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REPRESENTABILITY OF CONCRETE CATEGORIES BY NON-CONSTANT MORPHISMS

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Dedicated to the memory of Milan Sekanina

Abstract. We prove that the category of all compact Hausdorff spaces (or all metrizable spaces) admits a representation of every concrete category iff there does not exist a proper class of measurable cardinals.

Key words: almost universal category, compact Hausdorff space, metrizable space.

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In 1974, V. Koubek [4] proved that the category **Par** of paracompact Hausdorff spaces (and continuous maps) is *almost universal.* It means that any concrete category $\mathscr{K}$ has an embedding F (= one-to-one functor) into **Par** such that $g : FA \to FB$ is of the form $F(f)$ iff g is non-constant. Such embeddings F are called *almost full.* Due to constant maps, this is the strongest universality which topological spaces may offer. The second author proved that the categories **Metr** of metrizable spaces ([7]) and **Comp** of compact Hausdorff spaces ([8]) are almost universal (in both cases, morphisms are continuous maps) provided that the following statement is true

(M) It does not exist a proper class of measurable cardinals.

It remained open whether one really needs (M) for these results. We show that the answer is yes (for **Comp**, it solves Research Problem 12 in [6]).

Str(Δ) denotes the concrete category of structures of type Δ (= a set of possibly infinitary relation and operation symbols) and homomorphisms (maps preserving relations and operations). A full embedding of concrete categories is called a *realization* if it commutes with underlying set functors ([6]). A category $\mathscr{A}$ is called *universal* if any category can be fully embedded into $\mathscr{A}$. A basic (and deep) result is that the category **Graph** (= **Str**(Δ) where Δ consists of one binary relation) is universal iff (M) is fulfilled (see [6]). The mentioned results of [7] and [8] are proved by constructing almost full embeddings **Graph** – **Metr** and **Graph**^{op} – **Comp**.

Proposition 1. *The existence of an almost universal concrete category admitting a realization into* **Str**(Δ) *implies the universality of* **Graph**.

Proof: Let $(\mathcal{K}, U)$ be an almost universal concrete category and $F : \mathcal{K} \to \mathbf{Str}(\Delta)$ a realization. We will show that any concrete category $\mathcal{H}$ can be fully embedded into **Graph**.

Let $\mathcal{H}^+$ be the category obtained from $\mathcal{H}$ by adding an initial object I and a terminal object T; i.e. $obj\mathcal{H}^+ = obj\mathcal{H} \cup \{I, T\}$, $I \neq T$ and $obj\mathcal{H} \cap \{I, T\} = \emptyset$, $\mathcal{H}$ is a full subcategory of $\mathcal{H}^+$, $\mathcal{H}^+(I, H)$ and $\mathcal{H}^+(H, T)$ are one-element for any $H \in obj\mathcal{H}^+$, $\mathcal{H}^+(H, I) = \mathcal{H}^+(T, H) = \emptyset$ for any $H \in obj\mathcal{H}$. The underlying set functor of $\mathcal{H}$ can be easily extended to $\mathcal{H}^+$. Hence $\mathcal{H}^+$ is concrete and there is an almost full embedding $G : \mathcal{H}^+ \to \mathcal{K}$. Since F is a realization, the composition $E = F . G : \mathcal{H}^+ \to \mathbf{Str}(\Delta)$ is an almost full embedding. Therefore $E(m_T) : E(I) \to E(T)$ is non-constant (m_H is a unique morphism $I \to H$) and we can find $x, y \in E(I)$ such that their images in $E(m_T)$ are distinct. Then $x_H = E(m_H)(x)$, $y_H = E(m_H)(y)$ are distinct for any $H \in obj\,\mathcal{H}$ and $E(f)(x_H) = x_{\bar{H}}$, $E(f)(y_H) = y_{\bar{H}}$ for any $f : H \to \bar{H}$. Consequently, $g : E(H) \to E(\bar{H})$ is non-constant iff $g(x_H) = x_{\bar{H}}$ and $g(y_H) = y_{\bar{H}}$. Hence E gives a full embedding of $\mathcal{H}$ into $\mathbf{Str}(\Delta')$ where Δ' is obtained from Δ by adding two new constants interpreted as x_H and y_H. But $\mathbf{Str}(\Delta')$ has a full embedding into **Graph** (see [6]).

Theorem 1. **Metr** *is almost universal iff* (M) *holds.*

Proof: As already mentioned in the introduction, (M) implies the almost universality of **Metr**. For the converse, we realize **Metr** into structures with one ω-are relation; $(x_0, x_2, \ldots, x_n, \ldots)$ belongs to the relation iff the sequence $x_1, \ldots, x_n, \ldots$ converges to x_0. Proposition 1, **Graph** is universal. As stated in the introduction, it implies (M) (see [5]).

Remark 1: The same result is true for metrizable spaces with morphisms taken as

(a) uniformly continuous maps,

(b) non-expanding maps.

In case (a), we represent metrizable spaces by structures with an ω-ary relation again; but $(x_0, x_1, \ldots, x_n, \ldots)$ belongs to the relation iff $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$ where d is the distance. In case (b) we use ω binary relations R_n, $n > 0$ an integer; xR_ny iff $d(x, y) < 1/n$. The opposite implications are proved in [7].

