpha(a), \qquad v(b) - v(d) = \alpha(b) - \alpha(d),$$
$$(w(c) - w(a) = \beta(c) - \beta(a), \qquad w(b) - w(d) = \beta(b) - \beta(d)),$$

then

(15) $$g(t) \geqq 0, \qquad h(t) \leqq 0 \text{ in } [a, b]$$

and by using the identita $(x(t)\, x'(t))' = -Kx^2(t) + x'^2(t)$ which is true in $[a, b]$ for each solution $x(t)$ of (3) we get that

(16) $$v(t) \leqq 0, \qquad (w(t) \geqq 0) \text{ in } [a, b].$$

Hence if $G(t, s)$ is the Green function for the problem (3), (2_j), then the lower solution $\alpha(t)$ and the upper solution $\beta(t)$ for that problem satisfy the relations

$$\alpha(t) = v(t) + \int_a^b G(t, s) [f(s, \alpha(s), \alpha'(s)) + K\alpha(s) + g(s)]\, ds,$$

$$\beta(t) = w(t) + \int_a^b G(t, s) [f(s, \beta(s), \beta'(s)) + K\beta(s) + h(s)]\, ds,$$

and in view of Lemma 3, (15), (16), we have

(17) $$\alpha(t) \leqq T\alpha(t), \qquad \beta(t) \geqq T\beta(t), \qquad t \in [a, b],$$

where T: $C^1([a, b], R) \to C^2([a, b], R)$ is the operator defined by

(18) $$Tx(t) = \int_a^b G(t, s)\,[f(s, x(s), x'(s)) + Kx(s)]\,ds, \qquad a \leqq t \leqq b.$$

The meaning of T is based on the equivalence of the problem (1), (2_j) to the integro-differential equation

(19) $$x(t) = \int_a^b G(t, s)\,[f(s, x(s), x'(s)) + Kx(s)]\,ds, \qquad a \leqq t \leqq b.$$

The existence of the BVP (1), (2_j) will be proved by using the method developed by K. Schmitt in [7]. First we shall deal with a modified problem (2_j),

(20) $$x'' + Kx = F(t, x, x'),$$

where $K < 0$ and F is continuous on $[a, b] \times R^2$.

Lemma 5. *Let there exist a constant $L > 0$ such that*

$$|F(t, x, y)| \leqq L$$

for all $(t, x, y) \in [a, b] \times R^2$. Then the BVP (20), ($2_j$) *has a solution.*

Proof. Let $C^1 = C^1([a, b], R)$ be endowed with the norm $\|x\|_1 = \sup_{a \leqq t \leqq b} |x(t)| + \sup_{a \leqq t \leqq b} |x'(t)|$. Then $(C^1, \|\cdot\|_1)$ is a Banach space. Define the mapping T_1: $C^1 \to C^1$ by setting for each $x \in C^1$

$$T_1x(t) = \int_a^b G(t, s)\,F(s, x(s), x'(s))\,ds, \qquad a \leqq t \leqq b,$$

where G is the Green function for (3), (2_j). If

$$N = \sup_{[a,b]\times[a,b]} |G(t, s)|\,(b-a), \qquad N_1 = \sup_{[a,b]\times[a,b]} \left|\frac{\partial G(t, s)}{\partial t}\right|(b-a),$$

then we have that $|T_1x(t)| \leqq$ NL, $|(T_1x)'(t)| \leqq N_1L$. Therefore T_1 maps the closed, bounded and convex set

$$B_1 = \{x \in C^1 : |x(t)| \leqq NL, |x'(t)| \leqq N_1L, a \leqq t \leqq b\}$$

into itself. Furthermore T_1B_1 is compact. Hence, by the Schauder fixed point theorem T_1 has a fixed point in B_1. This is a solution of (20), (2_j).

