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A FOUR-POINT PROBLEM FOR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. The paper deals with the four-point problem $u'' = f(t, u, u')$, $u(c) - u(a) = A$, $u(b) - u(d) = B$, where $a, b, c, d, A, B \in R$, $a < c < d < b$. The sufficient conditions for the existence of solutions of this problem are established.

Key words. Four-point boundary value problem, a priori estimate, Carathéodory conditions.

MS Classification 34 B 10.

The questions of existence and uniqueness of solutions of the two-point boundary value problem for differential equations of the second order have a long history, going back to Picard (1893).

The boundary problems

$$u'' = f(t, u, u'), \tag{0.1}$$

$$\sum_{j=1}^{2} (a_{ij}u^{(j-1)}(a) + b_{ij}u^{(j-1)}(b)) = c_i, \qquad i = 1, 2, \tag{0.2}$$

where $a, b, a_{ij}, b_{ij}, c_i \in (-\infty, +\infty)$, $a < b$, and f is a continuous function or satisfies the local Carathéodory conditions, are solved for example in [3], [5], [7], [8], [12],. In [10], [12] the linear conditions (0.2) are generalized for the case of nonlinear ones.

The three-point problems for differential equations of the second order were studied in [1], [2], [9], and [11]. The problem of existence of solutions of the equation

$$u'' = f(t, u),$$

satisfying the conditions

$$u(0) = u(a) = u(2a), \qquad a \in (-\infty, +\infty)$$

is solved in [1], [2].

The theorems of existence and uniqueness of solutions of the equation (0.1) satisfying the conditons

$$u(a) = c_1, \qquad u(b) = u(t_0) + c_2, \qquad a, b, t_0, c_1, c_2 \in (-\infty, +\infty), a < t_0 < b,$$

are proved in [11] and for the linear differential equation in [9].

I

Our paper deals with the problem of existence of solutions of the equation

$$u'' = f(t, u, u'), \tag{1.1}$$

defined on the interval $[a, b]$ and satisfying the conditions

$$u(c) - u(a) = A, \qquad u(b) - u(d) = B, \tag{1.2}$$

where $A, B \in (-\infty, \infty)$, $-\infty < a < c < d < b < +\infty$.

We shall use the following notations:

$$R = (-\infty, +\infty), \qquad R_+ = [0, +\infty), \qquad D = [a, b] \times R^2, \qquad D_+ = [a, b] \times R_+^2,$$

$$\tau = \begin{cases} \max\{c - a, b - c\} & \text{for } d - a > b - c, \\ \max\{d - a, b - d\} & \text{for } d - a \leqq b - c, \end{cases} \qquad g_0(t) = \alpha t^2 + \beta t + \gamma, \text{ where}$$

$$\alpha = (B/(b - d) - A/(c - a))(b - c + d - a)^{-1},$$

$$\beta = (A(b + d)/(c - a) - B(c + a)/(b - d))(b - c + d - a)^{-1},$$

$$\gamma \in R, \qquad r_0 = \max\{|g_0(t)| : a \leqq t \leqq b\}, \qquad r_1 = \max\{|g_0'(t)| : a \leqq t \leqq b\}.$$

$AC^1(a, b)$ is the set of all real functions which are absolutely continuous with their first derivatives on $[a, b]$.

$\mathrm{Car}_{\mathrm{loc}}(D)$ is the set of all real functions satisfying the local Carathéodory conditions on D, i.e. $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ iff

$f(., x, y) : [a, b] \to R$ is measurable for every $(x, y) \in R^2$,
$f(t, ., .) : R^2 \to R$ is continuous for almost every $t \in [a, b]$,
$\sup\{|f(., x, y)| : |x| + |y| \leqq \varrho\} \in L(a, b)$ for any $\varrho \in (0, +\infty)$.

Definition. A function $u \in AC^1(a, b)$ which fulfils (1.1) for almost every $t \in [a, b]$ will be called a solution of the equation (1.1). Each solution of (1.1) which satisfies the conditions (1.2) will be called a solution of the problem (1.1), (1.2).

In the whole paper we suppose that $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ and $\lambda \in \{-1, 1\}$.

