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NECESSARY AND SUFFICIENT CONDITIONS FOR FINALLY VANISHING OSCILLATORY SOLUTIONS IN SECOND ORDER DELAY EQUATIONS

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Abstract. Necessary and sufficient conditions have been found to ensure that all oscillatory solutios of the equation

(1) $$(r(t)\,y'(t))' + a(t)\,y(g(t)) = f(t), \qquad g(t) \leqq t$$

approach zero. By way of several theorems it is shown that this behavior of equation (1) is associated with the presence of nonoscillatory solutions with certain properties.

Key words. Oscillatory, nonoscillatory, asymptotic, delay, forced.

MS Classification. 34 K 25.

1. INTRODUCTION

In [9], this author found conditions on $a(t)$, $r(t)$, $f(t)$ and $g(t)$ to ensure that all nontrivial oscillatory solutions of the equation

(1) $$(r(t)\,y'(t))' + a(t)\,y(g(t)) = f(t)$$

approach zero asymptotically. It was shown that an oscillatory solution $y(t)$ of (1) satisfies $\lim_{t\to\infty} y(t) = 0$ subject to:

$$\int^{\infty} 1/r(t)\,dt < \infty, \qquad \int^{\infty} |\,a(t)\,|\,dt < \infty \qquad \text{and} \qquad \int^{\infty} |\,f(t)\,|\,dt < \infty.$$

In section 3 of this work, we would present necessary and sufficient conditions to achieve asymptotic approach to zero of all oscillatory solutions of (1). This behavior of the oscillatory solutions of (1) is closely associated with (1) having a nonoscillatory solution with certain properties. The connection between oscillation and nonoscillation becomes very interesting under rather restrictive constraint $a(t) > 0$, in which case the ratio $|\,f(t)\,|/a(t)$ (Wallgren [14]) plays a significant role. This connection is examined in several theorems in this section without the restriction that $a(t) > 0$.

The proof of [9, Theorem 2] is lengthy and requires a stringent condition that the retardation $g(t)$ be slight by requiring $t - g(t) \leq B$, $B > 0$ a constant. We remedy this situation by giving an alternative proof based on $t > g(t)$ and $g(t) \to \infty$ entirely.

It turns out that we can deduce restrictions on the growth of oscillatory solutions from growth condition on $r(t)$. We examine this in section 4 and come up with alternative theorem to ensure asymptotic decay to zero of the oscillatory trajectories of (1).

Even though a voluminous literature exists about many oscillatory and non-oscillatory criteria for homogeneous and nonhomogeneous equations such as (1), the asymptotic nature of nonoscillatory or oscillatory solutions of these equations has not been so extensively studied, and for that matter the literature is very scanty with regard to oscillatory solutions. For asymptoticity on nonoscillation, the reader will find a good account in the works of Hammett [5], Londen [6] and this author [8, 10, 11, 12]. An excellent reference list is included by Graef [3] and Graef and Spikes [4] for any interested reader.

Throughout this study, all theorems proven are supported by examples to show that they are not vacuous. Although the results found apply well to ordinary differential equations, the presence of retarded term makes application of common techniques which work for ordinary differential equations a nontrivial matter. Travis [13] shows how a theorem of Bhatia [1] fails in such passage to retarded equations (cf, [9]). In what follows all results are easily extendable to the nonlinear equation

$$(r(t)\, y'(t))' + a(t)\, h(y(g(t))) = f(t). \tag{2}$$

2. DEFINITION AND ASSUMPTIONS

It will be assumed for the rest of this paper that

(i) $r(t)$, $a(t)$, $g(t)$, $f(t) : R \to R$ and continuous; R is real line,

(ii) $r(t) > 0$, $r'(t) \geqq 0$, $g(t) > 0$ on some positive half real line R^+.

(iii) $g(t) \to \infty$ as $t \to \infty$, $g(t) \leq t$ and $g'(t) > 0$ for $t \geqq t_0$ where $t_0 > 0$.

