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REMARKS ON HAMILTONIAN PROPERTIES OF SQUARES OF GRAPHS

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In memory of Milan Sekanina

Abstract. This paper deals with problems concerning the existence of such Hamiltonian cycles or paths in squares of graphs containing some edges of the original graphs. Using a method due to Řiha several results on blocks could be found generalizing previous ones.

Key words. Squares of graphs, blocks, Hamiltonian cycles, Hamiltonian paths.

MS Classification. 05 C

0. It has been M. Sekanina who 25 years ago posed the question for the structure of those graphs G the square of which has an open or a closed Hamiltonian line (i.e. G^2 is traceable or Hamiltonian, resp.), cf. [7]. Since that time many results concerning this problem could be obtained; to the most important and well-known ones among them certainly belong the Theorem of Fleischner [2], [3] verifying a conjecture of Plummer and Nash–Williams [4] (Every block G with at least 3 vertices has a Hamiltonian square) and its generalization by Chartrand, Hobbs, Jung, Kapoor, Nash–Williams [1] (For every block G its square is Hamiltonian-connected and, if G has at least 4 vertices, G^2 is 1-Hamiltonian as well). Recently, St. Řiha, a young former co-worker of Sekanina's succeeded in finding an excellent proof of the following statement (cf. [6]) which implies Fleischner's theorem and its generalization mentioned above.

Theorem 0: *Let G be a block with at least* 3 *vertices and x any vertex of G. Then there are two different G-neighbours a, b of x and a Hamiltonian path in $G^2 - x$ joining a and b.* □

Using Řiha's proof-method and his theorem, in the next sections of this paper we shall get several results on the existence of Hamiltonian cycles in G^2 containing some edges of G, especially a partial answer to the question, if the Hamiltonicity of G^2 always implies the existence of a Hamiltonian cycle in G^2 containing an edge of G. (In case that this conjecture were true it can be easily shown [8] that for any such G even there is a Hamiltonian cycle in G^2 containing at least two edges of G.)

All graphs considered here are supposed to be undirected, simple and finite (possibly empty). Let $G = (V, E)$ be a graph with the vertex-set $V(G) := V$ and the edge-set $E(G) := E$. If $x, y \in V$, $x \neq y$, are the end-vertices of an edge $l \in E$ we denote this edge l by the couple $\{x, y\}$. We say that $x \in V$ is a *G-neighbour* of $y \in V$ iff $\{x, y\} \in E$. The vertex x is called a G-neighbour of $M \subseteq V$ iff $x \notin M$ and x is a G-neighbour of some $y \in M$. If X is a vertex (a subgraph or a vertex-subset) of G then $N(X : G)$ denotes the set of all G-neighbours of the vertex X (of the set of all vertices belonging to X), and $G - X$ is defined to be the subgraph arising from G by deleting the vertex X (all vertices of X) and all edges incident with X (with some vertices of X). By $G(M)$ we denote the induced subgraph of G generated by $M \subseteq V$. The valency (degree) of the vertex $x \in V(H)$ in the subgraph H of G is denoted by $v(x : H)$. The *square* G^2 of G is the graph with $V(G^2) := V(G)$ and $\{x, y\} \in E(G^2)$ iff the distance of x and y in G is 1 or 2. A *block* is a graph which is 2-connected (*non-trivial block*) or a path of length 1 (*trivial block*). A block G is *minimal* iff there is no edge $l \in E(G)$ such that the graph arising from G by deleting l is a block. Paths and cycles w are comprehended to be special graphs (possibly subgraphs of a given graph); as usual they are represented by sequences of the vertices passed by w. Generally we shall not distinguish between a path (or a cycle) w and its representation by a vertex-sequence. A path of length 0 is called *trivial*. If $p = (x_0, x_1, \ldots, x_{r-1}, x_r)$ is a vertex-sequence the inverse sequence $(x_r, x_{r-1}, \ldots, x_1, x_0)$ is denoted by p^{-1}, and if $q = (y_0, y_1, \ldots, y_s)$ is another vertex-sequence then (p, q) is defined to be the vertex-sequence $(x_0, x_1, \ldots, x_{r-1}, x_r, y_0, y_1, \ldots, y_s)$; analogously in similar cases. The number of elements of a set M is denoted by $|M|$.