Theorem 1. *Let there exist $r \in (0, +\infty)$ such that on the set D the inequalities*

$$\lambda[f(t, x, y) - 2\alpha] \operatorname{sgn} x \geqq 0 \qquad \text{for } |x| > r, \tag{1.3}$$

$$|f(t,x,y)| \leqq \omega(t,|x|,|y|) \tag{1.4}$$

are fulfilled, where $\omega \in \mathrm{Car}_{\mathrm{loc}}(D_+)$ *is a non-negative function, non-decreasing with respect to its second and third variables and satisfying the conditions*

$$\limsup_{\varrho\to+\infty} \frac{1}{\varrho} \int_a^b \omega(t,\varrho(b-a),\varrho)\,\mathrm{d}t < 1. \tag{1.5}$$

Then the problem (1.1), (1.2) has at least one solution.

Corollary. *Let there exist* $r \in (0,+\infty)$ *such that on the set D the inequalities* (1.3) *and*

$$|f(t,x,y)| \leqq h_1(t)\,|x| + h_2(t)\,|y| + \omega(t,|x|+|y|) \tag{1.6}$$

are fulfilled, where $h_1, h_2 \in L(a,b)$ *are non-negative functions satisfying*

$$(b-a)\int_a^b h_1(t)\,\mathrm{d}t + \int_a^b h_2(t)\,\mathrm{d}t < 1 \tag{1.7}$$

and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a,b]\times R_+)$ *is a non-negative function, non-decreasing with respect to its second variable and satisfying the condition*

$$\lim_{\varrho\to+\infty} \frac{1}{\varrho} + \int_a^b \omega(t,\varrho)\,\mathrm{d}t = 0. \tag{1.8}$$

Then the problem (1.1), (1.2) has at least one solution.

Theorem 2. *Let there exist* $r \in (0,+\infty)$ *such that on the set D the inequalities* (1.3) *and*

$$|f(t,x,y)| \leqq a_1|x| + a_2\,|y| + \omega(t,|x|+|y|) \tag{1.9}$$

are fulfilled, where $a_1, a_2 \in (0,+\infty)$ *satisfy*

$$a_1(2(b-a)/\pi)^2 + a_2(2(b-a)/\pi) < 1 \tag{1.10}$$

and ω *is the function from Corollary.*

Then the problem (1.1), (1.2) has at least one solution.

Theorem 3. *Let there exist* $r \in (0,+\infty)$ *such that on the set D the inequalities* (1.3) *and* (1.9) *are fulfilled, where* $a_1, a_2 \in (0,+\infty)$ *satisfy*

$$a_1\tau(b-a)\,(2/\pi)^2 + a_2\tau\, 2/\pi < 1 \tag{1.11}$$

and $\omega : [a,b]\times R_+ \to R_+$ *is a function such that*

$$(1.12)\qquad \begin{cases} \omega(.,\varrho)\in L^2(a,b) & \textit{for any } \varrho\in R_+, \\ \omega(t,.)\in C(R_+) & \textit{is non-decreasing,} \\ \lim\limits_{\varrho\to+\infty}\dfrac{1}{\varrho}(\int\limits_a^b \omega^2(t,\varrho)\,\mathrm{d}t)^{1/2}=0. \end{cases}$$

Then the problem (1.1), (1.2) has at least one solution.

II

Lemma 1. ([6], Theorem 256, p. 219). *If $f\in AC(t_1, t_2)$, $f'\in L^2(t_1, t_2)$ and $f(t_0) = 0$, where $-\infty < t_1 < t_2 < +\infty$, $t_0\in[t_1, t_2]$, then*

$$\int\limits_{t_1}^{t_2} f^2(t)\,\mathrm{d}t = (2(t_2-t_1)/\pi)^2\int\limits_{t_1}^{t_2} f'^2(t)\,\mathrm{d}t.$$

Lemma 2. *Let $\varepsilon\in(0,+\infty)$ satisfy the inequality*

$$(2.1)\qquad \varepsilon\tau(b-a)(2/\pi)^2 < 1.$$

Then the problem

$$(2.2)\qquad v'' = \lambda\varepsilon v,$$

$$(2.3)\qquad v(c)-v(a)=0,\qquad v(b)-v(d)=0$$

has only the trivial solution.