Lemma 6. *Assume that the assumption of Lemma 5 is fulfilled and that there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of the problem* (20), (2_j) *such that $\alpha(t) \leqq \beta(t)$, $a \leqq t \leqq b$. Then there exists a solution $x(t)$ of* (20), (2_j) *with the property*

(21) $$\alpha(t) \leqq x(t) \leqq \beta(t), \qquad \textit{for every } t \in [a, b].$$

Proof. Define the function $H(t, x, y)$ on $[a, b] \times R^2$ by setting

$$H(t, x, y) = \begin{cases} F(t, \beta(t), y) + \dfrac{K}{2} \dfrac{x - \beta(t)}{1 + x^2} & \text{if } x > \beta(t), \\ F(t, x, y) & \text{if } \alpha(t) \leqq x \leqq \beta(t), \\ F(t, \alpha(t), y) + \dfrac{K}{2} \dfrac{x - \alpha(t)}{1 + x^2} & \text{if } x < \alpha(t). \end{cases}$$

Since F is bounded, H is also bounded. H is, together with F, continuous on $[a, b] \times R^2$. Hence, by Lemma 5, there exists a solution $x(t)$ of $x'' + Kx = H(t, x, x')$, (2_j). We now show that (21) is true. Denote $v(t) = x(t) - \beta(t)$, $t \in [a, b]$. If $v(t) \leqq 0$ on $[a, b]$ were not true, then there would exist a point $t_0 \in [a, b]$ at which $v(t)$ attains its positive absolute maximum in $[a, b]$.

If $t_0 \in (a, b)$, then $v(t_0) > 0$, $v'(t_0) = 0$, $v''(t_0) \leqq 0$. On the other hand, $v''(t_0) = x''(t_0) - \beta''(t_0) = -K(x(t_0) - \beta(t_0)) + \dfrac{K}{2} \dfrac{x(t_0) - \beta(t_0)}{1 + x^2(t_0)} > 0$ which is a contradiction. The case $t_0 = a$ or $t_0 = b$ also leads to contradiction, since the conditions (2_j), (11)–(14) imply that there is an inner point $t_1 \in (a, b)$ at which $v(t)$ attains its positive absolute maximum.

Similarly $x(t) \geqq \alpha(t)$, $a \leqq t \leqq b$, can be proved. This completes the proof of Lemma 6.

Definition 2 ([2], p. 174). We say that the function f satisfies a Bernstein–Nagumo condition if for each $M > 0$ there exists a continuous function h_M: $[0, \infty) \to [a_M, \infty)$ with $a_M > 0$ and $\int^{\infty} \dfrac{s\,ds}{h_M(s)} = +\infty$ such that for all x, $|x| \leqq M$, all $t \in [a, b]$ and all $y \in R$

$$|f(t, x, y)| \leqq h_M(|y|).$$

Lemma 7 ([3], p. 503, [2], p. 174). *Let f satisfy a Bernstein–Nagumo condition. Let $x(t)$ be any solution of (1) on $[a, b]$ satisfying the condition $|x(t)| \leqq M$, $a \leqq t \leqq b$. Then there exists a number $N > 0$ depending only on M, h_M such that $|x'(t)| \leqq N$ on $[a, b]$. More exactly, N can be taken as the root of the equation*

$$\int_{2M/(b-a)}^{N} \frac{s\,ds}{h_M(s)} = 2M.$$

Theorem 3 (Compare with [5], pp. 20–30). *If $\alpha(t)$, $\beta(t)$ are lower and upper solutions for the* BVP (1), (2_j) *such that $\alpha(t) \leqq \beta(t)$ on $[a, b]$ and f satisfies a Bernstein–Nagumo condition, then there exists a solution $x(t)$ of* (1), (2_j) *with $\alpha(t) \leqq x(t) \leqq \beta(t)$, $a \leqq t \leqq b$.*

Proof. Let $M = \max [\sup_{t\in[a,b]} |\alpha(t)|, \sup_{t\in[a,b]} |\beta(t)|]$. By Lemma 7, there exists an $N > 0$ such that for each solution $x(t)$ of (1) the implication holds: If $|x(t)| \leqq M$ on $[a, b]$, then $|x'(t)| \leqq N$ on the same interval. Let N be such that $N > |\alpha'(t)|$, $N > |\beta'(t)|$ for every $t \in [a, b]$.