1. Let G be a graph, w a non-trivial path in G and x an endvertex of w.

Definition. $\mathbf{S}$ is a (G^2, w, x)*-basic-set* iff $\mathbf{S}$ is a set of pairwise vertex-disjoint paths in $G^2 - w$ with $\bigcup_{p \in \mathbf{S}} V(p) = V(G) - V(w)$, and there is a mapping f from $\mathbf{S}$ into the power-set of $V(w)$ with the following properties:

(1) $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$ where $\mathbf{S}_i := \{p \in \mathbf{S} : |f(p)| = i\}$, $i = 1, 2$;

(2) for each $p \in \mathbf{S}_2$, if $\{a_1, a_2\} = f(p)$ and $\{e_1, e_2\}$ is the set of the endvertices of p (possibly $e_1 = e_2$), it holds: $\{a_1, e_1\}, \{a_2, e_2\} \in E(G)$ or $\{a_1, e_2\}, \{a_2, e_1\} \in E(G)$;

(3) for each $p \in \mathbf{S}_1$, if $\{a\} = f(p)$, then $\{a, e\} \in E(G)$ holds for every endvertex e of p;

(4) $f(p) \cap f(p') = \emptyset$ for any different $p, p' \in \mathbf{S}$;

(5) if $\mathbf{S}_2 = \emptyset$ then there is a $z \in V(w)$, $z \neq x$ such that $z \notin f(p)$ for each $p \in \mathbf{S}$.

The construction given by Říha in [6] – it is the main point of his proof of Theorem 0 – verifies the following

Lemma. *Let G be a graph, w a non-trivial path in G, x an endvertex of w and* $\mathbf{S}$ *a (G^2, w, x)-basic-set. Then there is a G-neighbour x' of x and a Hamiltonian path in G^2 joining the endvertices of w and containing the edge $\{x, x'\}$ and each $p \in S$ as a subpath.* □

2. Using this Lemma we shall prove some generalizations of Řiha's Theorem 0. For this end we introduce the following notations. Let G be a non-trivial block (i.e. $|V(G)| \geqq 3$) and w a path in G. By $C_1(w)$ and $C_2(w)$ we denote the set of all components C of the graph $G - w$ with $|V(C)| = 1$ and $|V(C)| \geqq 2$, respectively, and we define $C(w) := C_1(w) \cup C_2(w)$. Let $C \in C(w)$. Then $N(C : G) \subsetneqq V(w)$ and $|N(C : G)| \geqq 2$; for $C \in C_2(w)$ at least two vertices of C have G-neighbours in $N(C : G)$ and therefore there are two different vertices in C having a pair of different G-neighbours in $N(C : G)$. For every $C \in C(w)$ we form the graph G_C arising from $G(V(C) \cup N(C : G))$ by contracting all vertices of $N(C : G)$ to a new vertex $0 \notin \notin V(G)$ (the edges between C and $N(C : G)$ in G become edges between C and 0 in G_C, of course), where resulting multiple edges are replaced by a simple edge with the same endvertices and resulting loops are removed. Obviously, G_C is a block, and $|V(G_C)| \geqq 3$ if $C \in C_2(w)$. Let us suppose:

(6) For each $C \in C_2(w)$ there is given a Hamiltonian path h_C in $G_C^2 - 0$ joining two G_C-neighbours of 0.

Furthermore, for each $C \in C_1(w)$ we define $h_C := C$ (the trivial path consisting of the single vertex of C). Then it follows that h_C and $h_{C'}$ are vertex-disjoint if $C \neq C'$, $C, C' \in C(w)$. Denote by $\mathbf{P}$ the set of all subpaths arising from the family $(h_C : C \in C(w))$ by deleting, for each h_C, all edges in h_C which do not belong to $E(C^2)$. We remark that $\mathbf{P}$ consists of pairwise vertex-disjoint paths in $(G - w)^2$ the endvertices of which are G-neighbours of some vertices of w, that every edge belonging to $E(h_C) \cap E(G)$ for a $C \in C(w)$ is also an edge of some $p \in \mathbf{P}$, and that the (disjoint) union of all sets $V(p)$ with $p \in \mathbf{P}$ results in $V(G) - V(w)$. Now the following algorithm (*) is applied to $\mathbf{P}$ (see Řiha [6]):

(*) If there exist different paths $p, p' \in \mathbf{P}$ with the property that there is a $z \in V(w)$ which is a G-neighbour of an endvertex x of p as well as of an endvertex x' of p', we take such a pair $p = (a, \ldots, x)$, $p' = (x', \ldots, b)$ with $x, x' \in N(z : G)$ for a $z \in \in V(w)$, form the path $p'' = (p, p') = (a, \ldots, x, x', \ldots, b)$ which is a path of $G^2 - w$ whose endvertices a, b are G-neighbours of some vertices of w (possibly of only one vertex of w), and replace the elements p, p' in $\mathbf{P}$ by p''. We obtain the set $\mathrm{P}' := := (\mathbf{P} - \{p, p'\}) \cup \{p''\}$ and repeat this procedure with respect to $\mathbf{P}'$, and so on. After a finite number of steps – say r – this algorithm stops, and the resulting set $\mathbf{S} := \mathbf{P}^{(r)}$ has the properties:

(7) $\mathbf{S}$ consists of pairwise vertex-disjoint paths in $G^2 - w$ the endvertices of which are G-neighbours of some vertices of w;

(8) for any different elements $p = (x, \ldots, x')$ and $q = (y, \ldots, y')$ of $\mathbf{S}$, the end-

vertices of these paths satisfy

$$N(\{x, x'\} : G) \cap N(\{y, y'\} : G) \cap V(w) = \emptyset;$$

(9) $\bigcup_{p \in \mathbf{S}} V(p) = V(G) - V(w);$

(10) for any $C \in C_2(w)$ every $l \in E(h_C) \cap E(G)$ is also an edge of some $p \in \mathbf{S}$.