We call a function $Q(t) \in C[t_0, \infty)$ *oscillatory* if it has arbitrarily large zeros in $[t_0, \infty)$. Otherwise $Q(t)$ is called *nonoscillatory*. In this work, the term "solution" applies to those solutions (of equations under consideration) which can be extended to the right of some positive point on R, say t_0.

3. MAIN RESULTS

Theorem (1). *Suppose*

$$\int^{\infty} |a(t)|\, dt < \infty, \tag{3}$$

$$\int^{\infty} |f(t)|\,dt < \infty \tag{4}$$

and

$$\int^{\infty} 1/r(t)\,dt < \infty, \tag{5}$$

than all oscillatory solutions of (1) *approach zero as* $t \to \infty$.

Proof. Let $y(t)$ be an oscillatory solution of equation 1. Let $T > t_0$ be large enough so that

$$\int_T^{\infty} |a(t)|\,dt < 1/4 \tag{6}$$

and

$$\int_T^{\infty} |f(t)|\,dt < 1/4. \tag{7}$$

Suppose to the contrary that $\limsup_{t\to\infty} |y(t)| > d > 0$. Let $T_1 > T$ be so large that $g(T_1) \geqq T$ and $y(T_1) = 0$. There is $T' > T_2 > T_1$ such that $y(T_2) = 0$ and

$$\operatorname{Max}\{|y(t) : T \le t \le T'\} = |y(T')| > d > 0. \tag{8}$$

Let $[x_1, x_2]$ designate the smallest closed interval containing T' such that $y(x_1) = y(x_2) = 0$. Designate $M = \operatorname{Max}\{|\dot{y}(t)| : x_1 \le t \le x_2\}$. Note that $T_2 \le x_1$. It is clear that $M \geqq d$ and

$$|y(t)| \le M \tag{9}$$

and

$$|y(g(t))| \le M \tag{10}$$

for $t \in [x_1, x_2]$. Also let $T_M \in [x_1, x_2]$ be such that $M = |y(T_M)|$. Now

$$M = \int_{x_1}^{T_M} y'(t)\,dt,$$

which gives

$$M \leqq \int_{x_1}^{T_M} |y'(t)|\,dt. \tag{11}$$

also

$$-M = \int_{T_M}^{x_2} y'(t)\,dt,$$

which gives

$$M \leqq \int_{T_M}^{x_2} |y'(t)|\,dt. \tag{12}$$

From (11) and (12) we get

$$2M \leqq \int_{x_1}^{x_2} |y'(t)|\,dt = \int_{x_1}^{x_2} |y'|^{1/2} (r(t)\,|y'(t)|)^{1/2} (r(t))^{-1/2}\,dt.$$

By Schwarz's inequality

(13) $$4M^2 \leq \left(\int_{x_1}^{x_2} 1/r(t)\,\mathrm{d}t\right)\left(\int_{x_1}^{x_2} (r(t)\,y'(t))\,y'(t)\,\mathrm{d}t\right).$$

Integration by parts yield

(14) $$4M^2 \leq \left(\int_{x_1}^{x_2} 1/r(t)\,\mathrm{d}t\right)\left(-\int_{x_1}^{x_2} (r(t)\,y'(t))'y(t)\,\mathrm{d}t\right).$$

From (14) by using equation 1 we have

(15) $$4M^2 \leq \int_{x_1}^{x_2} 1/r(t)\,\mathrm{d}t\left(\int_{x_1}^{x_2} a(t)\,y(g(t))\,y(t)\,\mathrm{d}t - \int_{x_1}^{x_2} y(t)\,f(t)\,\mathrm{d}t\right).$$

From (9), (10) and (15)

$$4 \leq \left(\int_{x_1}^{x_2} 1/r(t)\,\mathrm{d}t\right)\left(\int_{x_1}^{x_2} |\,a(t)\,|\,\mathrm{d}t + \frac{1}{M}\int_{x_1}^{x_2} |\,f(t)\,|\,\mathrm{d}t\right),$$

i.e.

(16) $$\frac{4}{\int_{x_1}^{x_2} 1/r(t)\,\mathrm{d}t} \leq \frac{1}{4} + \frac{1}{4d}.$$

Unless $d = 0$, (16) yields a contradiction since $\int_{x_1}^{x_2} 1/r(t)\,\mathrm{d}t$ can be made arbitrarily small by choice of large T. The proof is complete.