Define $F(t, x, y)$ on the set $w \times R$ where $w = \{(t, x) \in R^2: \alpha(t) \leqq x \leqq \beta(t), t \in [a, b]\}$ by setting

$$F(t, x, y) = \begin{cases} f(t, x, N) + Kx, & \text{if } y > N, \\ f(t, x, y) + Kx, & \text{if } |y| \leqq N, \\ f(t, x, -N) + Kx, & \text{if } y < -N \end{cases}$$

and extend to $[a, b] \times R^2$ by the relation

$$F(t, x, y) = \begin{cases} F(t, \beta(t), y), & \text{if } x > \beta(t), \\ F(t, \alpha(t), y), & \text{if } x < \alpha(t). \end{cases}$$

Then F is bounded and $F(t, \alpha(t), \alpha'(t)) = f(t, \alpha(t), \alpha'(t)) + K\alpha(t)$, $F(t, \beta(t), \beta'(t)) = f(t, \beta(t), \beta'(t)) + K\beta(t)$, hence $\alpha(t)$ is a lower solution and $\beta(t)$ is an upper solution of (20), (2_j). By Lemma 6 there exists a solution $x(t)$ of that problem such that $\alpha(t) \leqq x(t) \leqq \beta(t)$, $t \in [a, b]$. In view of the definition of the function F, $x(t)$ is the solution of the equation $x'' = f_1(t, x, x')$ where

$$f_1(t, x, y) = \begin{cases} f(t, x, N), & \text{if } y > N, \\ f(t, x, y), & \text{if } -N \leqq y \leqq N, \\ f(t, x, -N), & \text{if } y < -N \end{cases}$$

and

$$|f_1(t, x, y)| \leqq h_M(|y|) \qquad \text{for all } t \in [a, b], |x| \leqq M, \qquad \text{and} \qquad |y| \leqq N.$$

By Lemma 7, each solution $z(t)$ of the equation $x'' = f_1(t, x, x')$ satisfying $|z(t)| \leqq M$ fulfils $|z'(t)| \leqq N$ and thus $x(t)$ satisfies the inequality $|x'(t)| \leqq N$ in $[a, b]$ which implies that $x(t)$ is a solution of (1), (2_j). The theorem is proved.

Denote

$$(22) \qquad \varphi(c) = \min_{a\leqq t\leqq b} f(t, c, 0), \qquad \psi(c) = \max_{a\leqq t\leqq b} f(t, c, 0) \qquad \text{for each } c \in R.$$

The functions φ, ψ are continuous and $\varphi(c) \leqq \psi(c)$ for every $c \in R$.

A necessary condition for the existence of a solution to (1), (2_j) is given by the lemma.

Lemma 8. *The following statements are true:*

1. $x(t) \equiv c$, $a \leqq t \leqq b$, *is a solution of* (1), (2_j) *if and only if* $\varphi(c) = \psi(c) = 0$.
2. *If there exists a solution* $x(t)$ *of* (1), (2_j), *then*

(23) $$\psi(c_3) \geqq 0, \qquad \varphi(c_4) \leqq 0,$$

where $c_3 = \min_{a \leqq t \leqq b} x(t)$, $c_4 = \max_{a \leqq t \leqq b} x(t)$.

3. *If* $\psi(c) < 0$ *in an interval* $[c_1, c_2]$ *or* $\varphi(c) > 0$ *in that interval, then there is no solution* $x(t)$ *of* (1), (2_j) *such that*

(24) $$c_1 \leqq x(t) \leqq c_2 \quad \textit{for all } t \in [a, b].$$

The proof of the statement 1 is trivial. The second statement follows from the fact that for each solution $x(t)$ of (1), (2_j) there exists a point $t_0 \in [a, b]$ such that $x(t) \geqq x(t_0) = c_3$ ($x(t) \leqq x(t_0) = c_4$) for every $t \in [a, b]$ and $x'(t_0) = 0$, $x''(t_0) \geqq 0$ ($x'(t_0) = 0$, $x''(t_0) \leqq 0$). The third statement follows from the second one.

A sufficient condition for the existence of a solution to (1), (2_j) is established in the following corollary to Theorem 3.

Corollary 1. *If f satisfies a Bernstein–Nagumo condition and there exists a pair* $c_1 \leqq c_2$ *such that*

(25) $$\psi(c_1) \leqq 0 \leqq \varphi(c_2),$$

then there exists a solution $x(t)$ *of* (1), (2_j) *satisfying* (24).

Proof. By (10)–14), $\beta(t) \equiv c_2$, $a \leqq t \leqq b$, is an upper solution of (1), (2_j) iff $f(t, c_2, 0) \geqq 0$ in $[a, b]$ and $\alpha(t) \equiv c_1$, $t \in [a, b]$, is a lower solution of (1), (2_j) iff $f(t, c_1, 0) \leqq 0$ in the same interval. Both inequalities are satisfied in $[a, b]$ when (25) is true.