Proof. Let v be a solution of the problem (2.2), (2.3). By (2.3), there exist $t_1\in(a,c)$, $t_2\in(d,b)$ such that $v'(t_1)=v'(t_2)=0$. Therefore, in view of (2.2), we have $t_0\in(t_1,t_2)$ such that $v''(t_0)=v(t_0)=0$. It follows from Lemma 1, that

$$\int\limits_a^b v'^2(t)\,\mathrm{d}t \leqq (2\tau/\pi)^2\int\limits_a^b v''^2(t)\,\mathrm{d}t$$

and

$$(2.4)\qquad \int\limits_a^b v^2(t)\,\mathrm{d}t \leqq (2/\pi)^4(\tau(b-a))^2\int\limits_a^b v''^2(t)\,\mathrm{d}t.$$

Hence, by (2.2), (2.4), we find, that

$$\int\limits_a^b v''^2(t)\,\mathrm{d}t \leqq (\varepsilon(2/\pi)^2\tau(b-a))^2\int\limits_a^b v''^2(t)\,\mathrm{d}t$$

and by (2.1) (2.4), we find, that $v(t)=0$ for $t\in[a,b]$.

Lemma 3. *Let $a_1, a_2\in(0,+\infty)$ satisfy* (1.10) *and $h_1, h_2\in L(a,b)$ be such that*

$$(2.5)\qquad |h_i(t)|\leqq a_i,\qquad i=1,2,\qquad a\leqq t\leqq b.$$

Then the problem

$$v'' = h_1(t)\, v + h_2(t)\, v', \tag{2.6}$$

$$v(t_0) = v'(t_1) = 0, \qquad t_0, t_1 \in (a, b), \tag{2.7}$$

has only the trivial solution.

Proof. Let v be a solution of the problem (2.6), (2.7). Then, by Lemma 1, we have

$$\int_a^b v'^2(t)\, dt \leqq (2(b-a)/\pi)^2 \int_a^b v''^2(t)\, dt \tag{2.8}$$

and

$$\int_a^b v^2(t)\, dt \leqq (2(b-a)/\pi)^4 \int_a^b v''^2(t)\, dt. \tag{2.9}$$

Therefore, by (2.5), (2.6), (2.8), (2.9), we obtain

$$\left(\int_a^b v''^2(t)\, dt\right)^{1/2} \leqq ((a_1\, 2(b-a)/\pi)^2 + a_2\, 2(b-a)/\pi) \left(\int_a^b v''^2(t)\, dt\right)^{1/2}.$$

From the last inequality, according to (1.10) and (2.9), it follows $v(t) = 0$ for $t \in [a, b]$.

Lemma 4. *Let* $g \in \mathrm{Car}_{\mathrm{loc}}(D)$ *and* $\varepsilon \in (0, +\infty)$ *satisfy* (2.1). *If there exists* $g^* \in L(a, b)$ *such that*

$$| g(t, x, y) | \leqq g^*(t) \qquad \text{on } D,$$

then the problem

$$v'' = \lambda \varepsilon v + g(t, v, v'), \qquad (2.3)$$

is solvable.

Proof. See [4] or [8], Theorem 2.4, p. 25.

Lemma 5. *Let* $a_1, a_2 \in (0, +\infty)$ *and let for any* $h_1, h_2 \in L(a, b)$ *satisfying* (2.5) *the problem* (2.6), (2.7) *have only the trivial solution. Then there exists such* $\gamma \in (0, +\infty)$, *that for any* $h_1, h_2 \in L(a, b)$ *satisfying* (2.5), *the inequality*

$$\left| \frac{\partial G(t, s)}{\partial t} \right| + | G(t, s) | \leqq \gamma, \qquad a \leqq t, s \lesseqqgtr b \tag{2.10}$$

is fulfilled, where G *is the Green function of the problem* (2.6), (2.7).

Proof. See [8], Lemma 2.2, p. 12.

III

Lemmas for a priori estimates

Lemma 6. *Let* $r \in (0, +\infty)$ *and* $\omega \in \mathrm{Car}_{\mathrm{loc}}(D_+)$ *be a non-negative function, non-decreasing with respect to its second and third variables and satisfying* (1.5).

