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REFLECTIONS IN LOCALLY PRESENTABLE CATEGORIES

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Dedicated to the memory of Milan Sekanina

Abstract. For each locally presentable category it is proved that all full subcategories closed under limits and α-filtered colimits are reflective.

Key words. Locally presentable category, reflective subcategory.

MS Classification. 18 A 40

M. Makkai and A. M. Pitts have recently proved that each locally finitely presentable category $\mathscr{K}$ has the following property: all full subcategories closed under limits and filtered colimits are reflective in $\mathscr{K}$, see [8]. One is impelled to ask: 1. does this hold for locally presentable categories of infinite rank?, and 2. can filtered colimits be substituted by α-filtered colimits? The proof presented in [8] does not seem to give an answer. We were particularly interested in the latter question since the affirmative answer is the best result possible absolutely (i.e., independently of set theory). We have namely proved in [9] that the well-known Vopěnka's principle (which is a large cardinal principle much stronger than the existence of measurable cardinals, see [7]) is logically equivalent to the following statement: if $\mathscr{K}$ is a locally presentable category then each full subcategory of $\mathscr{K}$ closed under limits is closed under α-filtered colimits for some regular cardinal α.

The aim of the present paper is to answer both of the above questions affirmatively:

Theorem. *Let $\mathscr{K}$ be a locally presentable category, and α a regular cardinal. Then each full subcategory of $\mathscr{K}$ closed under limits and α-filtered colimits is reflective in $\mathscr{K}$.*

Proof. We can suppose that $\mathscr{K} = \mathbf{Set}^M$ for a small category M. This will not lose generality since for each locally presentable category $\mathscr{K}$ there exists a small category M such that $\mathscr{K}$ is equivalent to a full, reflective subcategory $\mathscr{K}'$ of $\mathbf{Set}^M$ closed under β-filtered colimits in $\mathbf{Set}^M$ for some β. (See 8.5 in [6]).

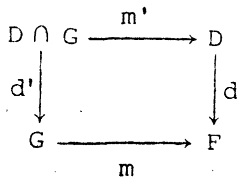
Then $\mathscr{H}'$ has the above property: each full subcategory $\mathscr{L}$ of $\mathscr{H}'$ closed under limits and α-filtered colimits in $\mathscr{H}'$ is closed under $(\alpha + \beta)$-filtered colimits in $\mathbf{Set}^M$, thus is reflective in $\mathbf{Set}^M$, and hence is reflective in $\mathscr{H}'$. It follows immediately that the equivalent category $\mathscr{H}$ has that property too.

Thus, we are to prove that each full subcategory $\mathscr{L}$ of $\mathbf{Set}^M$ closed under limits and α-filtered colimits is reflective in $\mathbf{Set}^M$. Without loss of generality, we may suppose that

$$\alpha > \operatorname{card}(\operatorname{mor} M).$$

It follows that each object F of $\mathbf{Set}^M$ is an α-directed union of all of its α-small subobjects, where an object D is *α-small* provided that $\sum_{X \in \operatorname{obj} M} \operatorname{card} DX < \alpha$. (In fact, for each $X \in \operatorname{obj} M$ and each $A \subset FX$, $\operatorname{card} A < \alpha$, consider the subfunctor $D_A \subset F$ defined on objects by $D_A Y = \bigcup_{f: X \to Y} Ff(A)$.)

A subobject $G \xrightarrow{m} F$ in $\mathbf{Set}^M$ is said to be *α-pure* provided that for each subobject $D \xrightarrow{d} F$ with D α-small, in the pullback



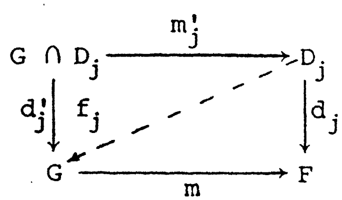
there exists a morphism $f: D \to G$ with $d' = f . m'$.