Corollary 2. *If f satisfies a Bernstein–Nagumo condition and there exists a sequence of pairs* $\{c_{1k}\}$, $\{c_{2k}\}$, $k = 1, 2, \ldots$, *such that*

$$c_{1k} \leqq c_{2k}, \qquad c_{2k} < c_{1,k+1}, \qquad \psi(c_{1k}) \leqq 0 \leqq \varphi(c_{2k}), \qquad k = 1, 2, \ldots,$$

then there exist infinitely many solutions of (1), (2_j).

BOUNDARY VALUE PROBLEM WITH A PARAMETER

Consider the problem (2_j),

(1_s) $$x'' = f(t, x, x') + s,$$

with a real parameter s.

Then the following statements are true:

1. If $\beta(t)$ is an upper solution of the problem (1_{s_1}), (2_j), then $\beta(t)$ is also an upper solution for (1_s), (2_j) for each $s \geqq s_1$.

2. If $\alpha(t)$ is a lower solution for the problem (1_{s_1}), (2_j), then $\alpha(t)$ is also a lower solution for (1_s), (2_j) for each $s \leqq s_2$.

3. Let $f(t, ., y)$ be nondecreasing in R for each $(t, y) \in [a, b] \times R$. Then the following statements holds: If $\beta(t)$ is an upper solution and $\alpha(t)$ a lower solution of (1_s), (2_j), then for each $c > 0$ the function $\beta(t) + c$ is also an upper solution and $\alpha(t) - c$ is a lower solution for the same problem.

4. Let $f(t, ., y)$ be nondecreasing in R for each $(t, y) \in [a, b] \times R$. If $s_1 \leqq s_2$ and there exists an upper solution $\beta_1(t)$ for the problem (1_{s_1}), (2_j) and a lower solution $\alpha_1(t)$ for the problem (1_{s_2}), (2_j), then for each s, $s_1 \leqq s \leqq s_2$, there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of $(1\ )$, (2_j) such that $\alpha(t) \leqq \beta(t)$ on $[a, b]$.

Proof. By the statements 1 and 2, β_1 is an upper solution and α_1 is a lower solution of (1_s), (2_j) for each s, $s_1 \leqq s \leqq s_2$. Then by taking sufficiently great $c > 0$, on the basis of the statement 3, we get that $\alpha(t) = \alpha_1(t) - c$ and $\beta(t) = = \beta_1(t) + c$, $a \leqq t \leqq b$, are a lower and an upper solution for (1_s), (2_j) with the desired property.

Let $\varphi(c)$ and $\psi(c)$ be defined by (22). Then the following statements hold:

5. $\beta(t) = c$, $a \leqq t \leqq b$, is an upper solution for (1_s), (2_j) for each $s \geqq -\varphi(c)$. $\alpha(t) \equiv c$, $a \leqq t \leqq b$, is a lower solution for (1_s), (2_j) for each $s \leqq -\psi(c)$.

6. If $c_1 < c_2$ and $\psi(c_1) \leqq \varphi(c_2)$, then for each s such that

$$-\varphi(c_2) \leqq s \leqq -\psi(c_1),$$

c_1 is a lower solution, c_2 is an upper solution for (1_s), (2_j).

On the basis of the last statement we prove the theorem.

Theorem 4. *If f satisfies a Bernstein–Nagumo condition and is such that there exist two sequences*

$$c_1 > c_2 > \dots > c_n > \dots \to -\infty, \qquad d_1 < d_2 < d_3 < \dots < d_n < \dots \to \infty$$

as $n \to \infty$ where $c_1 < d_1$ and there exists a number s_1 with the property

$$-\varphi(d_n) < s_1 < -\psi(c_n), \qquad n = 1, 2, \dots, \tag{26}$$

then the set of all s for which there exists a solution for (1_s), (2_j) is an interval containing s_1 as an inner point.

Proof. Since c_1 is a lower solution and d_1 is an upper solution for (1_{s_1}), (2_j), there exists a solution $x_{s_1}(t)$ to (1_{s_1}). (2_j). Clearly s_1 can vary in the open interval $(-\varphi(d_1), -\psi(c_1))$. Suppose that $\tilde{s} < s_1$ and that there exists a solution $x_{\tilde{s}}(t)$ to $(1_{\tilde{s}})$, (2_j). Then for s, $\tilde{s} < s < s_1$, $x_{\tilde{s}}(t)$ is an upper solution to (1_s), (2_j) and, in view of the statement 6 and (26) c_n with sufficiently great n, is a lower solution whereby $c_n < x_{\tilde{s}}(t)$ for each $t \in [a, b]$. Hence by Theorem 3 there exists a solution $x_s(t)$ of the problem (1_s), (2_j). Similar considerations for $\tilde{s} > s > s_1$ can be carried out.