Then there exists $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions*

$$v(a) = v(c), \qquad v(d) = v(b), \tag{3.1}$$

$$\lambda v''(t) \operatorname{sgn} v(t) > 0 \qquad \text{for } |v(t)| > r,\ t \in [a, b], \tag{3.2}$$

$$|v''(t)| \leqq \omega(t, |v|, |v'|), \qquad \text{for } a < t < b \tag{3.3}$$

imply the estimate

$$|v(t)| + |v'(t)| \leqq r^* \qquad \text{for } a \leqq t \leqq b. \tag{3.4}$$

Proof. The condition (3.1) implies the existence of $t_1, t_2 \in (a, b)$ such that $v'(t_1) = v'(t_2) = 0$. If $|v(t)| > r$ on (a, b) then, by (3.2), v' has to be strictly monotonous on (a, b) and we get the contradiction. Therefore there exists $t_0 \in (a, b)$ such that $|v(t_0)| \leqq r$.

Put $\varrho_0 = \max\{|v'(t)| : a \leqq t \leqq b\}$. Integrating the inequality $|v'(t)| \leqq \varrho_0$ from t_0 to t, we have $|v(t)| \leqq r + (b - a)\varrho_0$. Let $t^* \in [a, b]$ be such that $|v'(t^*)| = \varrho_0$. Integrating (3.3) from t_1 to t^*, we get

$$\varrho_0 \leqq \int_a^b \omega(t, r + (b - a)\varrho_0, \varrho_0)\, dt. \tag{3.5}$$

Hence, by (1.5), there exists $\delta > 0$ such that

$$(1 + \delta) \limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b - a), \varrho)\, dt < 1.$$

Consequently there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequalities

$$r + \varrho(b - a) \leqq (1 + \delta)\varrho(b - a)$$

and

$$\frac{1}{\varrho} \int_a^b \omega(t, (1 + \delta)(b - a)\varrho, \qquad (1 + \delta)\varrho)\, dt < 1 \tag{3.6}$$

are satisfied. By (3.5) and (3.6), we obtain $\varrho_0 \leqq \varrho^*$. Putting

$$r^* = r + (b - a + 1)\varrho^*,$$

we get the estimate (3.4).

Lemma 7. *Let* $r \in (0, +\infty)$, $a_1, a_2 \in (0, +\infty)$ *satisfy* (1.10) and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times$

$\times R_+)$ *is a non-negative function, non-decreasing with respect to its second variable and satisfying* (1.8).

Then there exists $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions* (3.1), (3.2) *and*

$$|v''(t)| \leqq a_1 |v(t)| + a_2 |v'(t)| + \omega(t, |v| + |v'|), \qquad t \in (a, b) \tag{3.7}$$

imply the estimate (3.4).

Proof. From (3.1), (3.2) it follows that there exist $t_0, t_1, t_2 \in (a, b)$ such that $v'(t_1) = v'(t_2) = 0$ and $v(t_0) = c_0$, where $|c_0| \leqq r$. Put $y(t) = v(t) - c_0$ for $a \leqq t \leqq b$ and consider the equation

$$y'' = h_1(t)\, y + h_2(t)\, y' + h_0(t), \tag{3.8}$$

where $h_i(t) = a_i \,.\, k(t)\, v''(t) \operatorname{sgn} v^{(i-1)}(t), i = 1, 2, h_0(t) = \omega(t, |v| + |v'|)\, k(t)\, v''(t) + h_1(t)\, c_0, k(t) = (a_1 |v| + a_2 |v'| + \omega(t, |v| + |v'|))^{-1}$. Since $|h_i(t)| \leqq a_i$, $i = 1, 2$, it follows from Lemma 3 that the problem

$$y'' = h_1(t) y + h_2(t)\, y', \tag{3.9}$$

$$y(t_0) = y'(t_1) = 0 \tag{3.10}$$

has only the trivial solution. Consequently, by Lemma 5, the solution

$$y(t) = \int_a^b G(t, s)\, h_0(s)\, \mathrm{d}s$$

of the problem (3.8), (3.10) satisfies

$$|y(t)| + |y'(t)| \leqq \gamma \int_a^b |h_0(s)|\, \mathrm{d}s \leqq \gamma(1 + r) \int_a^b (|h_1(s)| + \omega(s, r + |y| + |y'|))\, \mathrm{d}s.$$

