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ASYMPTOTIC BEHAVIOUR OF THE EQUATION $\dot{z} = G(t, z)[h(z) + g(t, z)]$

JOSEF KALAS

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. Asymptotic properties of the solutions of an equation $\dot{z} = G(t, z)[h(z) + g(t, z)]$ with real-valued function G and complex-valued functions h, g are studied. The technique of the proofs of results is based on the modified Liapunov function method. The results are applied to the generalized Riccati equation $\dot{z} = q(t, z) - p(t) z^2$.

Key words. Asymptotic behaviour, Liapunov function, Riccati equation.

MS Classification. 34 D 05, 34 D 20.

1. INTRODUCTION

Consider the equation

$$\dot{z} = h(z),$$

where h is a holomorphic function in a simply connected region Ω containing zero which satisfies the conditions $h(z) = 0 \Leftrightarrow z = 0$, $h^{(j)}(0) = 0$ $(j = 1, \ldots, n - 1)$, $h^{(n)}(0) \neq 0$, where $n \geqq 2$ is an integer. The paper is concerned with the asymptotic behaviour of the solutions of the perturbed equation

$$\dot{z} = G(t, z)[h(z) + g(t, z)], \tag{1.1}$$

where G is a real-valued function and h, g are complex-valued functions, t or z being a real or complex variable, respectively. The general results for the equation (1.1) are formulated in Section 2. The last section is devoted to the application of these results to the equation

$$\dot{z} = q(t, z) - p(t)\, z^2. \tag{1.2}$$

This application gives the generalization of some results of M. Ráb [6]. The technique of the proofs is based on the Liapunov function method with "Liapunov-like" function $W(z)$ defined in [1].

The case $n = 1$, which is qualitatively different from the case $n \geqq 2$, was investigated in several papers; for the list of these papers see [1] or [2]. The asymptotic behaviour of the solutions of the Riccati equation

$$\dot{z} = q(t) - p(t)\, z^2, \tag{1.3}$$

which is a special case of (1.2) was studied by M. Ráb and Z. Tesařová. Some results dealing with the asymptotic properties of the solutions of (1.1) under the assumption $n \geqq 2$ were published in [2] or [3]. Unfortunately, the assumptions of these results make necessary the existence of the trivial solution of (1.1). Moreover, the inequalities of the type (2.3) were assumed to be satisfied at some points arbitrarily close to the point $z = 0$. This fact is very restrictive and the results are not applicable to the equations (1.2), (1.3). In the present paper and in [4] we attempt to remove this limitation.

In contradistinction to the present paper the paper [4] deals with the sufficient conditions assuring the existence of the solutions $z(t)$ of (1.1) for $t \to \infty$ and

$$\liminf_{t \to \infty} |\, z(t) \,| \leqq \delta, \tag{1.4}$$

where $\delta \geqq 0$ is a given nonnegative number. Then the conditions which guarantee

$$\limsup_{t \to \infty} |\, z(t) \,| \leqq \delta \tag{1.5}$$

for any solution $z(t)$ of (1.1) satisfying (1.4) are obtained. Even though these results generalize several results of [8], they do not allow to get the results of the type of Theorem 3 and 4 of the present paper.

In the whole paper we use the following notation:

C	– set of all complex numbers
N	– set of all positive integers
R	– set of all real numbers
I	– interval $[t_0, \infty)$
Ω	– simply connected region in C such that $0 \in \Omega$
$C(\Gamma)$	– class of all continuous real-valued functions defined on the set Γ
$\tilde{C}(\Gamma)$	– class of all continuous complex-valued functions defined on the set Γ

$\mathscr{H}(\Omega)$	– class of all complex-valued functions holomorphic in the region Ω
Int Γ	– interior of a Jordan curve with the geometric image Γ
Cl Γ	– closure of a set $\Gamma \subset C$
Bd Γ	– boundary of a set $\Gamma \subset C$
$k, W(z)$	– see [1, pp. 66–67]
$\lambda_+, \lambda_- \mathscr{T}^+, \mathscr{T}^-, \varphi$	– see [1, pp. 73–74]
$B(0, \delta)$	– the set $\{z \in C : \lvert z \rvert \leqq \delta\}$.