Let $\mathbf{S}(h_C : C \in C(w))$ denote the set of all such path-sets $\mathbf{S}$ which can be obtained if we apply algorithm (*) to $\mathbf{P}$ in any possible way. Then it is easy to see that every $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ fulfils (10) and all properties of a (G^2, w, x)-basic-set with the exception of (5), where x is either endvertex of w. The mapping f is chosen as follows: If for a $p \in \mathbf{S}$ with $|V(p)| \geqq 2$ the endvertices e_1, e_2 of p satisfy $m := := |V(w) \cap (N(e_1 : G) \cup N(e_2 : G))| \geqq 2$, we take arbitrary $a_i \in N(e_i : G) \cap V(w)$, $i = 1, 2$, with $a_1 \neq a_2$ and define $f(p) := \{a_1, a_2\}$; if $m = 1$ we have to put $f(p) := \{a\} = N(e_1 : G) \cap V(w)$. For a $p \in \mathbf{S}$ with $|V(p)| = 1$ it follows $|N(e : G) \cap V(w)| \geqq 2$ for the vertex e of p if $p \in C_1(w)$, and we take $a_1, a_2 \in \in N(e : G) \cap V(w)$, $a_1 \neq a_2$, and $f(p) := \{a_1 a_2\}$; if $p \notin C_2(w)$ and $|N(e : G) \cap \cap V(w)| \geqq 2$ we proceed as before; if $p \notin C_2(w)$ and $|N(e : G) \cap V(w)| = 1$ we define $f(p) := \{a\}$ with $\{a\} = N(e : G) \cap V(w)$.

3. Now we suppose G to be a minimal block with $(V(G)) \geqq 3$, and let x and y be different vertices. Then there exists a cycle in G containing x and y. Because this cycle has two different vertices a, b with $v(a : G) = v(b : G) = 2$ (see Plummer [5], Řiha [6]), at least one of the two independent paths joining x and y which form a separation of this cycle must contain a vertex $z \neq x$ with $v(z : G) = 2$. A path p satisfying this property (i.e. p joins x and y and contains a vertex $z \neq x$ with $v(z : G) = 2$) is called an *admissible* (x, y)-path in G and x its *initial* vertex. (Obviously, an admissible (x, y)-path is not necessarily an admissible (y, x)-path.) Note that for any $x \neq y$ there is an admissible (x, y)-path in the minimal block G; if $\{x, y\} \in E(G)$ then every path in G of length $\geqq 2$ joining x and y is an admissible (x, y)-path, and if $\{x, y\} \notin E(G)$ then there is an admissible (x, y)-path which is not a Hamiltonian path. Let w be an admissible (x, y)-path in G and assume (6) for this w. Then there is a $z \in V(w)$, $z \neq x$ with $v(z : G) = 2$. Assume that the family $(h_C : C \in C_2(w))$ satisfies the additional property:

(6a) If $v(y : G) = 2$ and y is not a G-neighbour of x and the (only) G-neighbour $y^* \notin V(w)$ of y belongs to a component C^* of $G - w$ fulfilling $C^* C_2(w)$, then $h_C{}^*$ contains an edge $\{y^*, z\} \in E(G)$ with some $z \in N(y^* : C^*)$. Now consider an $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ and a mapping f described at the end of section 2. Then it follows that the set $\mathbf{S}_2 = \{p \in \mathbf{S} : |f(p)| = 2\}$ is empty only in the case that for each p the premise $p \in \mathbf{S}$ implies $|V(p)| \geqq 2$ and $|V(w) \cap (N(e_1 : G) \cup N(e_2 : G)| = = 1$, for the endvertices e_1, e_2 of p or $|V(p)| = 1$ and $|N(e : G) \cap V(p)| = 1$ with $(e) = p$. In the first case we conclude $f(p) = N(e_1 : G) \cap V(w) = N(e_2 : G) \cap$

$\cap V(w) = \{a_p\}$ because of (7); obviously, $v(a_p : G) \geqq v(a_p : w) + v(a_p : G(\{e_1, e_2, a_p\})) \geqq 3$, and thus we have $a_p \neq z$, i.e. $z \notin f(p)$.