Let $\varrho_0 = \max\{|y(t)| + |y'(t)| : a \leqq t \leqq b\}$. Then

$$\varrho_0 \leqq \gamma(r + 1) \int_a^b (|h_1(s)| + \omega(s, r + \varrho_0))\, \mathrm{d}s. \tag{3.11}$$

In view of (1.8) there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequality

$$\gamma(1 + r) \int_a^b (|h_1(t)| + \omega(t, r + \varrho))\, \mathrm{d}t < \varrho \tag{3.12}$$

is satisfied. From (3.11), (3.12) it is clear that $\varrho_0 \leqq \varrho^*$. Putting $r^* = \varrho^* + r$, we get the estimate (3.4).

Lemma 8. *Let* $r \in (0, +\infty)$, $a_1, a_2 \in (0, +\infty)$ *satisfy* (1.11) *and* $\omega : [a, b] \times + R_+ \to R_+$ *satisfy* (1.12).

Then there exists $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions* (3.1), (3.2) *and* (3.7) *imply the estimate* (3.4).

Proof. In the same way as in the proof of Lemma 6 we can find zeros of v' and the point t_0 such that $|v(t_0)| \leqq r$. By Lemma 1 we obtain

$$(3.13)\qquad \left(\int_a^b v'^2(t)\,dt\right)^{1/2} \leqq 2\tau/\pi\left(\int_a^b v''^2(t)\,dt\right)^{1/2}$$

and

$$(3.14)\qquad \left(\int_a^b (v(t) - v(t_0))^2\,dt\right)^{1/2} \leqq \tau(b - a)(2/\pi)^2\left(\int_a^b v''^2(t)\,dt\right)^{1/2}.$$

Let us put $\varrho_0 = \left(\int_a^b v''^2(t)\,dt\right)^{1/2}$. Then, by the Hölder inequality, we get

$$(3.15)\qquad |v'(t)| = \left|\int_{t_1}^t v''(s)\,ds\right| \leqq \varrho_0(b - a)^{1/2}$$

and

$$(3.16)\qquad |v(t)| \leqq \left|\int_{t_0}^t v'(s)\,ds\right| + r \leqq \varrho_0(b - a)^{3/2} + r.$$

From (3.7) it follows, by virtue of (3.13), 3.14), (3.15) and (3.16)

$$\varrho_0 \leqq (a_1\tau(b - a)(2/\pi)^2 + a_2\,2\tau/\pi)\,\varrho_0 + a_1 r\sqrt{b - a} + \\ + \left(\int_a^b \omega^2(t, r + \varrho_0(b - a + 1)^2)\,dt\right)^{1/2}.$$

In view of (1.11) and (1.12), there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequality

$$(a_1\tau(b - a)(2/\pi)^2 + a_2\,2\tau/\pi)\,\varrho + a_1 r\sqrt{b - a} + \\ + \left(\int_a^b \omega^2(t, r + \varrho(b - a + 1)^2\,dt\right)^{1/2} < \varrho$$

is valid and consequently $\varrho_0 \leqq \varrho^*$. Putting

$$r^* = r + \varrho^*((b - a)^{1/2} + (b - a)^{3/2}),$$

in accordance to (3.15), (3.16), we get the estimate (3.4).

IV

Proofs of Theorems

Proof of Theorem 1. Let $\varepsilon_0 \in (0. +\infty)$ satisfy

$$(4.1)\qquad \varepsilon_0(b - a)^2 + \limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b - a), \varrho)\,dt < 1$$

and r^* be the constant constructed by means of Lemma 6 for the function

$\tilde{\omega}(t, |x|, |y|) = \omega(t, |x| + r_0, |y| + r_1) + \varepsilon_0 |x| + 2|\alpha|$ and for the constant $\tilde{r} = r + r_0$. Put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq r^*, \\ 2 - s/r^* & \text{for } r^* < s < 2r^*, \\ 0 & \text{for } s \geqq 2r^*, \end{cases}$$

$$g(t, x, y) = f(t, x + g_0(t), y + g_0'(t)) - 2\alpha,$$
$$\tilde{g}(t, x, y) = \chi(r^*, |x| + |y|)\, g(t, x, y)$$

and consider the equation

$$v'' = \lambda\varepsilon v + \tilde{g}(t, v, v'), \qquad \varepsilon \in (0, \varepsilon_0]. \tag{4.2}$$

