Maria Cristina Pedicchio; Walter Tholen
Multiplicative structures over sup-lattices

Archivum Mathematicum, Vol. 25 (1989), No. 1-2, 107--114

Persistent URL: http://dml.cz/dmlcz/107346

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 25, No. 1–2 (1989), 107–114

MULTIPLICATIVE STRUCTURES OVER SUP-LATTICES

MARIA CRISTINA PEDICCHIO and WALTER THOLEN*

(Received May 31, 1988)

Dedicated to the memory of Professor Milan Sekanina

Abstract. Modules over a not necessarily commutative multiplicative sup-lattice A are described as the Eilenberg–Moore algebras of a fairly elementary monad (T, η, μ) over **Set** with $TX = A^X$ which was considered before for commutative A, in particular when A is a frame. These modules are shown to carry a generalized metric structure, inducing another monadic functor.

Key words: sup-lattice, multiplicative sup-lattice, frame, locale, quantale, module, monadic functor.

MS Classification: 06 D 99; 06 A 23, 18 C 15, 18 C 20

INTRODUCTION

For a *frame* A (= complete lattice with $x \wedge \bigvee y_i = \bigvee x \wedge y_i$) Machner [4] gave a rather technical description of the algebras of the following monad $\tau_A = (T, \eta, \mu)$ on **Set**:

$$TX = A^X, (Tf)(\varphi)(y) = \bigvee \{\varphi(x) \mid x \in f^{-1}y\} \ (f: X \to Y, \varphi \in A^X, y \in Y),$$

$$\eta_X : X \to A^X \quad \text{with} \quad \eta_X(x)(x') = \delta_{xx'} \quad \text{(Kronecker's delta)},$$

$$\mu_X : A^{A^X} \to A^X \quad \text{with} \quad \mu_X(\Phi)(x) = \bigvee \{\Phi(\varphi) \wedge \varphi(x) \mid \varphi \in A^X\} \ (\Phi \in A^{A^X}, x \in X).$$

However, from Joyal's and Tierney's work [3] one now has a nice characterization of these algebras: interpreting A as a commutative monoid (with $\wedge$ as multiplication) over the *sup-lattice* (= complete lattice in which one considers $\bigvee$ the only structural element) A, Eilenberg–Moore algebras with respect to τ_A are nothing but modules over the monoid A, i.e. sup-lattices M which come equipped with an associative and unary action $A \otimes M \to M$ of sup-lattices.

* Partial support by the Université Catholique de Louvain (Belgium) and by NSERC (Canada) is gratefully acknowledged.

In this short note we present this observation in the non-commutative case. More precisely, we show that the above monad exists for every sup-lattice A which comes equipped with an associative, but not necessarily commutative multiplication and a one-sided unit (so in particular for every quantale in the sense of [1], and that the algebras are the same as in the localic case described above. We also observe that they carry a generalized metric structure which we discuss in terms of adjoint functors.

1. SUP-LATTICES

The category **SupLat** has as its objects partially ordered sets X which admit arbitrary suprema (in particular, one has $0 = \bigvee\emptyset$ and $1 = \bigvee X$), and as its morphisms $f: X \to Y$ mappings which preserve suprema. Every such morphism has a right adjoint $f_* : Y \to X$, given by the formula

$$\frac{f(x) \leq y}{x \leq f_*(y)},$$

(or $f_*(y) = \bigvee\{x \mid f(x) \leq y\}$); f_* preserves all infima, so it can be interpreted as a morphism $f^0 : Y^0 \to X^0$ in **SupLat** with X^0 the sup-lattice provided with the partial order opposite to that one of X. (Recall that the existence of arbitrary suprema implies the existence of arbitrary infima.) Obviously,

$$(-)^0 : \textbf{SupLat} \to \textbf{SupLat}$$

is a contravariant isomorphism of categories, yielding a strong self-duality of the category **SupLat**.

A *bimorphism* $f: X \times Y \to Z$ of sup-lattices satisfies the laws

$$f(\bigvee x_i, y) = \bigvee f(x_i, y), \qquad f(x, \bigvee y_i) = \bigvee f(x, y_i).$$

The *tensor product* of two sup-lattices X, Y is given by a universal bimorphism

$$X \times Y \to X \otimes Y, \qquad (x, y) \mapsto x \otimes y,$$

so that $\mathrm{Bihom}(X \times Y, Z) \cong \mathrm{Hom}(X \otimes Y, Z)$. Therefore, bimorphisms can be always written as **SupLat**-morphisms on the tensor product.