Let $\mathscr{S}^+ \in \mathscr{T}^+/\varphi$ and $\mathscr{S}^- \in \mathscr{T}^-/\varphi$ be fixed. Then $\mathscr{S}^+ = \{\mathring{K}(\lambda) : 0 < \lambda < \lambda_+\}$, $\mathscr{S}^- = \{\mathring{K}(\lambda) : \lambda_- < \lambda < \infty\}$, where $\mathring{K}(\lambda)$ are the geometric images of Jordan curves such that: $0 \in \mathring{K}(\lambda)$, the equality $W(z) = \lambda$ holds for $z \in \mathring{K}(\lambda) - \{0\}$ and $\mathring{K}(\lambda_1) - \{0\} \subset \text{Int}\, \mathring{K}(\lambda_2)$ for $0 < \lambda_1 < \lambda_2 < \lambda_+$ or $\mathring{K}(\lambda_2) - \{0\} \subset \text{Int}\, \mathring{K}(\lambda)$ for $\lambda_- < \lambda_1 < \lambda_2 < \infty$. Define

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \mathring{K}(\mu) - \{0\} \qquad \text{for } 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_+$$

and

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_2 < \mu < \lambda_1} \mathring{K}(\mu) - \{0\} \qquad \text{for } \lambda_- \leqq \lambda_2 < \lambda_1 \leqq \infty.$$

2. MAIN RESULTS

Suppose $G(t, z)[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$, $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$, $h \in \mathscr{H}(\Omega)$. Assume that $h(z) = 0 \Leftrightarrow z = 0$ and $h^{(j)}(0) = 0$ $(j = 1, 2, \ldots, n - 1)$, $h^{(n)}_{(0)} \neq 0$, where $n \geqq 2$ is an integer.

Theorem 1. *Let $\delta \geqq 0$, $\vartheta_1 > 0$, $\vartheta \leqq \lambda_+$. Suppose there is a function $E(t) \in C(I)$ such that*

$$\sup_{t_0 \leqq s \leqq t < \infty} \int_s^t E(\xi)\, d\xi = \varkappa < \infty, \tag{2.1}$$

$$\vartheta_1 e^{\varkappa} < \vartheta \tag{2.2}$$

and

$$G(t, z)\, \text{Re}\left\{k h^{(n)}_{(0)} \left(1 + \frac{g(t, z)}{h(z)}\right)\right\} \leqq E(t) \tag{2.3}$$

holds for $t \geqq t_0$, $z \in K(\vartheta_1, \vartheta)$, $\lvert z \rvert > \delta$.

If a solution $z(t)$ of (1.1) satisfies

$$z(t_1) \in \text{Cl}\, K(0, \gamma),$$

where $t_1 \geqq t_0$, $0 < \gamma e^{\varkappa} < \vartheta$ *and*

$$z(t) \notin B(0, \delta) - K(0, \vartheta_1)$$

for all $t \geqq t_1$ *for which* $z(t)$ *exists, then*

$$z(t) \in \mathrm{Cl}\, K(0, \beta) \qquad \textit{for } t \geqq t_1,$$

where $\beta = e^{\varkappa} \max [\gamma, \vartheta_1]$.

Proof. Let $\mathcal{M} = \{t \geqq t_1 : | z(t) | > \delta,\ z(t) \in K(\vartheta_1, \vartheta)\}$. For $t \in \mathcal{M}$ we have

$$\dot{W}(z) = G(t, z)\, W(z)\, \mathrm{Re}\left\{kh_{(0)}^{(n)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\},$$

where $z = z(t)$. Using (2.3) we obtain

$$\dot{W}(z(t)) \leqq E(t)\, W(z(t)) \tag{2.4}$$

for $t \in \mathcal{M}$. Suppose there is a $t^* > t_1$ such that $z(t^*) \in K(\beta, \vartheta)$. Without loss of generality it may be assumed that $z(t) \in K(0, \vartheta)$ for $t \in [t_1, t^*]$. There exists a γ_1 such that $\beta < \gamma_1 e^{\varkappa} < W(z(t^*))$. Obviously $\vartheta_1 < \gamma_1 < W(z(t^*))$, $\gamma_1 > \gamma$. Put $t_2 = \sup \{t \in [t_1, t^*] : z(t) \in \mathrm{Cl}\, K(0, \gamma_1)\}$. From (2.4) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \{W(z(t)) \exp [-\int_{t_2}^{t} E(s)\, \mathrm{d}s]\} \leqq 0, \qquad t \in [t_2, t^*].$$