Remark (1). The above theorem improves our theorem 2 in [9] by eliminating the requirement that $g(t) = t - \tau(t)$ with $\tau(t)$ bounded. If conditions (3) and (4) hold then condition (5) is necessary as the following example shows.

Example (1). The equation

(17) $$y''(t) + \frac{1}{t^2}\,y(t) = -\frac{\cos(\log t)}{t^2}, \qquad t > 0,$$

has $y(t) = \sin(\log t)$ as a solution.

The decomposition of $a(t)$ as $a(t) = a_1(t) + a_2(t)$ can be effectively used by assuming conditions on $a_1(t)$ and $a_2(t)$. Our next theorem uses such a decomposition toward obtaining necessary and sufficient condition for all oscillatory solutions of (1) to approach zero asymptotically.

Theorem. (2). *Suppose $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $|\,a_2/a_1\,| \leq k_1$ for some $k_1 > 0$ and large t. Further suppose that $|\,f(t)\,|/a_1(t)$ approaches a limit as $t \to \infty$.* Let

$$\int^{\infty} 1/r(t)\,\mathrm{d}t < \infty, \quad \text{and} \quad \int^{\infty} a_1(t)\,\mathrm{d}t < \infty.$$

Then

$$\lim_{t \to \infty} (| f(t) | / a_1(t)) = 0$$

is a necessary and sufficient further condition for all oscillatory solutions of (1) *to approach zero as* $t \to \infty$.

Proof. The sufficiency is obvious. To prove necessity we rewrite (1) as

$$\frac{(r(t)\,y'(t))'}{a_1(t)} + y(g(t)) + \frac{a_2(t)}{a_1(t)}\,y(g(t)) = \frac{f(t)}{a_1(t)}. \tag{18}$$

This yields

$$\frac{|(r(t)\,y'(t))'|}{a_1(t)} \geqq \frac{|f(t)|}{a_1(t)} - (1 + k_1)\,|y(g(t))|, \tag{19}$$

where $y(t)$ is an oscillatory solution such that $y(t) \to 0$ as $t \to \infty$. Now if

$$\liminf_{t \to \infty} \frac{|f(t)|}{a_1(t)} > 0,$$

then $\dfrac{(r(t)\,y'(t))'}{a_1(t)}$ is bounded away from zero. Thus $(r(t)\,y'(t))'$ assumes a constant sign making $y(t)$ nonoscillatory. This contradiction completes the proof of the theorem.

Example (2). The equation

$$(t^2 y'(t))' + \frac{1 - 2\sin(\log t)}{t^2}\,y(t) = \frac{6}{t^3} + \frac{10}{t^3}(\sin(\log t) - \cos(\log t)) +$$

$$+ \frac{1 - 4\sin^2(\log t)}{t^5}, \qquad t > 0, \tag{20}$$

has $y = \dfrac{1 + 2\sin(\log t)}{t^3}$ as an eventually vanishing solution. Here all conditions of Theorem 2 are easily verified. Hence all oscillatory solutions of (20) approach zero as $t \to \infty$.

Corollary (1). *Suppose* $a(t) > 0$, $\int^{\infty} 1/r \, \mathrm{d}t < \infty$, *and* $\int^{\infty} a(t) \, \mathrm{d}t < \infty$. *Further suppose* $\lim_{t \to \infty} \dfrac{|f(t)|}{a(t)}$ *exists. Then a necessary and sufficient condition for* (1) *to have all oscillatory solutions approaching zero is* $\lim_{t \to \infty} \dfrac{|f(t)|}{a(t)} = 0$.

Proof. Follows from Theorem 2.

Sufficiency part of the proof of Theorem 2 leads us to the following theorem.

Theorem (3). *Suppose* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $a_2(t)/a_1(t)$ *bounded for large* t, $\int^{\infty} a_1(t) \, \mathrm{d}t < \infty$, *and* $\int^{\infty} 1/r \, \mathrm{d}t < \infty$. *Further suppose that* $f(t)/a_1(t)$ *is bounded. Then all oscillatory solutions of* (1) *approach zero as* $t \to \infty$.