Corollary 3. *If f satisfies a Bernstein–Nagumo condition, $f(t, ., 0)$ is nondecreasing in R for each $t \in [a, b]$ and there are numbers $c_1 < d_1$, s_1 such that*

$$-\varphi(d_1) < s_1 < -\psi(c_1), \tag{27}$$

then the conclusion of Theorem 4 is true.

Proof. Since both functions $\varphi(c)$, $\psi(c)$ are nondecreasing, the inequalities (27) imply the inequalities (26) and the result follows.

Remark 3. In the proof of Theorem 4 we have shown the following implications: If $\tilde{s} \leqq s \leqq s_1$, then for each solution $x_{\tilde{s}}(t)$ of $(1_{\tilde{s}})$, (2_j) and each constant $c_n \leqq x_{\tilde{s}}(t)$, $a \leqq t \leqq b$, satisfying (26), there exists a solution $x_s(t)$ of (1_s), (2_j) such that

$$c_n \leqq x_s(t) \leqq x_{\tilde{s}}(t), \qquad a \leqq t \leqq b.$$

If $s_1 \leqq s \leqq \tilde{s}$, then for each solution $x_{\tilde{s}}(t)$ of $(1_{\tilde{s}})$, (2_j) and each constant $d_n \geqq x_{\tilde{s}}(t)$, $a \leqq t \leqq b$, for which (26) is true there exists a solution $x_s(t)$ of (1_s), (2_j) with the property

$$x_{\tilde{s}}(t) \leqq x_s(t) \leqq d_n, \qquad a \leqq t \leqq b.$$

By this remark and by Corollary 3 we get the following theorem. In this theorem the Banach space $C^1 = C^1([a, b], R)$ is provided with the same norm as above.

Theorem 5 (Comparison theorem). *If f satisfies a Bernstein–Nagumo condition, $f(t, ., y)$ is increasing in R for each $(t, y) \in [a, b] \times R$ and the condition (27) is fulfilled, then there exists an interval I such that for each $s \in I$ there exists a unique solution $x_s(t)$ for (1_s), (2_j) whereby*

$$s_1 < s_2 \text{ implies that } x_{s_1}(t) \geqq x_{s_2}(t) \text{ in } [a, b] \text{ for any two } s_1, s_2 \in I \tag{28}$$

and the solution $x_s(t)$ continuously depends in C^1 on $s \in I$.

Proof. The existence and uniqueness of the solution to (1_s), (2_j) for each s from an interval I follows from Corollary 3 and Theorem 2. The last remark gives the implication (28).

Fix a constant $K < 0$ and denote $G(t, u)$ the Green function for (3), (2_j). Then for each $s \in I$ the solution $x_s(t)$ of (1_s), (2_j) satisfies the integral equation

$$x_s(t) = \int_a^b G(t, u)\,[f(u, x_s(u), x_s'(u)) + Kx_s(u) + s]\,du = \tag{29}$$

$$= \frac{s}{K} + \int_a^b G(t, u)\,[f(u, x_s(u), x_s'(u)) + Kx_s(u)]\,du, \qquad a \leqq t \leqq b.$$

Then

$$x_s'(t) = \int_a^b \frac{\partial G(t, u)}{\partial t}\,[f(u, x_s(u), x_s'(u)) + Kx_s(u)]\,du, \qquad a \leqq t \leqq b. \tag{30}$$