Integration over $[t_2, t^*]$ yields

$$W(z(t^*)) \exp [-\int_{t_2}^{t^*} E(s)\, \mathrm{d}s] - W(z(t_2)) \leqq 0.$$

Using (2.1) and $W(z(t_2)) = \gamma_1$, we get

$$W(z(t^*)) \leqq \gamma_1 \exp [\int_{t_2}^{t^*} E(s)\, \mathrm{d}s] \leqq \gamma_1 e^{\varkappa} < W(z(t^*))$$

and we have a contradiction. Therefore

$$z(t) \in \mathrm{Cl}\, K(0, \beta) \qquad \text{for } t \geqq t_1.$$

Theorem 2. *Let* $\vartheta_j > 0$, $\vartheta \leqq \lambda_+$, $s_j \in I$, $\delta_j \geqq 0$ *for* $j \in N$. *Suppose there are functions* $E_j(t) \in C(I)$ *such that*

$$\int_{t_0}^{\infty} E_j(s)\, \mathrm{d}s = -\infty \qquad (j = 2, 3, \ldots),$$

$$\sup_{s_j \leqq s \leqq t < \infty} \int_{s}^{t} E_j(\xi)\, \mathrm{d}\xi = \varkappa_j < \infty \qquad (j = 1, 2, \ldots),$$

$$\vartheta_j e^{\varkappa_j} < \vartheta \qquad (j = 1, 2, \ldots), \tag{2.5}$$

and,

$$G(t, z) \operatorname{Re}\left\{kh^{(n)}_{(0)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t) \tag{2.6}$$

holds for $t \geqq s_j$, $z \in K(\vartheta_j, \vartheta)$, $|z| > \delta_j$, $j \in N$. *Denote*

$$\vartheta^* = \inf_{j \in N} [\vartheta_j e^{\varkappa_j}].$$

If a solution $z(t)$ *of* (1.1) *satisfies*

$$z(t_1) \in K(0, \vartheta e^{-\varkappa_1}),$$

where $t_1 \geqq s_1$, *and*

$$z(t) \notin B(0, \delta_j) - K(0, \vartheta_j) \tag{2.7}$$

for all $t \geqq t_1$ *for which* $z(t)$ *exists and all* $j \in N$, *then to any* ε, $\vartheta^* < \varepsilon < \lambda_+$, *there is a* $T > 0$ *such that*

$$z(t) \in K(0, \varepsilon)$$

for $t \geqq t_1 + T$.

Proof. Put $\mathscr{M}_j = \{t \geqq s_j: |z(t)| > \delta_j, z(t) \in K(\vartheta_j, \vartheta)\}$. For $t \in \mathscr{M}_j$ we obtain

$$\dot{W}(z) = G(t, z) W(z) \operatorname{Re}\left\{kh^{(n)}_{(0)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}.$$

Using (2.6) we get

$$\dot{W}(z(t)) \leqq E_j(t) W(z(t)). \tag{2.8}$$

By Theorem 1 we have $z(t) \in K(\vartheta)$ for $t \geqq t_1$. Let ε, $\vartheta^* < \varepsilon < \lambda_+$ be given. Without loss of generality it may be supposed that $\varepsilon < \vartheta$. Choose a fixed positive integer j such that

$$\vartheta_j e^{\varkappa_j} < \varepsilon.$$

Put $\sigma = \max [s_j, t_1]$. Let $T > |s_j - s_1|$ be such that

$$\int_\sigma^t E_j(s) \, ds < \ln \frac{\varepsilon}{2\vartheta}$$

for $t \geqq t_1 + T$. Clearly $t_1 + T > \sigma$.