Proof. It is clear that $\int^{\infty} | a(t) | \, dt < \infty$. Since $\int^{\infty} a_1(t) \, dt < \infty$, and $f(t)/a_1(t)$ is bounded as $t \to \infty$, we have $\int^{\infty} | f(t) | \, dt < \infty$. The conclusion now follows by Theorem 1.

Example (3). Consider the equation

$$(21) \qquad (t^2 y'(t))' + \frac{1 - 2\sin(\log(t))}{t^3} y(t) = \frac{6}{t^3} +$$
$$+ \frac{10}{t^3}(\sin(\log t) - \cos(\log t)) + \frac{1 - 4\sin^2(\log t)}{t^6}, \qquad t > 0,$$

which has $y(t) = \dfrac{1 + 2\sin(\log t)}{t^3}$ as a vanishing oscillatory solution. In fact, since all conditions of Theorem 3 are satisfied all oscillatory solutions of (21) tend to 0 as $t \to \infty$.

Remark (2). Theorem 3 and Example 3 show that the existence of the limit

$$\lim_{t \to \infty} \frac{| f(t) |}{a_1(t)}$$

is essential in Theorem 2. In fact, if all oscillatory solutions of (1) approach 0, then (19) in the proof of Theorem 2 shows that $\liminf (| f(t) | / a_1(t)) = 0$. In example 3 we see that $\dfrac{f(t)}{a_1(t)}$ is bounded, $\liminf\limits_{t\to\infty} \dfrac{| f(t) |}{a_1(t)} = 0$ but $\lim\limits_{t\to\infty} \dfrac{| f(t) |}{a_1(t)}$ does not exist.

Our next theorem gives sufficient conditions when oscillatory solutions do not have limits.

Theorem (4). *Suppose* $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$ *and* $a_2(t)/a_1(t)$ *is bounded for large t. Further suppose that* $\liminf\limits_{t\to\infty} | f(t) | / a_1(t) > 0$. *Let* $y(t)$ *be an oscillatory solution of* (1). *Then* $\limsup\limits_{t\to\infty} | y(t) | > 0$.

Proof. Suppose to the contrary that $y(t) \to 0$ as $t \to \infty$. From (1), we get inequality (19)

$$\frac{| (r(t) \, y'(t))' |}{a_1(t)} \geqq \frac{f(t)}{a_1(t)} - (1 + k_1) | y(g(t)) | \qquad \text{where } \frac{| a_2(t) |}{a_1(t)} \leqq k_1,$$

$k_1 > 0$. A contradiction is immediately reached, since $(r(t) \, y'(t))'$ assumes a constant sign. The proof is complete. The following example satisfies the conditions and conclusion of Theorem 4.

Example (4). Consider the equation

$$(22) \qquad y''(t) + y(t - 2\pi) = 1.$$

All oscillatory solutions of (22) satisfy $\limsup\limits_{t\to\infty} | y(t) | > 0$ since all conditions of theorem 4 are satisfied. In fact $y(t) = 1 + \cos t$ is one such solution.

Remark (3). Note that Theorem 4 does not require $\int^{\infty} 1/r(t) < \infty$.

Theorem (5). *Suppose $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $a_2(t)/a_1(t)$ bounded for large t, $\int^{\infty} a_1(t)\,dt < \infty$, $\int^{\infty} 1/r\,dt < \infty$, and $f(t)/a_1(t)$ is bounded for large t. Then all solutions of* (1) *are nonoscillatory if* $\liminf\limits_{t\to\infty} \dfrac{|f(t)|}{a_1(t)} > 0$.

Proof. It is easily seen that $\int^{\infty} |f(t)|\,dt < \infty$ and $\int^{\infty} |a(t)|\,dt < \infty$. Since $\int^{\infty} 1/r(t)\,dt < \infty$, by Theorem 1, all oscillatory solutions approach zero. Let now $y(t)$ be a solution of (1), If $y(t)$ is oscillatory then $y(t) \to 0$ as $t \to \infty$. From (1), we obtain (in a manner of inequality (19))

$$\frac{|(r(t)\,y'(t))'|}{a_1(t)} \geqq \frac{|f(t)|}{a_1(t)} - (1 + k_1)\,|y(g(t))|,$$

which clearly gives a contradiction by making $y(t)$ nonoscillatory.