Since ε and $\tilde{g}$ satisfy the assumptions of Lemma 4, the problem (4.2), (2.3) has a solution v. Clearly v satisfies (3.1). Let $v(t) > \tilde{r}$ for some $t \in [a, b]$. Then $v(t) + g_0(t) > r$ and

$$\lambda v''(t) = \lambda\chi(r^*, |v| + |v'|)(f(t, v + g_0(t), v' + g_0'(t)) - 2\alpha) + \varepsilon v(t) > 0.$$

Analogously, if $v(t) < -\tilde{r}$, then $v(t) + g_0(t) < -r$ and $\lambda v''t) < 0$. Consequently v satisfies (3.2) with the constant $\tilde{r}$. Further

$$|v''(t)| \leqq |f(t, v + g_0(t), v' + g_0'(t)) - 2\alpha| + \varepsilon |v(t)| \leqq$$
$$\leqq \omega(t, |v| + r_0, |v'| + r_1) + 2|\alpha| + \varepsilon_0 |v(t)| = \tilde{\omega}(t, ||v|, |v'|).$$

According to (4.1) there exists $\delta > 0$ such that

$$\varepsilon_0(b - a)^2 + (1 + \delta) \limsup_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho(b - a), \varrho)\, dt < 1. \tag{4.3}$$

It follows from (4.3) that there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequalities

$$r_0 + \varrho(b - a) \leqq (1 + \delta)\, \varrho(b - a), \qquad r_1 + \varrho \leqq (1 + \delta)\, \varrho,$$

$$\varepsilon_0(b - a)^2 + \frac{1}{\varrho} \int_a^b (\omega(t, (1 + \delta)\, \varrho(b - a), (1 + \delta)\, \varrho) + 2|\alpha|)\, dt < 1.$$

The latter inequality implies that $\tilde{\omega}$ satisfies (1.5). Hence, by Lemma 6, the estimate (3.4) is valid and v is a solution of the equation $v'' = \lambda\varepsilon v + g(t, v, v')$. Thus $u = v + g_0$ is a solution of the equation

$$u'' = \lambda\varepsilon(u - g_0(t)) + f(t, u, u') \tag{4.4}$$

and satisfies the conditions (1.2). Therefore for any $\varepsilon \in (0, \varepsilon_0]$ there exists a solution u_ε of the problem (4.4), (1.2) satisfying the estimate $|u_\varepsilon| + |u_\varepsilon'| \leqq r^* + r_0 + r_1$ for $a \leqq t \leqq b$. From this it follows that all functions of the set $\{u_\varepsilon : \varepsilon \in (0, \varepsilon_0]\}$ are uniformly bounded with their derivatives and so also equi-continuous on

$[a, b]$. Therefore, by the Arzelà–Ascoli lemma, there exists a sequence $(\varepsilon_k)_{k=1}^{\infty}$, $\varepsilon_k \to 0$ for $k \to \infty$, and a sequence $(u_{\varepsilon_k})_{k=1}^{\infty}$ uniformly converging together with $(u'_{\varepsilon_k})_{k=1}^{\infty}$ on $[a, b]$ such that $u_0(t) = \lim_{k\to\infty} u_{\varepsilon_k}(t)$ is a solution of the problem (1.1), (1.2).

Proof of Theorem 2. Let $\varepsilon_0 \in (0, +\infty)$ satisfy the inequality $\varepsilon_0(2/\pi)^2 (b - a)^2 + a_1(2/\pi)^2 (b - a)^2 + a_2(2/\pi)(b - a) < 1$ and r^* be the constant constructed by means of Lemma 7 for the function $\tilde{\omega}(t, |x| + |y|) = \omega(t, |x| + |y| + r_0 + r_1) + a_1 r_0 + a_2 r_1 + 2|\alpha|$ and for the constants $a_1 + \varepsilon_0, a_2, \tilde{r} = r + r_0$. Then, using Lemma 7, we can prove Theorem 2 in a similar way as Theorem 1.

Proof of Theorem 3. Theorem 3 can be proved in the same way as Theorem 2, only by means of Lemma 8.

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