We claim that $z(t) \in K(\varepsilon)$ for $t \geqq t_1 + T$. If it is not the case, there exists a $t^* \geqq$ $\geqq t_1 + T$ for which

$$z(t^*) \notin K(\varepsilon). \tag{2.9}$$

Using Theorem 1 we have

$$z(t) \in K(\varepsilon e^{-\varkappa_j}, \vartheta) \cup [\mathring{K}(\varepsilon e^{-\varkappa_j}) - \{0\}] = K(\vartheta_j, \vartheta)$$

for $t \in [\sigma, t^*]$. In view of (2.7), $|z(t)| > \delta_j$. The inequality (2.8) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t))\exp[-\int_\sigma^t E_j(s)\,\mathrm{d}s]\} \leqq 0, \qquad t \in \mathcal{M}_j.$$

Integration over $[\sigma, t^*]$ yields

$$W(z(t^*))\exp[-\int_\sigma^{t^*} E_j(s)\,\mathrm{d}s] - W(z(\sigma)) \leqq 0.$$

Therefore

$$W(z(t^*)) \leqq W(z(\sigma))\exp[\int_\sigma^{t^*} E_j(s)\,\mathrm{d}s] \leqq \vartheta\frac{\varepsilon}{2\vartheta} = \frac{\varepsilon}{2} < \varepsilon,$$

which contradicts (2.9) and proves $z(t) \in K(\varepsilon)$ for $t \geqq t_1 + T$.

Analogously we can prove the following two theorems corresponding to the case $\vartheta \geqq \lambda_-$:

Theorem 1'. *Let $\delta \geqq 0$, $\vartheta \geqq \lambda_-$. Suppose there is a function $E(t) \in C(I)$ such that*

$$\sup_{t_0 \leqq s \leqq t < \infty} \int_s^t E(\xi)\,\mathrm{d}\xi = \varkappa < \infty,$$

$$\vartheta e^{\varkappa} < \vartheta_1 < \infty$$

and

$$-G(t, z)\,\mathrm{Re}\left\{kh_{(0)}^{(n)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

holds for $t \geqq t_0$, $z \in K(\vartheta_1, \vartheta)$, $|z| > \delta$.

If a solution $z(t)$ of (1.1) satisfies

$$z(t_1) \in \mathrm{Cl}\,K(\infty, \gamma),$$

where $t_1 \geqq t_0$, $\vartheta < \gamma e^{-\varkappa} < \infty$ and

$$z(t) \notin B(0, \delta) - K(\infty, \vartheta_1)$$

for all $t \geqq t_1$ for which $z(t)$ exists, then

$$z(t) \in \mathrm{Cl}\,K(\infty, \beta) \qquad \text{for } t \geqq t_1,$$

where $\beta = e^{-\varkappa}\min[\gamma, \vartheta_1]$.

Theorem 2'. *Let $\vartheta \geqq \lambda_-$, $\vartheta_j < 0$, $s_j \in I$, $\delta_j \geq 0$ for $j \in N$. Suppose there are functions $E_j(t) \in C[t_0, \infty)$ such that*

$$\int_{t_0}^{\infty} E_j(s)\,\mathrm{d}s = -\infty \qquad (j = 2, 3, \ldots),$$

$$\sup_{s_j \leqq s \leqq t < \infty} \int_s^t E_j(\xi)\,\mathrm{d}\xi = \varkappa_j < \infty \qquad (j = 1, 2, \ldots),$$

$$\vartheta e^{\varkappa_j} < \vartheta_j \qquad (j = 1, 2, \ldots)$$

and,

$$-G(t, z)\,\mathrm{Re}\left\{k h_{(0)}^{(n)}\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t)$$

holds for $t \geqq s_j$, $z \in K(\vartheta_j, \vartheta)$, $|z| > \delta_j$, $j \in N$. *Denote*

$$\vartheta^* = \sup_{j \in N} [\vartheta_j e^{-\varkappa_j}].$$

If a solution $z(t)$ *of* (1.1) *satisfies*

$$z(t_1) \in K(\infty, \vartheta e^{\varkappa_1}),$$

where $t_1 \geqq s_1$, *and*

$$z(t) \notin B(0, \delta_j) - K(\infty, \vartheta_j)$$

for all $t \geqq t_1$ *for which* $z(t)$ *exists and all* $j \in N$, *then to any* ε, $\lambda_- < \varepsilon < \vartheta^*$, *there is a* $T > 0$ *such that*

$$z(t) \in K(\infty, \varepsilon)$$

for $t \geqq t_1 + T$.