A. $\mathscr{L}$ is closed under α-pure subfunctors; i.e., if $G \xrightarrow{m} F$ is α-pure and $F \in \mathscr{L}$, then we will prove that $G \in \mathscr{L}$. Let $(D_j)_{j \in J}$ be the diagram of all α-small subfunctors $D_j \subseteq F$. Observe that J is an α-directed poset and F is the colimit of that diagram (where the colimit morphisms are the inclusion maps $D_j \xrightarrow{d_j} F$). For each j we have, by the α-purity, a morphism $f_j: D_j \to G$ with

$$f_j . m_j' = d_j' \tag{1}$$

in the following pullback of inclusion maps

(2)



Observe that

(3) $$G \cap G_j \xrightarrow{m_j'} D_j \overset{d_j}{\underset{m.f_j}{\rightrightarrows}} F \quad \text{is an equalizer}$$

since $f_j(x) = x$ iff x is an element of $G \cap D_j$. (Unfortunately, f_j' s need not be compatible with the diagram (D_j).)

Define a diagram $H : J \to \mathscr{L}$ as follows. For each $j \in J$, H_j is a product of copies of F indexed by $\{k \in J \mid j \leqq k\}$, and if $j \leqq j'$ then $H_{jj'}: H_j \to H_{j'}$ is the canonical projection. Since H is α-filtered, the colimit *colim* $H = (H_j \xrightarrow{h_j} \hat{H})_{j \in J}$ belongs to $\mathscr{L}$. For each $j \in J$ define $\Delta_{j'} \delta_j : D_j \to H_j$ by the following compositions with the projections $\pi_k: H_j \to F, j \leqq k$:

(4) $$\pi_k . \Delta_j = d_j$$

and

(5) $$\pi_k . \delta_j = m . f_k . d_{jk}$$

where d_{jk}: $D_j \to D_k$ is the inclusion map. It is easy to verify that if $j \leqq j'$, then $\Delta_{j'} . d_{jj'} = H_{jj'} . \Delta_j$ and hence, we have a compatible collection $h_j . \Delta_j: D_j \to \hat{H}$ yielding

$$\Delta = colim\, \Delta_j: F \to \hat{H}, \qquad \Delta . d_j = h_j . \Delta_j.$$

Analogously, $\delta_{j'} . d_{jj'} = H_{jj'} . d_j$ and thus we have

$$\delta = colim\, \delta_j: F \to \hat{H}, \qquad \delta . d_j = h_j . \delta_j.$$

We will prove that

$$G \xrightarrow{m} F \overset{\Delta}{\underset{\delta}{\rightrightarrows}} \hat{H}$$

is an equalizer—since $F, \hat{H} \in \mathscr{L}$ and $\mathscr{L}$ is closed under limits, this implies $G \in \mathscr{L}$. Since in $\mathbf{Set}^M$ finite limits commute with α-filtered colimits, it is sufficient to show that each

$$G \cap D_j \xrightarrow{m_j'} D_j \overset{\Delta_j}{\underset{\delta_j}{\rightrightarrows}} H_j$$

is an equalizer. First $\Delta_j . m_j' = \delta_j . m_j'$ because, for each $k \geqq j$,

$$\begin{aligned} \pi_k . \Delta_j . m_j' &= d_j . m_j' && \text{by (4)}, \\ &= m . d_j' && \text{by (2)}, \\ &= m . d_k' . d_{jk}', \end{aligned}$$

where d_{jk}': $G \cap D_j \to G \cap D_k$ is the inclusion map, and

$$\begin{aligned} \pi_k . \delta_j . m_j' &= m . f_k . d_{jk} . m_j' && \text{by (5)}, \\ &= m . f_k . m_k' . d_{jk}' \\ &= m . d_k' . d_{jk}' && \text{by (1)}. \end{aligned}$$

Further, suppose that $p\colon P \to D_j$ fulfils $\Delta_j \,.\, p = \delta_j \,.\, p$. By composing with $\pi_j\colon H_j \to$ $\to F$ we obtain, via (4) and (5),

$$d_j \,.\, p = m \,.\, f_j \,.\, p$$

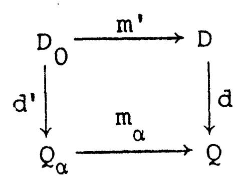
and thus, by (3), p factors through m_j'.