Let $\{s_n\}$ be a nonincreasing sequence in I converging to $s \in I$. Then $x_{s_n}(t)$ is a nondecreasing sequence converging to a function $x(t) \leqq x_s(t)$ pointwise in $[a, b]$. Further both sequences $\{x_{s_n}\}$, $\{x'_{s_n}\}$ are uniformly bounded on $[a, b]$. The uniform boundedness of $\{x_{s_n}(t)\}$ follows from the inequalities $x_{s_1}(t) \leqq x_{s_n}(t) \leqq \ldots \leqq x_s(t)$ for each $n = 1, 2, \ldots$, and each $t \in [a, b]$. The uniform boundedness of $\{x'_{s_n}(t)\}$ follows on the basis of the Bernstein–Nagumo condition from that of $\{x_{s_n}(t)\}$. As $x''_{s_n}(t) = f(t, x_{s_n}(t), x'_{s_n}(t)) + s_n$, the sequence $\{x''_{s_n}(t)\}$ is uniformly bounded on $[a, b]$, too and hence, by the Ascoli theorem, there is a subsequence $\{x_{s_{n(k)}}(t)\}$ such that $\{x_{s_{n(k)}}(t)\}$ converges uniformly to $x(t)$ and $\{x'_{s_{n(k)}}(t)\}$ to $x'(t)$ on $[a, b]$. From (29), (30), by the limit process for $s = s_{n(k)}$ we get that

$$x(t) = \frac{s}{K} + \int_a^b G(t, u)\,[f(u, x(u), x'(u)) + Kx(u) + s]\,du, \qquad a \leqq t \leqq b.$$

This implies that $x(t)$ is a solution of (1_s), (2_j) which, on the basis of the uniqueness result, gives that $x(t) \equiv x_s(t)$, $a \leqq t \leqq b$, and the proof in this case is complete. Similarly we can proceed when $\{s_n\}$ is a nondecreasing sequence. In both cases the whole sequences $\{x_{s_n}(t)\}$, $\{x'_{s_n}(t)\}$ converge uniformly (to the function $x_s(t)$ and $x'_s(t)$, respectively). Since any convergent sequence $\{s_n\} \subset I$ contains a monotonic convergent subsequence, the proof by contradiction gives that also in the general case $\{x_{s_n}(t)\}$ converges uniformly on $[a, b]$ to $x_s(t)$ and $\{x'_{s_n}(t)\}$ to $x'_s(t)$ what we had to prove.

Theorem 6. *If f satisfies a Bernstein–Nagumo condition and is such that there exist two sequences*

$$s_1 < s_2 < \ldots < s_n < \ldots \to \infty, \qquad s_{-1} > s_{-2} > \ldots > s_{-n} > \ldots \to -\infty,$$

as $n \to \infty$ with $s_{-1} \leqq s_1$ and the sequences

$$d_1 < d_2 < \ldots < d_n < \ldots \to \infty, \qquad c_1 > c_2 > \ldots > c_n > \ldots \to -\infty,$$

as $n \to \infty$ where $c_1 < d_1$, with the property

$$(31) \qquad s_n \leqq -\psi(c_n), \qquad s_{-n} \geqq -\varphi(d_n), \qquad n = 1, 2, \ldots,$$

then the problem (1_s), (2_j) has a solution for each $s \in R$.

Proof. By (31), and the statement 6, for each $s \in [s_{-n}, s_n]$ c_n is a lower solution and d_n is an upper solution of (1_s), (2_j). Hence by Theorem 3, there exists a solution $x_s(t)$ for (1_s), (2_j) such that $c_n \leqq x_s(t) \leqq d_n$, $a \leqq t \leqq b$.

A SPECIAL CASE OF f

When $f = f(t, x)$, then this function satisfies a Bernstein – Nagumo condition. Now the functions $\varphi(c)$, $\psi(c)$ will mean

$$\varphi(c) = \min_{a \leqq t \leqq b} f(t, c), \qquad \psi(c) = \max_{a \leqq t \leqq b} f(t, c). \tag{32}$$

Consider the case

$$f(t, .) \text{ is nondecreasing in } R \text{ for each } t \in [a, b].$$

Then $\varphi(c)$ and $\psi(c)$ are nondecreasing, too. Since the conditions of Lemma 4 are fulfilled, Peano's phenomenon can occur for the problem (2_j),

$$x'' = f(t, x). \tag{33}$$

Further, by the statement 4, if there exist a lower and an upper solution for (33), (2_j), then there exist a lower solution $\alpha(t)$ and an upper rolution $\beta(t)$ for that problem such that $\alpha(t) \leqq \beta(t)$ on $[a, b]$ and by Theorem 3 we get the following theorem.

Theorem 7. *If $f(t, .)$ is nondecreasing in R for each $t \in [a, b]$ and there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ for the problem* (33), (2_j), *then there exists a solution $x(t)$ of that problem satisfying*

$$\alpha(t) - c \leqq x(t) \leqq \beta(t) + c, \qquad a \leqq t \leqq b,$$

for a $c \geqq 0$ such that $\alpha(t) - c \leqq \beta(t) + c$ for all $t \in [a, b]$.