3. APPLICATION TO THE EQUATION $\dot{z} = q(t, z) - p(t)\,z^2$

Supposing that $q \in \tilde{C}(I \times C)$, $p \in \tilde{C}(I)$ and $a \in C$, $a \neq 0$, the equation

$$\dot{z} = q(t, z) - p(t)\,z^2 \tag{3.1}$$

can be written in the form

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)], \tag{3.2}$$

where $h(z) = -az^2$, $G(t, z) = 1$ and $g(t, z) = q(t, z) + az^2 - p(t)\,z^2$. In view of [1, Example 1], where $\Omega = C$, $b = -a$, we get $h'(z) = -2az$, $h''(z) = -2a$, $n = 2$, $W(z) = \exp[\mathrm{Re}\,(2\bar{a}z^{-1})]$, $\lambda_+ = \lambda_- = 1$, $k = -\bar{a}$. The sets $\hat{K}(\lambda)$, where $0 < \lambda < \lambda_+ = 1$ or $1 = \lambda_- < \lambda < \infty$, are circles with centres $\dfrac{\bar{a}}{\ln \lambda}$ and radii $\dfrac{|a|}{|\ln \lambda|}$, $K(0, 1) = \{z \in C : \mathrm{Re}\,(az) < 0\}$, $K(\infty, 1) = \{z \in C : \mathrm{Re}\,(az) > 0\}$.

For $a \in C$, $a \neq 0$, $A > 0$, $B > 0$, $\delta \in \left(0, \dfrac{\pi}{4}\right]$ denote

$$\Omega_{A,B}(a) = \{z \in C : -A\,\mathrm{Re}\,[a^2 z^2] - B\,|\,\mathrm{Im}\,[a^2 z^2]\,| > 0\},$$

$$\Omega_\delta(a) = \left\{z = \mu e^{i\varphi} : \mu \in \boldsymbol{R} - \{0\}, \operatorname{Arg} \bar{a} + \frac{\pi}{2} - \delta < \varphi < \operatorname{Arg} \bar{a} + \frac{\pi}{2} + \delta\right\}.$$

Obviously,

$$\Omega_{A,B}(a) \subset \Omega_{\pi/4}(a) = \{z \in \boldsymbol{C} : \operatorname{Re}(a^2 z^2) < 0\}$$

for any $A, B > 0$, and, to any $A, B > 0$ there exists a $\delta_0 \in \left(0, \frac{\pi}{4}\right)$ such that

$$\Omega_\delta(a) = \Omega_{A,B}(a) \qquad \text{for } \delta \in (0, \delta_0].$$

First we shall prove the following lemma:

Lemma 1. *Assume there are $a \in \boldsymbol{C}$ and $C \geqq 0$ such that*

(3.3) $$\operatorname{Re}[\bar{a}p(t)] > 0 \qquad \textit{for } t \in I,$$

(3.4) $$\liminf_{t\to\infty} \operatorname{Re}[\bar{a}p(t)] > 0, \qquad \limsup_{t\to\infty} |\operatorname{Im}[\bar{a}p(t)]| < \infty,$$

(3.5) $$\operatorname{Re}[aq(t, z)] \geqq -C\,|\operatorname{Im}[a^2z^2]| \qquad \textit{for } t \in I, z \in \Omega_{\pi/4}(a)$$

and

(3.6) $$q(t, 0) \neq 0 \qquad \textit{for } t \in I.$$

Then every solution $z(t)$ of (3.1) *satisfying at $t_1 \geqq t_0$ the condition* $\operatorname{Re}[az(t_1)] \geqq 0$ *fulfils* $\operatorname{Re}[az(t)] \geqq 0$ *for all $t > t_1$ for which $z(t)$ exists.*