Proposition 1 is not applicable to **Comp** because **Comp** cannot be fully embedded into **Str**(Δ) without (M) (see [5]).

Proposition 2. *Let there exist an almost universal concrete category* $\mathcal{K}$ *admitting a full embedding* $F : \mathcal{K}^{op} \to \mathbf{Str}(\Delta)$ *with the property:*

For every $K \in obj\mathcal{K}$ *there is a subset* Y_K *of (the underlying set of)* $F(K)$ *such that for any* $f : \bar{K} \to K$ *in* $\mathcal{K}$

(i) *$F(f)$ maps Y_K into $Y_{\bar{K}}$.*

(ii) *$F(f)$ maps the whole $F(K)$ into $Y_{\bar{K}}$ iff f is constant.*

Then **Graph** *is universal.*

Proof: We will follow the proof of Proposition 1. Let $\mathscr{L}$ be a concrete category. Since $\mathscr{H} = \mathscr{L}^{op}$ is concrete ([6], p. 33), we may take $\mathscr{H}^+$, an almost full embedding $G : \mathscr{H}^+ \to \mathscr{K}$ and the composition $E = F . G : (\mathscr{H}^+)^{op} \to \mathbf{Str}(\Delta)$. Then E is an embedding and $E(h)$ maps $Y_{G(H)}$ into $Y_{G(\bar{H})}$ for any $h : \bar{H} \to H$ in $\mathscr{H}^+$. Moreover $g : E(H) \to E(\bar{H})$ does not map the whole $E(H)$ into $Y_{G(\bar{H})}$ iff $g = E(h)$ for rome $h : \bar{H} \to H$.

Choose $x \in E(T)$ such that $y = E(m_T)(x) \notin Y_{G(I)}$. Then $x_H = E(n_H)(x)$ (n_H is a unique morphism $H \to T$) does not belong to $Y_{G(H)}$. We have $E(h)(x_H) = x_{\bar{H}}$ for any $h : \bar{H} \to H$. Hence E gives a full embedding of $\mathscr{L} = \mathscr{H}^{op}$ into $\mathbf{Str}(\Delta')$ where Δ' is obtained from Δ adding a new constant interpreted as x_H.

Theorem 2. Comp *is almost universal iff (M) holds.*

Proof: As already mentioned in the introduction, (M) implies the almost universality of **Comp**. Let $F : \mathbf{Comp}^{op} \to \mathbf{Ring}$ send a compact Hausdorff space X to its ring of continuous real-valued functions (**Ring** is the category of rings with unit and with unit preserving homomorphisms). It is well known that F is a full embedding (cf. e.g. [3], p. 152). Taking for Y_X the set of all constant real-valued functions on X, it is easy to check that (i) and (ii) of Proposition 2 are fulfilled. Hence the almost universality of **Comp** implies (M).

Remark 2. The almost universality of **Comp** implies the existence of a *stiff* proper class $\mathscr{S}$ of compact Hausdorff spaces (i.e. if $S, \bar{S} \in \mathscr{S}$ and $f : S \to \bar{S}$ is a morphism then either f is constant or $S = \bar{S}$ and f is the identity). One does not need the full force of (M) for it, the existence of a *rigid* proper class $\mathscr{R}$ of graphs is sufficient (cf. [8]), $\mathscr{R}$ is rigid if $X, Y \in \mathscr{R}$ and $f : X \to Y$ is a morphism then $X = Y$ and f is the identity).

Our method yields that, conversely, the existence of a stiff proper class of compact Hausdorff spaces implies the existence of a rigid propeı class of graphs (not to enlarge $\mathscr{S}$ to $\mathscr{S}^+$ but kill constant maps by choosing $x_S \notin Y_S$, $S \in \mathscr{S}$).

For metrizable spaces, the following statements are equivalent:

(a) **Metr** contains a stiff proper class of objects,

(b) The category of metrizable spaces and uniformly continuous maps contains a stiff proper class of objects.

(c) The category of metrizable spaces and non-expanding maps contains a stiff proper class of objects.

(d) **Graph** contains a rigid proper class of objects.

Remark 3. The existence of a rigid proper class $\mathscr{R}$ of graphs is really weaker than (M). Indeed, it is easy to show (cf. [1]) that the existence of $\mathscr{R}$ is exactly the negation of the *Vopěnka's Principle* which is well known in set theory (see [2], *VP* [= Vopěnka's Principle] says that, for each first-order language, every class of models such that none of them has an elementary embedding into another is a set). Hence

$$VP \Rightarrow non(M).$$

It is known in set theory that *VP* is stronger than *non*(*M*) (even, it cannot be shown that *VP* is consistent with *ZFC* + *non*(*M*)). It follows by Gödel's second incompleteness theorem and by the fact that *VP* yields a model of *ZFC* + *non*(*M*). Indeed, *VP* implies the existence of a supercompact cardinal $\varkappa$ ([2], 33.15, 33.14 (a)) and the set V_K of all sets of rank less than $\varkappa$ is a model of *ZFC* + *non*(*M*) (by [2], the Corollary to 33.10).

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