2. MODULES OVER MULTIPLICATIVE SUP-LATTICES

A sup-lattice A is called *multiplicative* when it comes equipped with a nullary operation $\varepsilon : 1 \to A$ (i.e. an element $\varepsilon \in A$) and a binary operation

$$A \otimes A \to A, \qquad \alpha \otimes \beta \mapsto \alpha\beta,$$

in **SupLat**. A *left A-module* M is a sup-lattice together with an action

$$A \otimes M \to M, \qquad \alpha \otimes x \mapsto \alpha x,$$

in **SupLat** such that

$$(\alpha\beta)\,x = \alpha(\beta x) \qquad \text{and} \qquad \varepsilon x = x \ (\alpha, \beta \in A, x \in M)$$

hold. The morphisms of the category A-**Mod** of left A-modules are morphisms $f: M \to N$ in **SupLat** such that $f(\alpha x) = \alpha f(x)$. A *right A-module M* is a left A^*-module where A^* has the multiplicative structure given by ε and $\alpha * \beta = \beta\alpha$. We write **Mod**-$A$ for A^*-**Mod**.

If A with its multiplicative structure is itself a left (right resp.) A-module, then A is called a *left* (*right* resp.) *monoid* over **SupLat**; it is a *monoid* if it is both a left and right A-module.

Every frame (= *locale*) is a monoid when putting $\alpha\beta = \alpha \wedge \beta$ and $\varepsilon = 1$; in fact, frames are those monoids over **SupLat** with $\varepsilon = 1$ and $\alpha^2 = \alpha$. (The Joyal–Tierney [3] proof survives dropping commutativity.) Prime examples of locales are the lattices of open sets of a topological space.

More generally, *quantales* in the sense of Borceux and van den Bossche [1] are, by definition, right monoids over **SupLat** with $\varepsilon = 1$ and $\alpha^2 = \alpha$. Those were introduced to describe, inter alia, the lattice of closed right ideals in a C^*-algebra.

For a multiplicative A, a left A-module M, and every $\alpha \in M$, the **SupLat**-morphism $\alpha(-) : M \to M$ has a right adjoint, denoted by $(-)^\alpha$, so

$$\frac{\alpha x \leq y}{x \leq y^\alpha}.$$

One has a **SupLat**-morphism

$$M^0 \otimes A \to M^0, \qquad y \otimes \alpha \mapsto y^\alpha,$$

which provides M^0 with a right A-module structure:

$$\frac{\dfrac{x \leq y^\varepsilon}{\varepsilon x \leq y}}{x \leq y} \qquad \frac{\dfrac{\dfrac{\dfrac{x \leq y^{\alpha\beta}}{(\alpha\beta)\,x \leq y}}{\alpha(\beta x) \leq y}}{\beta x \leq y^\alpha}}{x \leq (y^\alpha)^\beta}$$

This way one obtains a strong duality

$$(-)^0 : A\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}A.$$

For *A commutative* this gives a strong self-duality of A-**Mod** (which is the self-duality of **SupLat** mentioned before when taking A to be the 2-element chain).

3. MONADICITY OF LEFT A-MODULES

Theorem 1. *For a left monoid A over* **SupLat**, *A-**Mod** is monadic over* **Set**.

Proof: For every set X, $A^X = \mathbf{Set}(X, A)$ carries the structure of a left A-module, with $(\alpha\varphi)(x) = \alpha\varphi(x)\,(\alpha \in A,\ \varphi \in A^X,\ x \in X)$, which is simply a direct product of X copies of the left A-module A. It is indeed the free left A-module over X, since every **Set**-map $f: X \to M$ into a left A-module M factors through

$$\eta_X : X \to A^X, \qquad \text{with} \qquad \eta_X(x)(x) = \varepsilon \qquad \text{and} \qquad \eta_X(x)(x') = 0 \qquad \text{for} \qquad x \neq x',$$

by a unique morphism in **SupLat**, namely

$$g : A^X \to M \qquad \text{with} \qquad g(\varphi) = \bigvee\{\varphi(x)\, f(x) \mid x \in X\}$$

for all $\varphi \in A^X$.

It is elementary to show that the forgetful A-**Mod** $\to$ **Set** creates coequalizers of absolute pairs, so it is monadic (cf. [5]). But it is not difficult either to see directly how τ_A-algebras (M, m) correspond to left A-modules M (here τ_A is the monad induced by A-**Mod** $\to$ **Set** which may be described as in the Introduction, replacing $\wedge$ by the multiplication of A): for a left A-module M, the Eilenberg–Moore structure m is a morphism $A^M \to M$ in A-**Mod** with $m\eta_M = 1_M$, so

$$m(\varphi) = \bigvee\{\varphi(x)\, x \mid x \in M\};$$

on the other hand, given an Eilenberg–Moore structure m on a set M, A acts on M by

$$\alpha x = m(\alpha\eta_M(x)). \qquad \square$$

Analogously one can show that **Mod**-A is monadic over **Set** when A is a right monoid. So one has:

Corollary 1. *For a commutative monoid A over* **SupLat**, *both A-**Mod** and $(A$-**Mod**$)^{op}$ are monadic over* **Set**. $\square$

4. THE INDUCED HEYTING STRUCTURE

For a left monoid A and a left A-module M and every $x \in M$, the **SupLat**-Morphism $(-)\,x: A \to M$ has a right adjoint, denoted by $x \to (-)$, so

$$\frac{\alpha x \leq y}{\alpha \leq x \to y}.$$

One has a **SupLat**-morphism

$$M \otimes M^0 \to A^0, \qquad x \otimes y \mapsto (x \to y),$$

satisfying the following laws for all $x, y \in M$:

Proposition 1.