Moreover, $\operatorname{Re}[az(t)] > 0$ *provided $z(t) \neq 0$.*

Proof. Choose $A, B > 0$ so that

$$\operatorname{Re}[\bar{a}p(t)] \geqq |a|^2 A, \qquad |\operatorname{Im}[\bar{a}p(t)]| \leqq |a|^2 (B - C)$$

for $t \geqq t_1$. There exists $\delta_0 \in \left(0, \frac{\pi}{4}\right)$ with the property $\Omega_{\delta_0}(a) \subset \Omega_{A,B}(a)$. For $t \geqq t_1$ such that $z = z(t) \in \Omega_{\delta_0}(a)$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}[az(t)] &= \operatorname{Re}[a\dot{z}(t)] = \operatorname{Re}[aq(t, z)] - \operatorname{Re}[ap(t)\, z^2] = \\ &= \operatorname{Re}[aq(t, z)] - |a|^{-2} \operatorname{Re}[\bar{a}p(t)\, a^2z^2] = \\ &= \operatorname{Re}[aq(t, z)] - |a|^{-2} \{\operatorname{Re}[\bar{a}p(t)] \operatorname{Re}[a^2z^2] - \\ &- \operatorname{Im}[\bar{a}p(t)] \operatorname{Im}[a^2z^2]\} \geqq -C\,|\operatorname{Im}[a^2z^2]| - A \operatorname{Re}[a^2z^2] - \\ &- (B - C)\,|\operatorname{Im}[a^2z^2]| \geqq -A \operatorname{Re}[a^2z^2] - B\,|\operatorname{Im}[a^2z^2]| > 0.\end{aligned}$$

If $z(t) = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re}[az(t)] = \operatorname{Re}[aq(t, 0)] > 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}[az(t)] = \operatorname{Re}[aq(t, 0)] = 0. \tag{3.7}$$

Because of (3.6) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}[az(t)] = \operatorname{Im}[aq(t, 0)] \neq 0$$

in the case (3.7). In view of the fact that Re $[az] = 0$ implies $z \in \Omega_{\delta_0}(a) \cup \{0\}$, we get Re $[az(t)] \geqq 0$ for all $t \geqq t_1$ for which $z(t)$ is defined. Clearly, Re $[az(t)] > 0$ if $z(t) \neq 0$.

Remark. If the condition (3.6) of Lemma 2 is replaced by Re $[aq(t, 0)] > 0$, we get the assertion Re $[az(t)] > 0$ for all $t > t_1$ for which $z(t)$ exists.

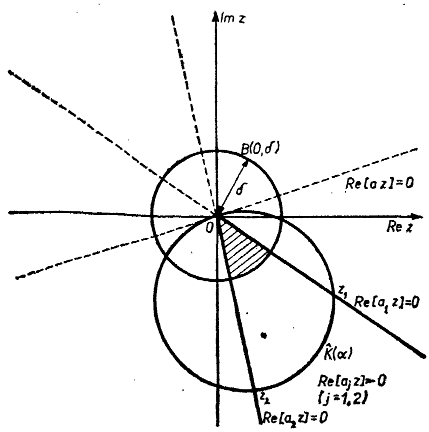
The next lemma will be useful in our further considerations.

Lemma 2. *Let $\delta > 0$, $a_1, a_2 \in C$ and let a_1, a_2 be linearly independent. If $a = (a_1 + a_2)/2$,*

$$1 < \alpha \leqq \exp\left[\delta^{-1} \min_{m=1,2}\left(|a_m|\left|\operatorname{Im}\frac{a_{3-m}}{a_m}\right|\right)\right] \tag{3.8}$$

and

$$\operatorname{Re}[a_m z] > 0 \qquad (m = 1, 2), \qquad \textit{then} \qquad z \notin B(0, \delta) - K(\infty, \alpha).$$



Proof. Since Re $[a_m z] > 0$ ($m = 1, 2$) implies Re $[az] > 0$, it is sufficient to prove that $\delta \leqq \min[|z_1|, |z_2|]$, where $z_m \neq 0$ is the intersection of $\mathring{K}(\alpha)$ with the line Re $[a_m z] = 0$.