B. A morphism $f\colon P \to Q$ in $\mathbf{Set}^M$ will be called a γ-epimorphism if $\sum_{X \in \mathrm{obj}\, M} \mathit{card}$ $[QX - f_X(PX)] < \gamma$. We are going to prove that for each object P in $\mathbf{Set}^M$ there exists a cardinal γ such that every morphism with the domain P factors as a γ-epimorphism followed by an α-pure monomorphism.

We will first show a standard process of enlarging any monomorphism m_0: $Q_0 \to Q$ in $\mathbf{Set}^M$ to an α-pure one. This is done inductively, by defining an α-chain of monomorphism $m_i\colon Q_i \to Q$ for $i \leqq \alpha$, with m_α α-pure. First, m_0 is the given monomorphism, and $m_i = \bigcup_{j<i} m_j$ for each limit ordinal i. Given m_i: $Q_i \to Q$ consider all pairs of subobjects $(D_0 \xrightarrow{m'} D,\ D_0 \xrightarrow{d'} Q_i)$ where D is α-small. Since $\mathbf{Set}^M$ is wellpowered and has, essentially, only a set of α-small subobjects, all such pairs have a (small) set of representatives. Let us choose a set T_i of representative pairs of monomorphisms $(D_0 \xrightarrow{m'} D,\ D_0 \xrightarrow{d'} Q_i)$ such that D is α-small and that there is a monomorphism $d\colon D \to Q$ with $d \,.\, m' = m_i \,.\, d'$ a pullback; choose one d and denote it $d = u_{m',d'}$. Then put

$$m_{i+1} = m_i \cup \bigcup_{(m',d') \in T_i} u_{m',d'} \,.$$

Let us verify that m_α is α-pure. Given a subobject $d\colon D \to Q$ with D α-small, we form the pullback of m_α and d:



Since D_0 is α-small, d' factors through some Q_i, $i < \alpha$. That is, if $m_{i\alpha}\colon Q_i \to Q_\alpha$ denotes the connecting monomorphism, there is $d''\colon D_0 \to Q_i$ with $d' = m_{i\alpha} \,.\, d''$. It is clear that $d \,.\, m' = m_i \,.\, d''$ is a pullback, and hence, we can suppose that $(m', d'') \in T_i$. Then we have $u_{m',d''} \subset m_{i+1}$, i.e. there is a morphism $f_0\colon D \to Q_{i+1}$ with $u_{m',d''} = m_{i+1} \,.\, f_0$, which composed with m_{i+1} yields $f\colon D \to Q_\alpha$ with $f \,.\, m' =$ $= d'$. Thus, m_α is α-pure.

Now, inspecting the construction of m_α we see that $m_{0\alpha}\colon Q_0 \to Q_\alpha$ is a δ-epimorphism for the following cardinal $\delta = \bigvee_{i<\alpha} \delta_i$ (independent of Q): $\delta_0 = 0$ and if i is a limit ordinal, then $\delta_i = \bigvee_{j<i} \delta_j$. Given δ_i, let δ_{i+1} be the cardinality of a set of representative pairs of subobjects $(D_0 \xrightarrow{m'} D,\ D_0 \xrightarrow{d'} Q')$ where D is any α-small object and Q' is any object for which a δ_i-epimorphism $Q_0 \to Q'$ exists. Thus, for each object Q_0 we have found a cardinal δ such that any monomorphism with domain Q_0 factors as a δ-epimorphism followed by an α-pure monomorphism.