Now we shall apply the theory of antitone operators (see [8]). Consider the vector space $C = C([a, b], R)$ with the sup-norm. Then C is a Banach space which can be ordered by the rule $x \leqq y$ iff $x(t) \leqq y(t)$ for every $t \in [a, b]$ for two functions $x, y \in C$. C with this ordering is an ordered Banach space. The positive cone in this space is made of all nonnegative continuous functions on $[a, b]$. P is normal. If $\alpha \leqq \beta$ are two points of C, then the subset $[\alpha, \beta] = \{z \in C: \alpha \leqq z \leqq \beta\}$ is called an ordered interval.

Suppose that $K < 0$ is a constant and consider the operator T defined by (18). Since

$$Tx(t) = \int_a^b G(t, s)\,[f(s, x(s)) + Kx(s)]\,\mathrm{d}s, \qquad a \leqq t \leqq b, \tag{34}$$

$T: C \to C$. We can easily show that T is a completely continuous operator. If the function $f(t, x) + Kx$ is nondecreasing in $x \in R$ for each fixed $t \in [a, b]$, then T is antitone, which means that for any two elements $x, y \in C$, $x \leqq y$ implies that $Tx \geqq Ty$. By Theorem 1 in [8], p. 533, we get the following theorem (compare with Theorem 10 in [8], p. 552).

Theorem 8. *Let there exist two numbers $K < 0$ and $c_1 \in R$ such that the function*

$$f(t, x) + Kx \leqq c_1 \qquad \text{for each } (t, x) \in [a, b] \times R, \tag{35}$$

or

$$f(t, x) + Kx \geqq c_1 \qquad \text{for each } (t, x) \in [a, b] \times R$$

and let the function $f(t, x) + Kx$ be nondecreasing in $x \in R$ for each $t \in [a, b]$. Then there exists a unique solution of (33), (2_j).

Proof. Since $G(t, s) \leqq 0$ for all $(t, s) \in [a, b] \times [a, b]$, the inequality $f(t, x) + Kx \leqq c_1$ implies that

$$Tx(t) \geqq \int_a^b G(t, s)\, c_1\, ds = \frac{c_1}{K} \qquad \text{for all } x(t) \in C.$$

Similarly in the second case of (35) T is bounded from above. Then the existence of a solution to (33), (2_j) follows from Theorem 1 cited above. As $f(t, .)$ is increasing for each $t \in [a, b]$, the uniqueness of that solution is implied by Theorem 2.

In case

the function $f(t, x) + Kx$ is nonincreasing in $x \in R$ for each $t \in [a, b]$,

the operator T given by (34) is isotone, i.e. if $x, y \in C$ and $x \leqq y$, then $Tx \leqq Ty$. By Corollary 2.2 ([1], p. 369) we get the following theorem.

Theorem 9. *Let there exist a number $K < 0$ such that the function $f(t, x) + Kx$ is nonincreasing in $x \in R$ for each fixed $t \in [a, b]$ and let there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of the problem* (33), (2_j) *whereby $\alpha(t) \leqq \beta(t)$, $a \leqq t \leqq b$. Then there exist a minimal solution $u(t)$ and a maximal solution $v(t)$ of* (33), (2_j) *in the order interval $[\alpha, \beta]$. Moreover, the sequences $\{\alpha_p\}_{p=0}^{\infty}$, $\{\beta_p\}_{p=0}^{\infty}$ defined by*

$$\alpha_0(t) = \alpha(t), \qquad \alpha_{p+1}(t) = T\alpha_p(t), \qquad \beta_0(t) = \beta(t), \qquad \beta_{p+1}(t) = T\beta_p(t),$$
$$a \leqq t \leqq b, \qquad p = 0, 1, 2, \ldots,$$

are such that

$$\alpha_0(t) \leqq \alpha_1(t) \leqq \ldots \leqq \alpha_p(t) \leqq \ldots \leqq u(t) \leqq v(t) \leqq \ldots \leqq \beta_p(t) \leqq \ldots \leqq$$
$$\leqq \beta_1(t) \leqq \beta_0(t), \qquad a \leqq t \leqq b,$$

and $\lim_{p\to\infty} \alpha_p(t) = u(t)$, $\lim_{p\to\infty} \beta_p(t) = v(t)$ uniformly on $[a, b]$.

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