(1) $$x \leq y \Leftrightarrow \varepsilon \leq x \to y,$$

(2) $$\bigvee_{z \in X} (z \to y)(x \to z) = x \to y.$$

Proof: (1) is trivial, and it implies

$$x \to y = \varepsilon(x \to y) \leq (y \to y)(x \to y) \leq \text{l.h.s. of (2).}$$

For the other inequality needed in (2), first observe that trivially

$$(x \to z)\, x \leq z \qquad (*)$$

for all $x, z \in M$; therefore,

$$((z \to y)(x \to z))\, x = (z \to y)((x \to z)\, x) \leq (z \to y)\, z \leq y,$$

hence $(z \to y)(x \to z) \leq x \to y$ for all $x, y, z \in M$. □

Passing to the induced Heyting structure causes no problems when forming direct products:

Proposition 2. *For families $(x_i)_i$, $(y_i)_i$ in the direct* product $\prod_i M_i$ *in A-***Mod** *one has*

$$(x_i)_i \to (y_i)_i = \bigwedge_i (x_i \to y_i).$$

Proof: Since the partial order in $\prod_i M_i$ is componentwise, we have

$$\frac{\alpha \leq (x_i) \to (y_i)}{\dfrac{\alpha(x_i) \leq (y_i)}{\dfrac{\forall\, i : \alpha x_i \leq y_i}{\dfrac{\forall\, i : \alpha \leq x_i \to y_i}{\alpha \leq \bigwedge_i (x_i \to y_i)}}}}$$
□

However, morphisms require more detailed considerations:

Proposition 3. *For left A-modules M, N and a* **Set**-*map $f: M \to N$ one has*

(1) *$x \to y \leq f(x) \to f(y)$ $(x, y \in M)$ holds if and only if f is monotone* (i.e. $x \leq y \Rightarrow f(x) \leq f(y)$) *and satisfies $\alpha f(x) \leq f(\alpha x)$ $(\alpha \in A, x \in M)$.*

(2) *For f monotone and onto, $f(x) \to f(y) \leq x \to y$ $(x, y \in M)$ implies $f(\alpha x) \leq \alpha f(x)$ $(\alpha \in A, x \in M)$.*

(3) *$f(\alpha x) \leq \alpha f(x)$ $(\alpha \in A, x \in M)$ implies $f(x) \to f(y) \leq x \to y$ $(x, y \in M)$ if and only if f reflects the order* (i.e. $f(x) \leq f(y) \Rightarrow x \leq y$).

Proof: (1) "$\Rightarrow$" f is monotone by Prop. 1 (1). From $\alpha \leq x \to \alpha x \leq f(x) \to f(\alpha x)$ one obtains $\alpha f(x) \leq f(\alpha x)$. "$\Leftarrow$" In $\alpha \leq f(x) \to f(\alpha x)$ we may substitute $\alpha = x \to y$ to obtain with (*)

$$x \to y \leq f(x) \to f((x \to y)\, x) \leq f(x) \to f(y)$$

since f is monotone.

(2) We may write, for $\alpha \in A$ and $x \in M$ given, $\alpha f(x) = f(y)$ and have $\alpha \leq f(x) \rightarrow f(y) \leq x \rightarrow y$, hence $\alpha x \leq y$, so $f(\alpha x) \leq f(y) = \alpha f(x)$.

(3) "$\Rightarrow$" Reflection of the order follows from Prop. 1 (1) again. "$\Leftarrow$" With $\alpha = f(x) \rightarrow f(y)$ one obtains from (*)

$$f((f(x) \rightarrow f(y))\, x) \leq (f(x) \rightarrow f(y)) f(x) \leq f(y),$$

hence $(f(x) \rightarrow f(y))\, x \leq y$, so $f(x) \rightarrow f(y) \leq x \rightarrow y$. □

5. THE METRIC POINT OF VIEW

If, for a left monoid A over **SupLat** with $\varepsilon = 1$ and for a left A-module M, we write

$$d(x, y) = x \rightarrow y, \quad \alpha + \beta = \beta\alpha, \quad \alpha \prec \beta \Leftrightarrow \beta \leq \alpha, \quad \Theta = \varepsilon;$$

then Prop. 1 gives

(1) $$d(x, y) = \Theta = d(y, x) \Leftrightarrow x = y,$$

(2) $$d(x, y) \prec d(x, z) + d(z, y)$$

for all $x, y, z \in M$.