Example (5). The equation

$$\left(\frac{1}{2}t^2 y'(t)\right)' + \frac{1 + \sin t}{t^2}\,y(t) = \frac{1}{t^2} + \frac{1 + \sin t}{t^4}, \tag{23}$$

has $y(t) = \dfrac{1}{t^2}$ a nonoscillatory solution. In fact, taking $a_1(t) = 1/t^2$, $a_2(t) = \sin t/t^2$, $r(t) = 1/2t^2$ and $f(t) = (t^2 + \sin t + 1)/t^4$ we find that all conditions of Theorem 5 hold. Hence all solutions of (23) are nonoscillatory.

Our next theorem generalizes Theorem 2.6 of Wallgren [14].

Theorem (6). *Suppose $r(t)$ is bounded, $a(t) = a_1(t) + a_2(t)$, $a_1(t) > \varrho > 0$, $|a_2(t)|/a_1(t) \leqq k_1$, for large t and $\lim\limits_{t\to\infty} |f(t)|/a_1(t) = \infty$. Then all solutions of* (1) *are unbounded.*

Proof. From equation 1

$$\frac{(r(t)\,y'(t))'}{a_1(t)} + \left(1 + \frac{a_2}{a_1}\right) y(g(t)) = f(t)/a_1(t).$$

Thus

$$\frac{|(r(t)\,y'(t))'|}{a_1(t)} \geqq \frac{|f(t)|}{a_1(t)} - (1 + k_1)\,|y(g(t))|.$$

If $y(t)$ is bounded, then above inequality shows that $\dfrac{|(r(t)\,y'(t))'|}{a_1(t)} \to \infty$ as $t \to \infty$. Since $a_1(t) \geqq \varrho > 0$ we get $|(r(t)\,y'(t))'| \to \infty$ as $t \to \infty$. Since $r(t)$ is bounded $y'(t) \to \pm\infty$ as $t \to \infty$. The conclusion follows by contradiction.

Example (6). The equation

$$y''(t) + 2y(t - \pi/2)\,e^{\pi/2} = e^{\pi/2}\,2(t - \pi/2), \tag{24}$$

has $y = e^t \sin t + t$ and $y = t$ as solutions. All conditions of Theorem 6 are satisfied.

Theorem (7). *Suppose* $\int^{\infty} |a(t)|\,\mathrm{d}t < \infty$ and $\int^{\infty} f(t)\,\mathrm{d}t = \infty$ *then all oscillatory solutions of* (1) *are unbounded.*

Proof. From (1) we get

$$r(t)\,y'(t) - r(T)\,y'(T) + \int_T^t a(s)\,y(g(s))\,\mathrm{d}s = \int_T^t f(s)\,\mathrm{d}s. \tag{25}$$

If $y(t)$ is oscillatory and bounded then (25) yields

$$r(t)\,y'(t) - r(T)\,y'(T) + m\int_T^t |a(s)|\,\mathrm{d}s \geqq \int_T^t f(s)\,\mathrm{d}s, \tag{26}$$

where $|y(t)| \geqq m$ for $t \leqq T$. (26) readily leads to a contradiction which proves this theorem.

Example (7). The following equation shows that under the conditions of Theorem 7, bounded nonoscillatory solutions can exist.

$$(t^{5/2}y'(t))' + \frac{1}{t^2}\,y(t) = \frac{1}{2\sqrt{t}} - \frac{1}{t^3}, \tag{27}$$

has $y(t) = -1/t$ as a solution.