Supposing

$$W(z_m) = \exp\{\mathrm{Re}\,[2\bar{a}z_m^{-1}]\} = \alpha \qquad \text{and} \qquad \mathrm{Re}\,[a_m z_m] = 0,$$

there exists a $\tau_m \in \boldsymbol{R}$ such that

$$z_m = i\bar{a}_m\tau_m \qquad \text{and} \qquad \mathrm{Re}\,\frac{2\bar{a}}{i\bar{a}_m\tau_m} = \ln\alpha.$$

Hence

$$\tau_m = [\ln\alpha]^{-1}\,\mathrm{Re}\,\frac{\bar{a}_{3-m}}{i\bar{a}_m}$$

and

$$z_m = \frac{i\bar{a}_m}{\ln\alpha}\,\mathrm{Re}\,\frac{\bar{a}_{3-m}}{i\bar{a}_m} = -\frac{i\bar{a}_m}{\ln\alpha}\,\mathrm{Im}\,\frac{a_{3-m}}{a_m}.$$

Therefore

$$|z_m| = \frac{|a_m|}{|\ln\alpha|}\left|\,\mathrm{Im}\,\frac{a_{3-m}}{a_m}\right|.$$

In view of (3.8) we obtain $\delta \leqq \min\,[|z_1|, |z_2|]$.

Applying Theorem 2′ and using Lemma 1 and Lemma 2 we obtain

Theorem 3. *Suppose there are $a_1, a_2 \in \boldsymbol{C}$ linearly independent such that the following inequalities are fulfilled for $m = 1, 2$:*

(3.9) $$\mathrm{Re}\,[\bar{a}_m p(t)] > 0 \qquad \text{for } t \in I,$$

(3.10) $$\liminf_{t\to\infty} \mathrm{Re}\,[\bar{a}_m p(t)] > 0, \qquad \limsup_{t\to\infty} |\mathrm{Im}\,[\bar{a}_m p(t)]| < \infty,$$

(3.11) $$\mathrm{Re}\,[a_m q(t, z)] \geqq 0 \qquad \text{for } t \in I,\ z \in \boldsymbol{C},$$

(3.12) $$\mathrm{Re}\,[a_m q(t, 0)] > 0 \qquad \text{for } t \in I.$$

Assume there exists $D(t) \in C(I)$ such that

$$|q(t, z)| \leqq D(t) \qquad \text{for } t \geqq t_0,\ z \in \boldsymbol{C}$$

and

(3.13) $$\lim_{t\to\infty} D(t) = 0.$$

Then any solution $z(t)$ of (3.1) *satisfying* $\mathrm{Re}\,[a_m z(t_1)] > 0$ $(m = 1, 2)$, *where* $t_1 \geqq t_0$ *satisfies the condition*

$$\lim_{t\to\infty} z(t) = 0.$$

Moreover, $\mathrm{Re}\,[a_m z(t)] > 0$ $(m = 1, 2)$ *for* $t \geqq t_1$.

Proof. Put $a = (a_1 + a_2)/2$. Choose $\vartheta = \lambda_- = 1$, $s_1 = t_1$,

$$\delta_1 = 2\sqrt{|a| \frac{\max_{t \in I} D(t)}{\min_{t \in I} \mathrm{Re}[\bar{a}p(t)]}}, \qquad \delta_j = j^{-1} \qquad (j = 2, 3, \ldots),$$

$\vartheta_j = \exp\{\delta_j^{-1} \min[|a_m| \, |\mathrm{Im}(a_{3-m}a^{-1})|]\}$, $\varkappa_j = 0$, $E_j(t) = 2|a|\delta_j^{-2}D(t) - 2\,\mathrm{Re}[\bar{a}p(t)]$. For $j \geqq 2$ let $s_j \geqq t_0$ be such that $E_j(t) < 0$ for $t \geqq s_j$. Then $-G(t, z)\,\mathrm{Re}\left\{kh''(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} = 2\,\mathrm{Re}[\bar{a}z^{-2}q(t, z)] - 2\,\mathrm{Re}[\bar{a}p(t)] \leqq E_j(t)$ foi $t \geqq s_j$, $z \in K(\vartheta_j, \vartheta)$, $|z| > \delta_j$. Further it holds that $\vartheta < \vartheta_j$ and $\vartheta^* = \sup_{j \in N} \vartheta_j = \infty$. In view of Lemma 1 and following Remark we have $\mathrm{Re}[a_m z(t)] > 0$ ($m = 1, 2$) for all $t \geqq t_1$ for which $z(t)$ exists. By use of Lemma 2 we infer that