In the second case we have $f(p) = N(e : G) \cap V(w) = \{a_p\}$ for the vertex e of p; further $v(a_p : G) \geqq v(a_p : w) + v(a_p : G(\{e, a_p\}) \geqq 2 + 1 = 3$ if a_p is an inner vertex of w, and if $a_p = y$ and $\{x, y\} \in E(G)$ then $v(y : G) \geqq v(y : G(w)) + v(y : G(\{e, y\}) \geqq 2 + 1 = 3$. Now let $a_p = y$, $\{x, y\} \notin E(G)$; if $v(y : G) = 2$ then because of (6a) and (10) it follows that $\{y^*, z\} \in E(p)$ for some $z \in N(y^*:G - w)$, where $y^* \in N(y : G) - V(w)$. This is a contradiction to $|V(p)| = 1$. Hence in every case $v(a_p : G) \geqq 3$, and thus $a_p \neq z$, i.e. $z \notin f(p)$. So we have proved $z \notin f(p)$ for each $p \in \mathbf{S}$ if $\mathbf{S}_2 = \emptyset$. Consequently, $\mathbf{S}$ and f fulfil (5); using the statements of section 2 and the notations introduced there we get

Corollary 1. *Let G be a minimal non-trivial block, $x, y \in V(G)$ with $x \neq y$, and w an admissible (x, y)-path in G. Furthermore, we assume that we are given a family $(h_C : C \in C_2(w))$ according to* (6) *and fulfilling* (6a). *Then every $S \in S(h_c$: $C \in C(w)$ is a (G^2, w, x)-basic-set satisfying property* (10). □

For any block H with $|V(H)| \geqq 3$ we define

$$s(H) := |V(H)| \sum_{x \in V(H)} (v(x : H) - 2) = 2|V(H)|| |C|(E(H)| - |V(H)|).$$

Obviously, $s(H) \geqq 0$ because of $v(x : H) \geqq 2$ for $x \in V(H)$, and $s(H) = 0$ iff H is a cycle. Referring to the notations of section 2 we can prove

Corollary 2. *Let G be a non-trivial block not being a cycle, and w a non-trivial path in G with the endvertices x, y. Then for every $C \in C_2(w)$ the graph G_C is a non-trivial block satisfying*

$$s(G_C) < s(G). \tag{11}$$

Proof: $C \in C_2(w)$ implies (see section 2) $|V(C)| \geqq 2$, $|N(C : G)| \geqq 2$, $N(C : G) \subseteq V(w)$, and $V(C) \cap N(w : G) = N(0 : G_C)$. Hence, $|V(G_C)| < |V(G)|$, and G_C is a block with $(V(G_C)) \geqq 3$. Let $N(C : G) - \{x, y\} = \{e_1, \ldots, e_k\}$, and write $e_0 = x$ and $e_{k+1} = y$. Obviously, for each $\bar{x} \in V(C)$ we have $v(\bar{x} : G_C) \leqq v(\bar{x} : G)$. If $x, y \notin N(C : G)$ we get $k \geqq 2$ and $2 \leqq v(0 : G_C) \leqq \sum_{i=1}^{k} (v(e_i : G) - 2)$; if $x \in N(C : G)$, $y \notin N(C : G)$ it follows $k \geqq 1$ and $2 \leqq v(0 : G_C) \leqq \sum_{i=0}^{k} (v(e_i : G) - 2) + 1$, analogously for $y \in N(C : G)$, $x \notin N(C : G)$; if $x, y \in N(C : G)$ we find $k \geqq 0$ and $2 \leqq v(0 : G_C) \leqq \sum_{i=0}^{k+1} (v(e_i : G) - 2) + 2$. In each of these cases we obtain

$$s(G_C) \leqq |V(G_C)| \sum_{\bar{x} \in V(G)} (v(\bar{x} : G) - 2) = \lambda s(G),$$

with $\lambda = \dfrac{|V(G_C)|}{|V(G)|} < 1$. This results in (11) because G is not a cycle and therefore $s(G) > 0$. □

Note that for blocks H, G with $|V(H)| \geqq 3$, where H is a subgraph of G and $H \neq G$, it follows $s(H) < s(G)$.

4. Generalizing Řiha's theorem (Theorem 0) we show

Theorem 1. *Let G be a block and x, y adjacent vertices. Then there is a G-neighbour x' of x and a Hamiltonian path in G^2 joining x and y and containing the edge $\{x, x'\}$.*