Theorem (8). *If under the hypothesis of theorem 7 we require* $r(t)$ *to be bounded, all other conditions being the same then all solutions of* (1) *are unbounded.*

Proof. We only need to prove it when $y(t)$ is nonoscillatory. Following the proof of Theorem 7, if $|y(t)| \leq m$ then (26) yields $r(t)\,y'(t) \to \infty$ as $t \to \infty$ and since $r(t)$ is bounded, we have $y'(t) \to \infty$ as $t \to \infty$ which forces $y(t)$ to be unbounded. The proof is now complete by contradiction.

4. EFFECT OF LARGE $r(t)$ AND NONOSCILLATION

Example (8). The equation

$$(e^t y'(t))' + e^{-2\pi}\,y(t - 2\pi) = e^{-t}\sin t - 3e^{-t}\cos t + e^{-2t}\sin t, \tag{29}$$

has $y = e^{-2t}\sin t$ as an oscillatory solution approaching zero. But this equation is not covered by Theorem 1 since $\int^{\infty} |a(t)|\,\mathrm{d}t = \infty$. However, it will be shown by our next theorem, that all oscillatory solutions of (29) approach zero as $t \to \infty$. In fact, Theorem 9 measures the growth of solutions of (1) in terms of $r(t)$. By taking $r(t)$ large enough, the sizes of $a(t)$ and $f(t)$ can be compensated for. As an outcome of this approach, we observe that oscillatory trajectories of (1) eventually vanish if (1) has a nonoscillatory solution satisfying certain properties.

Theorem (9). *Suppose* $\int^{\infty} 1/r(t)\,\mathrm{d}t < \infty$,

$$\int^{\infty} |a(x)| \left(\int_x^{\infty} 1/r(s)\,\mathrm{d}s \right) \mathrm{d}x < \infty \tag{30}$$

and

$$\int^{\infty} |f(x)| \left(\int_x^{\infty} 1/r(s)\,\mathrm{d}s \right) \mathrm{d}x < \infty; \tag{31}$$

then all oscillatory solutions of (1) tend to zero asymptotically.

Proof. We proceed as in Theorem 1 with $y(t)$ as an oscillatory nonvanishing solution of (1), and arrive at conclusions (9) and (10) namely $|y(t)| \leq M$ and $|y(g(t))| \leq M$ for $t \in [x_1, x_2]$, $y(x_1) = y(x_2) = 0$. Let $x_0 \in [x_1, x_2]$ such that $M = |y(x_0)|$. Integrating (1) for $t \in [x_0, x_2]$, we have

$$r(t)\,y'(t) + \int_{x_0}^{t} a(x)\,y(g(x))\,\mathrm{d}x = \int_{x_0}^{t} f(x)\,\mathrm{d}x, \tag{32}$$

since $y'(x_0) = 0$. Dividing (32) by $r(t)$ and integrating between $[x_0, x_2]$ we have

$$\pm M = -\int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} a(x)\,y(g(x))\,\mathrm{d}x\,\mathrm{d}t + \int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} f(x)\,\mathrm{d}x\,\mathrm{d}t,$$

which gives

$$M \leqq \int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} |a(x)|\;|y(g(x))|\,\mathrm{d}x\,\mathrm{d}t + \int_{x_0}^{x_2} 1/r(t) \int_{x_0}^{t} |f(x)|\,\mathrm{d}x\,\mathrm{d}t.$$

Since $|y(g(t))| \leq M$ for $t \in [x_0, x_2] \subset [x_1, x_2]$ we have

$$1 \leqq \int_{x_0}^{x_2} \left(\int_x^{x_2} 1/r(s)\,\mathrm{d}s \right) |a(x)|\,\mathrm{d}x + \frac{1}{M} \int_{x_0}^{x_2} |f(x)| \left(\int_x^{x_2} 1/r(s)\,\mathrm{d}s \right) \mathrm{d}x, \tag{33}$$

where, in (33), the integrals have been rearranged by change of order of integration. Unless M becomes arbitrarily small, (33) leads to a contradiction.