For a partially ordered (**Set**-based) semigroup $(S, +, \prec)$ (so $(S, +)$ is a not necessarily commutative semigroup and $(S, \prec)$ is a poset with the binary $+$ monotone in each variable) such that there is a bottom element Θ with $\Theta + \Theta = \Theta$, we consider the category

S-**Met**

whose objects are pairs (M, d) with a set M and a function $d : M \times M \rightarrow S$ that satisfies (1) and (2), and whose morphisms $f : (M, d) \rightarrow (M', d')$ are non-expanding maps, i.e.

$$d'(f(x), f(y)) \prec d(x, y).$$

Putting $(x \leq y \Leftrightarrow d(x, y) = \Theta)$ defines a functor S-**Met** $\rightarrow$ **PoSet** (the category of partially ordered sets and monotone maps).

If we denote by A^+ the partially ordered semigroup as described above (so A^+ is, as a semigroup, A^* and, as a poset, A^0) then Propositions 2 and 3 give immediately:

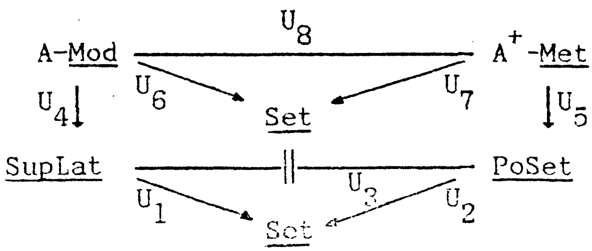
Corollary 2. *There is a faithful functor A-**Mod** $\rightarrow$ A^+-**Met** that preserves products and reflects isomorphisms.*

Next we shall point out that the functor is actually monadic.

6. SUMMARY IN TERMS OF ADJOINTS

For a left monoid A over **SupLat** with $\varepsilon = 1 \neq 0$ one has:

Theorem 2. *In the diagram*



of forgetful functors, each one has a left adjoint; U_1, U_3, U_4, U_6, U_8 *are monadic whereas* U_2, U_5 *and* U_7 *induce trivial monads.*

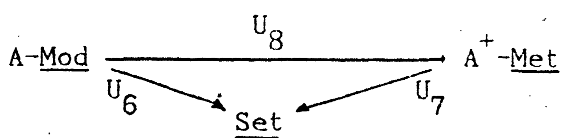
Proof: Denoting the left adjoint of U_i by F_i, one has F_1X the power set PX of the set X, $F_2X = X$ with the discrete order, and F_3X the system of down-sets in the poset X (cf. [2]). F_4 is tensoring with A, so F_4F_1 gives an alternative way of constructing the left adjoint F_6 as in Theorem 1, i.e.

$$A \otimes PX \cong A^X.$$

For a poset X, the metric structure of $F_5X = X$ is given by

$$d(x, y) = \begin{cases} \varepsilon & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(recall that 0 is the bottom element in A, i.e. the top element in A^+). Since $U_7 = U_2U_5$ trivially has a left adjoint, we just need to show existence of F_8: this can be derived from Corollary 2 above and Theorem 3 of [6], applied to the triangle



(we do not have an explicit construction of F_8).

Monadicity of U_1, U_3, U_4, U_6, U_8 is easily checked with the Beck–Paré criterion (cf. [5]); U_2, U_5 and U_7 obviously induce identical monads (to have $U_5F_5 = \mathrm{Id}$, one needs $1 \neq 0$ in A). □

REFERENCES

[1] F. Borceux and G. van den Bossche, *Quantales and their sheaves*. Order 3 (1986) 61 – 87.
[2] P. Johnstone, *Stone Spaces* (Cambridge University Press, Cambridge, 1982).
[3] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck*, Mem. Amer. Math. Soc. 51 (no. 309) (1984).
[4] J. Machner, *T-algebras of the monad L-Fuzz*, Czechoslovak Math. J. 35 (110) (1985) 515 – 528.
[5] S. MacLane, *Categories for the Working Mathematician* (Springer-Verlag, New York – Heidelberg – Berlin 1971).
[6] W. Tholen, *Adjungierte Dreiecke, Colimites und Kan-Erweiterungen*, Math. Ann. 217 (1975) 121 – 129.

M. C. Pedicchio
Dipartimento di Scienze Matematiche
Università degli Studi di Trieste
34100 Trieste
ITALY

W. Tholen
Department of Mathematics
York University
North York, Ontario, M3J 1P3
CANADA