Proof: Obviously, the assertion is true if $|V(G)| = 2$ and also if G is Hamiltonian. Assume, Theorem 1 fails to hold. Let G be a block with the least value of $s(G)$ such that G does not fulfil the property stated in this theorem for some adjacent vertices $x \neq y$. Hence it follows, that G is a minimal block with $|V(G)| \geqq 3$ being not Hamiltonian, i.e. G is not a cycle, and therefore $s(G) > 0$. Because G is a minimal block there is an admissible (x, y)-path w. Obviously w is a non-Hamiltonian path. According to section 2 we form the set $C(w) = C_1(w) \cup C_2(w)$, and for each $C \in C_2(w)$ we consider the graph G_C which is a non-trivial block. Owing to Theorem 0 (cf. [6]) there exists a Hamiltonian path h_C in $G_C^2 - 0$ joining two G_C-neighbours of 0; therefore we can find a family $(h_C : C \in C_2(w))$ realizing (6). (Note that (6a) is trivial because of $\{x, y\} \in E(G)$.) Owing to Corollary 1 every $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ is a (G^2, w, x)-basic-set. Because w is a non-Hamiltonian path, $\mathbf{S}(h_C : C \in C(w)) \neq \emptyset$. Taking an $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w))$ and using the Lemma of section 1 we get a Hamiltonian path in G^2 joining x and y and containing an edge $\{x, x'\}$ for some G-neighbour x' of x, which is a contradiction to the assumption on G. □

Theorem 2. *Let G be a non-trivial block, and x, y different vertices. Then there are different G-neighbours a, b of x, a G-neighbour z of y, and a Hamiltonian path in G^2-x joining a and b and containing the edge $\{y, z\}$.*

Proof: The assertion holds for Hamiltonian graphs, i.e. for all non-trivial blocks G with $s(G) = 0$. Assume Theorem 2 to be not true, and consider a block G with $|V(G)| \geqq 3$ and the least value of $s(G)$ such that the property stated in Theorem 2 is not fulfilled for some $x \neq y$. Then G is a minimal non-trivial block and not Hamiltonian (i.e. not a cycle), what implies $s(G) > 0$.

Case 1: Suppose that there is a cycle k in G with $x \in V(k)$ and $y \notin V(k)$. Let b be a k-neighbour of x. Deleting the edge $\{x, b\}$ in k we obtain a non-Hamiltonian path w which is an admissible (x, b)-path.

According to section 2 we form the set $C(w) = C_1(w) \cup C_2(w)$, and for each $C \in C_2(w)$ we consider the graph G_C which is a block with $|V(G_C)| \geqq 3$.

a) Let $y \in V(T)$ for some $T \in C_2(w)$. Then Corollary 2 yields $s(G_T) < s(G)$; hence it follows that there is a Hamiltonian path h_T in $G_T^2 - 0$ joining two G_T-neighbours of 0 and containing an edge $\{y, z\}$ with a suitable G_T-neighbour z of y. Then $y, z \neq 0$, and therefore z is a G-neighbour of y as well. Thus $\{y, z\} \in E(h_T) \cap E(G)$. For every $C \in C_2(w)$, $C \neq T$, Theorem 0 yields a Hamiltonian path h_C in $G_C^2 - 0$ joining two G_C-neighbours of 0. In this way we have succeeded in finding a family $(h_C : C \in C_2(w))$ realizing (6). Owing to Corollary 1 every $\mathbf{S} \in \mathbf{S}(h_C : C \in C(w)) \neq \emptyset$ (w is a non-Hamiltonian path) is a (G^2, w, x)-basic-set satisfying (10) and consequently, $\{y, z\} \in E(p)$ for some $p \in \boldsymbol{S}$. Using the Lemma of section 1 with such an $\mathbf{S}$ we obtain a G-neighbour a of x and a Hamiltonian path in G^2 joining x and b and containing the edges $\{x, a\}$ and $\{y, z\}$. Because of $|V(G)| \geqq 3$ we have $a \neq b$, and we have found a Hamiltonian path in $G^2 - x$, joining two different G-neighbours a, b of x and containing the edge $\{y, z\} \in E(G)$. This is a contradiction to the assumption on G.

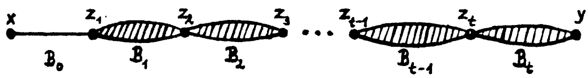
b) Let $y \in V(T)$ for some $T \in C_1(w)$. Then T consists of the vertex y, and y is a G-neighbour of exactly two vertices $z', z \in V(w) = V(k)$ which cannot be adjacent in G (note that k has not diagonals because G is a minimal block). We may assume $z \neq x$. Both paths w_1, w_2 joining z' and z and forming a separation of the cycle k must contain at least one inner vertex ($\neq z, z'$). Then it follows that $G - y$ is a block with $|V(G - y)| \geqq 4$ and $s(G - y) < s(G)$. Thus (because of $x \neq z$) there is a Hamiltonian path p in $(G - y)^2 - x$ joining two $(G - y)$-neighbours a, b' of x and containing an edge $\{z, t\} \in E(G - y) \subseteqq E(G)$. Replacing the subpath (z, t) (which corresponds to the edge $\{z, t\}$) in p by (z, y, t) which is a path of length 2 in G^2, we get a Hamiltonian path p' in $G^2 - x$ joining different G-neighbours a, b' of x and containing the edge $\{y, z\} \in E(G)$. But this is a contradiction to the assumption on G.