Our next theorem highlights nonoscillation in obtaining some results about oscillatory solutions. We will need the equation

$$(r(t)\,y'(t))' + a(t)\,y(g(t)) = 0. \tag{34}$$

Theorem (10). *Suppose equation* (34) *has a nonoscillatory solution* $y(t)$ *such that* $\operatorname{sgn}(y(t)) = \operatorname{sgn}(y'(t))$. *Further suppose that* $a(t) > 0$, $\int^{\infty} 1/r(t)\,\mathrm{d}t < \infty$ *and* $\int^{\infty} |f(x)| \left(\int_x^{\infty} 1/r(t)\,\mathrm{d}t \right) \mathrm{d}x < \infty$. *Then all oscillatory solutions of* (1) *approach zero as* $t \to \infty$.

Proof. Let T be large enough so that for $t \geqq T$, $y(t)$ and $y(g(t))$ are of the same sign. Without any loss of generality, we can assume that for $t \geqq T$ we have

$$y(t) > 0,\; y(g(t)) > 0,\; y'(t) > 0 \quad \text{and} \quad y'(g(t)) > 0. \tag{35}$$

Dividing by $y(g(t))$ and integrating between T and t we have

$$(36)\qquad \frac{r(t)\,y'(t)}{y(g(t))} - \frac{r(T)\,y'(T)}{y(g(T))} + \int_T^t \frac{r(x)\,y'(x)\,y'(g(x))\,g'(x)\,\mathrm{d}x}{y^2(g(x))} + \\ + \int_T^t a(x)\,\mathrm{d}x = 0.$$

(36) yields on further manipulation

$$\int_T^t \frac{y'(s)}{y(g(s))}\,\mathrm{d}s - \frac{r(T)\,y'(T)}{y(g(T))}\int_T^t 1/r(s)\,\mathrm{d}s + \int_T^t \frac{1}{r(t)}\int_T^s \frac{r(x)\,y'(x)\,y'(g(x)\,g'(x)\,\mathrm{d}x\,\mathrm{d}s}{y^2(g(x))} =$$

$$(37)\qquad = -\int_T^t 1/r(s)\int_T^s a(x)\,\mathrm{d}x\,\mathrm{d}s.$$

From (35), the first and third term on the left are positive; the second term is finite. Since the first term on the right hand side is negative, we arrive at the conclusion

$$\lim_{t\to\infty}\int_T^t 1/r(s)\int_T^s a(x)\,\mathrm{d}x\,\mathrm{d}s = \lim_{t\to\infty}\int_T^t \Big(\int_x^t 1/r(s)\,\mathrm{d}s\Big)\,a(x)\,\mathrm{d}x < \infty.$$

The proof is now complete by the application of Theorem 9.

The following example gives an application of this theorem.

Example (9). The equation

$$(38)\qquad (e^{t/2}y'(t))' + \frac{e^{t/2}}{2(e^t-1)}\,y(t) = 0,$$

has $y(t) = 1 - e^{-t}$ as a nonoscillatory solution. Hence all oscillatory solutions of

$$(39)\quad (e^{t/2}y'(t))' + \frac{e^{t/2}}{2(e^t-1)}\,y(t) = 4e^{-2t}\sin t - \frac{9}{2}e^{-2t}\cos t + \frac{e^{-2t}}{2(e^t-1)}\sin t$$

approach zero. In fact $y = (e^{-5/2\,t}\sin t)$ is one such solution of (39). It is easily verified that all conditions of Theorem 10 are satisfied. We also note that all conditions of Theorem 9 are satisfied. Indeed, Theorem 10 is a recapitulation of Theorem 9 in terms of the nonoscillatory solutions of the homogeneous part of (1).

Example 9 suggests the following theorem.

Theorem (11). *Suppose* (1) *has a nonoscillatory solution* $y(t)$ *such that* $\operatorname{sgn}(y(t)) = \operatorname{sgn}(y'(t))$. *Further suppose that* $a(t) > 0$, $\int^\infty 1/r(t)\,\mathrm{d}t < \infty$ *and* $\int^\infty f(x)\,|\big(\int^\infty 1/r(t)\,\mathrm{d}t\big)\mathrm{d}x < 0$. *Then all oscillatory solutions of* (1) *approach zero as* $t \to \infty$.