$$z(t) \notin B(0, \delta_j) - K(\infty, \vartheta_j)$$

for all $t \geqq t_1$ for which $z(t)$ exists and all $j \in N$. Applying Theorem 2′ we find out that to any ε, $1 < \varepsilon < \infty$ there is a $T > 0$ such that $z(t) \in K(\infty, \varepsilon)$ for $t \geqq t_1 + T$, which implies

$$\lim_{t \to \infty} z(t) = 0.$$

The replacement of the condition (3.13) by

$$\int_{t_0}^{\infty} D(t)\,\mathrm{d}t < \infty \tag{3.14}$$

leads to the following theorem:

Theorem 4. *Let the assumptions of Theorem 3 be fulfilled with the exception that* (3.13) *is replaced by* (3.14). *Then the conclusion of Theorem 3 remains true.*

Proof. Set $a = (a_1 + a_2)/2$, $\vartheta = \lambda_- = 1$, $s_1 = t_1$,

$$\delta_1 = \sigma\sqrt{2|a| \int_{t_0}^{\infty} D(t)\,\mathrm{d}t},$$

$$\delta_j = \min_{m=1,2} [|a_m| \, |\mathrm{Im}(a_{3-m}a_m^{-1})|]/j \qquad (j = 2, 3, \ldots),$$

$$\vartheta_j = \exp\{\delta_j^{-1} \min[|a_m| \, |\mathrm{Im}(a_{3-m}a_m^{-1})]\} \qquad (j = 1, 2, \ldots),$$

where

$$\sigma = 2\max\left\{\ln^{-1/2} W(z(t_1)), \frac{\sqrt{2|a| \int_{t_0}^{\infty} D(t)\,\mathrm{d}t}}{\min_{m=1,2} [|a_m| \, |\mathrm{Im}(a_{3-m}a_m^{-1})|]}\right\}. \tag{3.15}$$

For $j \geqq 2$ let $s_j \geqq t_0$ be such that

$$2|a| \sup_{s_j \leqq s \leqq t < \infty} \int_s^t D(\xi)\,d\xi \leqq \delta_j^2.$$

Put

$$E_j(t) = 2\,|\,a\,|\,\delta_j^{-2}D(t) - 2\,\mathrm{Re}\,[\bar{a}p(t)],$$

$$\varkappa_j = \sup_{s_j \leqq s \leqq t < \infty} \int_s^t E_j(\xi)\,d\xi.$$

Then $\vartheta e^{\varkappa_1} \leqq e^{\sigma^{-2}} < \vartheta_1$, $\vartheta e^{\varkappa_j} = e^{\varkappa_j} \leqq e < \vartheta_j$ $(j = 2, 3, \ldots)$,

$$\vartheta^* = \sup_{j \in N} [\vartheta_j e^{-\varkappa_j}] \geqq \sup_{j \in N} [\vartheta_j e^{-1}] = \sup_{j \in N} e^{j-1} = \infty$$

and

$$-G(t, z)\,\mathrm{Re}\left\{kh''(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} = 2\,\mathrm{Re}\,[\bar{a}z^{-2}q(t, z)] - 2\,\mathrm{Re}\,[\bar{a}p(t)] \leqq E_j(t).$$

In view of (3.15) we have $W(z(t_1)) > e^{\sigma^{-2}}$, whence $z(t_1) \in K(\infty, \vartheta e^{\varkappa_1})$. Analogously as in the proof of Theorem 3 we infer that $\mathrm{Re}\,[a_m z(t)] > 0$ and $z(t) \notin B(0, \delta_j) - K(\infty, \vartheta_j)$ for all $t \geqq t_1$ for which $z(t)$ exists. The application of Theorem 2′ yields the desired result.

Remark. In a special case $p(t) = 1$, $q(t, z) \equiv q(t)$ the conditions (3.9)–(3.12) are reduced to $\mathrm{Re}\,a_m > 0$, $\mathrm{Re}\,[a_m q(t)] > 0$ $(m = 1, 2)$ and we can put $D(t) = |\,q(t)\,|$. Thus we get some results of M. Ráb [6].

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