Finally, given an object P, let γ be the join of all δ's associated with quotient objects Q_0 of P. Then each morphism $f\colon P \to Q$ factors as an epimorphism $e\colon P \to Q_0$ followed by a monomorphism $m_0\colon P_0 \to Q$, and factoring the latter as a δ-epimorphism $m_{0,\alpha}\colon Q_0 \to Q_\alpha$ followed by an α-pure monomorphism $m_\alpha\colon Q_\alpha \to Q$, we obtain the required factorization: $m_{0,\alpha} . e$ is a γ-epimorphism (since $\gamma > \delta$) and $f = m_\alpha . (m_{0,\alpha} . e)$.

C. $\mathscr{L}$ is reflective in $\mathbf{Set}^M$. In fact, the embedding functor $\mathscr{L} \to \mathbf{Set}^M$ satisfies the solution set condition: for each object P of $\mathbf{Set}^M$ consider a representative set of γ-epimorphisms with the domain P and codomain in $\mathscr{L}$ (for γ as in B.). Then A. and B. show that this set is a solution set of the embedding functor.

Remarks. (1) The above proof shows that the theorem can be slightly strengthened: each full subcategory of $\mathscr{K}$ closed under limits and reduced powers modulo α-complete filters is reflective in $\mathscr{K}$. (If α is compact, the filters can be replaced by ultrafilters. In particular, each full subcategory of $\mathscr{K}$ closed under limits and ultraproducts is reflective.)

(2) We have been partly inspired by [5]. Some ingredients of our proof are not really new; see the characterization of α-algebraically closed ($= \alpha$-pure) embeddings in [4], (5–7), and the well-known procedure of constructing algebraic closures.

(3) The assumption that $\mathscr{K}$ be locally presentable cannot be omitted in the above theorem. For example, the dual $\mathbf{Ord}^{\mathrm{op}}$ of the usual category of ordinals is not reflective in its extension by an initial object. However, that extension is complete and complete, and $\mathbf{Ord}^{\mathrm{op}}$ is closed in it under (small) limits and nonempty colimits.

(4) We have shown in [2] that, under some set-theoretical assumptions, the collection *Ref*($\mathscr{K}$) of all full reflective subcategories can be badly behaved even if $\mathscr{K}$ is locally finitely presentable: for $\mathscr{K}$ = graphs we have exhibited two members of *Ref*($\mathscr{K}$) whose intersection is not a member of *Ref*($\mathscr{K}$).

The situation is different with the collection $Ref_\alpha(\mathscr{K})$ of all full reflective subcategories of $\mathscr{K}$ closed under α-filtered colimits.

Proposition. *For each locally presentable category $\mathscr{K}$ and each regular cardinal α, $Ref_\alpha(\mathscr{K})$ is a small complete lattice in which meets are intersections.*

Proof. Since $Ref_\alpha(\mathscr{K})$ coincides with the collection of all full subcategories of $\mathscr{H}$ closed under limits and α-filtered colimits, the only fact to be proved is that $Ref_\alpha(\mathscr{H})$ is small. This follows from an easy inspection of the above proof. First, since $\alpha \leqq \beta$ implies $Ref_\alpha(\mathscr{H}), \subseteq Ref_\beta(\mathscr{H})$ we can suppose, without loss of generality, that $\mathscr{H}$ is locally α-presentable. Then for each $\mathscr{L} \in Ref_\alpha(\mathscr{H})$ the reflector of $\mathscr{L}$ preserves α-filtered colimits and hence, $\mathscr{L}$ is determined by the reflections of α-presentable $\mathscr{H}$-objects in $\mathscr{L}$. There is, essentially, a set only of α-presentable objects P, and for each of them we have provided in our proof a solution set of morphisms with domain P and codomains in $\mathscr{L}$ the size of which was independent of $\mathscr{L}$. Thus $Ref_\alpha(\mathscr{H})$ is small.

The fact that $Ref_\alpha(\mathscr{H})$ is closed under intersections also follows by [3] (a remark before 5.3.).

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