Proof. We proceed as in Theorem 10 and arrive at conclusion (35). Dividing (1) by $y(g(t))$ and integrating we get

$$\frac{r(t)\,y'(t)}{y(g(t))} - \frac{r(T)\,y'(T)}{y(g(T))} + \int_T^t \frac{r(x)\,y'(x)\,y'(g(x)\,g'(x)\,\mathrm{d}x}{y^2(g(x))} +$$
$$+ \int_T^t a(x)\,\mathrm{d}x = \int_T^t \frac{f(x)}{y(g(x))}\,\mathrm{d}x.$$

Since $y(g(x)) > y(g(T)) > 0$, further integration yields

$$\int_T^t \frac{y'(s)}{y(g(s)}\,\mathrm{d}s - \frac{r(T)\,y'(T)}{y(g(T))}\int_T^t 1/r(s)\,\mathrm{d}s + \int_T^t 1/r(s)\int_T^s \frac{r(x)\,y'(x)\,y'(g(x))\,g'(x)\,\mathrm{d}x\,\mathrm{d}s}{y^2(g(x))} \leq$$

(40) $$\leq \frac{1}{y(g(T))}\int_T^t 1/r(s)\int_T^s |f(x)|\,\mathrm{d}x\,\mathrm{d}s.$$

In view of (35) and condition on $f(x)$, (39) yields

$$\lim_{t\to\infty}\int_T^t \frac{1}{r(t)}\int_T^s a(x)\,\mathrm{d}x\,\mathrm{d}s = \int_T^\infty a(x)\,[\int_x^\infty 1/r(s)\,\mathrm{d}s]\,\mathrm{d}x < \infty.$$

The conclusion follows by Theorem 9.

Example (10). In equation (39), the nonoscillatory solution $y(t) = 1 - e^{-t} + e^{-5/2\,t}\sin t$ for sufficiently large t satisfies the requirements of this theorem.

Theorem (12). *Suppose* $\int^\infty 1/r(t)\,\mathrm{d}t < \infty$ *and there exist nonnegative functions* $H_1(t)$, $H_2(t)$ *such that* $\operatorname{sgn}(H_i(t)) = \operatorname{sgn}(H_i'(t))$, $i = 1, 2$. *Further suppose that* H_1 *and* H_2 *satisfy*

(41) $$(r(t)\,H_1'(t))' + |a(t)|\,H_1(g(t)) \leq 0,$$

(42) $$(r(t)\,H_2'(t))' + |f(t)|\,H_2(g(t)) \leq 0.$$

Then all oscillatory solutions of (1) *approach zero as* $t \to \infty$.

Proof. Following identically the proof of Theorem 10 we obtain (cf. this author [7, Theorem 2])

$$\int^\infty |a(x)|\int_x^\infty 1/r(t)\,\mathrm{d}t\,dt < \infty$$

and

$$\int^\infty |f(x)|\int_x^\infty 1/r(t)\,\mathrm{d}t\,\mathrm{d}x < \infty,$$

which are the conditions of Theorem 9.

Our next theorem gives an alternative version of Theorem 4.

Theorem (13). *Suppose* $\liminf_{t\to\infty}\int^t (f(t) - |a(t)|)\,\mathrm{d}t > 0$. *Then any oscillatory solution* $y(t)$ *of* (1) *satisfies* $\limsup_{t\to\infty}|y(t)| > 0$.

Proof. Let $y(t)$ be an oscillatory solution. Then $y'(t)$ is oscillatory. Let T be large enough so that $y'(T) = 0$. From (1)

$$r(t)\, y'(t) + \int_T^t |\, a(x)\,|\ |\, y(g(x))\,|\, dx \geqq \int_T^t f(x)\, dx. \tag{43}$$

Suppose to the contrary that $y(t) \to 0$ as $t \to \infty$. Without any loss we can assume that T is large enough so that for $t \geqq T$, $|\, y(g(t))\,| \leq 1$. From (43) and this fact

$$(r(t)\, y'(t)) \geqq (\liminf_{t \to \infty} \int_T^t (f(x) - |\, a(x)\,|)\, dx) > 0. \tag{44}$$

But (44) implies that $y'(t)$ is eventually positive and $y(t)$ is nonoscillatory. This contradiction completes the proof.

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