Case 2: We have to suppose that every cycle containing x must contain y as well. Note that at least one such cycle exists. Each of the components of the graph $G - \{x, y\}$ is adjacent with x and with y in G and contains exactly one G-neighbour of x. If x and y are adjacent in G, then $G - \{x, y\}$ has exactly one component (G is a minimal block), say T_1; otherwise $G - \{x, y\}$ has at least two components, say $T_1, T_2, \ldots, T_r, r \geqq 2$.

a) Let $\{x, y\} \notin E(G)$. By $\bar{H}_i$ we denote the graph arising from $H_i := G(V(T_i) \cup \cup \{x, y\})$ by adding the new edge $\{x, y\}$, $i = 1, \ldots, r$. Obviously, $\bar{H}_i$ is a block with $|V(\bar{H}_i)| \geqq 3$ and $s(\bar{H}_i) < s(G)$ (because of $r \geqq 2$), and, furthermore, $v(x : H_i) = 1$, $v(x : \bar{H}_i) = 2$, $i = 1, \ldots, r$. Consider any $i \in \{1, \ldots, r\}$ and write $\bar{H}$ and H instead of $\bar{H}_i$ and H_i, respectively. Note that H arises from $\bar{H}$ by deleting the edge $\{x, y\}$. Let z denote the $\bar{H}$ neighbour of x being different from y, and let p be any path in $\bar{H}$ joining x and y and not containing the edge $\{x, y\}$; such a path exists, for $\bar{H}$ is a block. Then p is a path in H which contains all cutpoints of H. (A cutpoint z' of H with $z' \notin V(p)$ would imply that both x, y belong to the same component C of $H - z'$, and that there is at least another component $C' \neq C$; therefore the edge $\{x, y\} \in E(\bar{H})$ joins vertices of the same component C of $H - z'$, and we get at least two components of $\bar{H} - z'$, in contradiction to the fact that $\bar{H}$ is a block.) Obviously, one cutpoint of H is z. Hence it follows that p can be represented by the sequence

$$p = (x, z = z_1, \dots, z_2, \dots, z_3, \dots, z_t, \dots, y) = \\ = (x, p_1', p_2', \dots, p_{t-1}', p_t),$$

where $z_1, z_2, \dots, z_t (t \geqq 1)$ are all the (different) cutpoints of H, and $p_0 = (x, z = z_1)$, $p_k = (z_k, \dots, z_{k+1}) = (p_k', z_{k+1})$, $k = 1, \dots, t-1$, and $p_t = (z_t, \dots, y)$ are non-trivial subpaths of p forming a separation of p. (Of course, $z_t \neq y$ holds because $\bar{H}$ is a block.) With $z_0 := x$, $z_{t+1} := y$ the couple $\{z_k, z_{k+1}\}$ of the endvertices of p_k determines a (maximal) block B_k of H (B_k is the maximal subgraph in H being a block and containing z_k and z_{k+1}), $k = 0, 1, \dots, t$, and these $B_k's$ satisfy the properties: $V(B_k) \cap V(B_{k+1}) = \{z_{k+1}\}$, $k = 0, \dots, t-1$, $V(B_l) \cap V(B_k) = \emptyset$ for $0 \leqq l < k \leqq t$ with $k \neq l+1$, and $B_0, B_1, \dots, B_t$ are all the (maximal) blocks of H. (For otherwise we could find a path in H joining x and y and not containing every cutpoint of H, or we would get a cutpoint of $\bar{H}$, respectively, but we have seen that neither of these situations is possible. To put it concisely: The block-cutpoint-graph of H is a path, and x and y belong to its different end-blocks.) Of course, $B_0 = p_0 = (x, z)$.

H : 

Because of $s(\bar{H}) < s(G)$ we have $s(B_k) < s(G)$ if $|V(B_k)| \geqq 3$, $k = 0, 1, \dots, t$. Hence it follows that for such a B_k there is a Hamiltonian path in $B_k^2 - z_{k+1}$ joining two suitable B_k-neighbours z_{k+1}' and z_{k+1}'' of z_{k+1} and containing some edge $\{z_k, \bar{z}_k\} \in E(B_k)$. We can write this path in the form $(z_{k+1}', \dots, z_k, \bar{z}_k, \dots, z_{k+1}'')$ and consider the two subpaths $q_k' := (z_{k+1}', \dots, z_k)$ and $q_k'' := (\bar{z}_k, \dots, z_{k+1}'')$; note that $\{z_k, \bar{z}_k\}, \{z_{k+1}', z_{k+1}\}, \{z_{k+1}'', z_{k+1}\} \in E(H)$. In case that $|V(B_k)| = 2$, $k \geqq 1$, we have $\{z_k, z_{k+1}\} \in E(H)$ and we consider the maximal sequence $B_k, B_{k+1}, \dots, B_{k+l}$ with $|V(B_{k+j})| = 2$, $j = 0, 1, \dots, l$, and $k \leqq k + l \leqq t$ (that is: Either $k + l = t$ or if $k + l < t$ then it holds $|V(B_{k+l+1})| \geqq 3$); now we define $q_k' := \emptyset$, $q_k'' := (z_k)$ if l is even, and $q_k' := (z_k)$, $q_k'' := \emptyset$ if l is odd. Then the sequence

$$q := (q_t', q_{t-1}', \dots, q_1', q_1'', q_2'', \dots, q_t'')$$

is a Hamiltonian path in $(H - x)^2 - y$ satisfying the following property:

If $|V(B_t)| \geqq 3$ then q joins two B_t-neighbours (and therefore H-neighbours) $y' := z_{t+1}'$ and $y'' := z_{t+1}''$ of $y = z_{t+1}$;

if $|V(B_t)| = 2$ and $t \geqq 2$ then q joins some B_{t-1}-neighbour y' of z_t (namely $y' := z_t'$ if $|V(B_{t-1})| \geqq 3$, and $y' := z_{t-1}$ if $|V(B_{t-1})| = 2$) with the B_t-neighbour $y'' := z_t$ of y;

if $|V(B_t)| = 2$ and $t = 1$ then $q = (z_1)$ consists of the only B_t-neighbour $y' := \\ := y'' := z_1$ of y.

Thus we can write $q = (y', \ldots, y'')$, where y'' is an H-neighbour of y and y' is an H^2-neighbour of y.

Furthermore, from Theorem 1 it follows, that for each block B_k, $k = 0, \ldots, t$, there is a Hamiltonian path q_k^* in B_k^2 joining the vertices z_{k+1} and z_k and containing some edge $\{z_{k+1}, z_{k+1}^*\} \in E(B_k)$. (This is obvious if z_k and z_{k+1} are adjacent. If they are not adjacent we consider the block $\bar{B}$ consisting of B_k, a new vertex 0 and the edges $\{0, z_k\}$ and $\{0, z_{k+1}\}$. Then because of $r \geqq 2$ we have $s(\bar{B}) < s(G)$, and this remains valid also for $r = 1$ if $t \geqq 2$, i.e. in the next subcase **b)** only the situation for $t = 1$ must be considered separately. Hence it follows, that there is a Hamiltonian path in $\bar{B}^2 - 0$ joining the two $\bar{B}$-neighbours of 0 and containing some edge $\{z_{k+1}, z_{k+1}^*\} \in E(\bar{B})$.) We can write We can write $q_k^* = (z_{k+1}, z_{k+1}^*, \ldots, z_k)$, $k = 0, 1, \ldots, t$, and with $\bar{q}_k^* := (z_{k+1}^*, \ldots, z_k)$ – i.e. $q_k^* = (z_{k+1}, \bar{q}_k^*)$ –, $k = 0, 1, \ldots, t$, it is obvious that the sequence

$$\bar{q} := (\bar{q}_t^*, \bar{q}_{t-1}^*, \ldots, \bar{q}_0^*)$$

is a Hamiltonian path in $H^2 - y$ joining an H-neighbour $y^* := z_{t+1}^*$ of y and the vertex x and containing the edge $\{z, x\} \in E(H)$ (because z_1^* is a B_0-neighbour of $z_1 = z$ and therefore $z_1^* = x$).

Thus we have proved the following assertions for $i = 1, \ldots, r$:

There is a Hamiltonian path $q_i = (y_i', \ldots, y_i'')$ in $(H_i - x)^2 - y$ joining an H_i^2-neighbour y_i' of y and an H_i-neighbour y_i'' of y.

There is a Hamiltonian path $\bar{q}_i$ in $H_i^2 - y$ joining an H_i-neighbour y_i^* of y and the vertex x and containing the edge $\{z^i, x\} \in E(H_i)$, where z^i is the only H_i-neighbour of x; write $\bar{q}_i = (y_i^*, \ldots, z^i, x)$ and $\tilde{q}_i := (y_i^*, \ldots, z^i) = \bar{q}_i - x$.

Because $V(G - \{x, y\})$ and $E(G)$ are the disjoint unions of the sets $V(H_i - - \{x, y\})$ and $E(H_i)$, respectively, and $V(H_i) \cap V(H_j) = \{x, y\}$ if $i \neq j$ we obtain:

$$(\tilde{q}^{-1}, y, \tilde{q}_2, \tilde{q}_3^{-1}, \tilde{q}_4, \ldots, \tilde{q}_{r-1}^{-1}, \tilde{q}_r) \qquad \text{if } r \text{ is even and}$$

$$(\tilde{q}_1^{-1}, y, q_2, \tilde{q}_3, \tilde{q}_4^{-1}, \ldots, \tilde{q}_{r-1}^{-1}, \tilde{q}_r) \qquad \text{if } r \text{ is odd}$$

is a Hamiltonian path in $G^2 - x$ joining the two G-neighbours $a := z^1$and $b := z^r$ of x and containing the edge $\{y_1^*, y\} \in E(G)$ (and the edge $\{y, y_2^*\} \in E(G)$ if r is even as well). However, this is a contradiction to the assumption on G.

b) Let $\{x, y\} \in E(G)$. Then $G - \{x, y\}$ has exactly one component T_1. Write $\bar{H} := G$ and let H be the graph arising from $\bar{H}$ by deleting the edge $\{x, y\}$. Obviously, we have the same situation as considered in subcase **a)** with respect to the graphs $H, \bar{H}$ with the only exception that now $s(\bar{H}) < s(G)$ does not hold (because of $\bar{H} = G$). However, if $t \geqq 2$ (note that $t + 1$ is the number of the blocks of H) the construction of the path $\bar{q}$ remains valid. Now let $t = 1$. Then G consists of the block $B := B_1$ containing the two different vertices $z_1 = z$ and $z_2 = y$, of the vertex x and of the edges $\{x, z\}$ and $\{x, y\}$. Obviously, $s(B) < s(G)$ if B is a nontrivial block. To construct a path $\bar{q}$ wanted it suffices to construct a Hamiltonian path h in B^2 joining z and y and containing some edge $\{y, y^*\}$ with $y^* \in \in N(y : B)$. If $|V(B)| = 2$ or B is Hamiltonian (i.e. B is a cycle because G – and therefore B – is a minimal block) or B has a Hamiltonian (z, y)-path, the existence of such an h is obvious. So let B be a nontrivial block being not a cycle and therefore $0 < s(B) < s(G)$. Now consider an admissible (y, z)-path $\bar{w}$ in the minimal block B. Then $\{y, z\} \in E(B)$ is not possible because G is a minimal block. Thus $\{y, z\} \notin E(B)$, and we may suppose that $\bar{w}$ is not a Hamiltonian path in B. Then we proceed as in the proof of Theorem 1 (now for B instead of G, y instead of x, and z instead of y, of course) with the following modification: If $v(z : B) = 2$ and the only B-neighbour $z^* \notin V(\bar{w})$ of z belongs to a component C^* of $B - \bar{w}$ fulfilling $C^* \in \in C_2(\bar{w})$, we choose a Hamiltonian path h_{C^*} of $B_{C^*} - 0$ joining two B_{C^*} – neighbours of 0 and containing the edge $\{z^*, z'\} \in E(B)$ with some $z' \in N(z^* : B_{C^*})$; such an h_{C^*} exists because of $s(B_{C^*}) < s(B) < s(G)$. Hence, besides (6) also (6a) is fulfilled by the family $(h_C : C \in C_2(\bar{w}))$ having been chosen, and Corollary 1 and the Lemma of section 1 yield the required Hamiltonian path h.

So in every case there is a Hamiltonian path $\bar{q}$ in $H^2 - y$ joining an H-neighbour y^* of y and the vertex x and containing the edge $\{z, x\} \in E(H)$, where z denotes the only H-neighbour of x; write $\bar{q} = (y^*, \ldots, z, x)$ and $\tilde{q} := (y^*, \ldots, z) = \bar{q} - z$. Then $(\tilde{q}^{-1}, y)$ is a Hamiltonian path in $G^2 - x$ joining the G-neighbour $z \neq y$ of x with the G-neighbour y of x and containing an edge $\{y^*, y\} \in E(G)$. But this is a contradiction to the assumption on G. Thus Theorem 2 is proved. □

Now we can generalize Theorem 1 to

Theorem 1′ *Let G be a block and x, y, z vertices with $x \neq y$. Then there is a G-neighbour z' of z and a Hamiltonian path in G^2 joining x and y and containing the edge $\{z, z.\}$.*

Proof Form the graph H consisting of G, a new vertex 0 and the edges $\{0, x\}$ and $\{0, y\}$, and apply Theorem 2 to the nontrivial block H and the vertices 0 and z (instead of G and x and y, respectively). □

5. Let G be a connected graph, $z \in V(G)$ a cutpoint of G, further G_1 and G_2 two connected subgraphs of G forming a *non-trivial separation* of G with $V(G_1) \cap \cap V(G_2) = \{z\}$ (that means: $V(G_1) \cup V(G_2) = V(G), E(G_1) \cap E(G_2) = E(G(\{z\})) = = \emptyset$, $E(G_1) \cup E(G_2) = E(G)$, and $G_1, G_2 \neq G$) and h_1 and h_2 two paths in G_1^2 and G_2^2, respectively. Now we consider the following properties:

(12) h_1 is a Hamiltonian path in G_1^2 joining two different G-neighbours of z, and h_2 is a Hamiltonian path in $G_2^2 - z$ joining two different G-neighbours of z if $|V(G_2 - z)| \geqq 2$ and consisting of the only G-neighbour of z in G_2 if $|V(G_2 - z)| = 1$.

(13) h_1 is a Hamiltonian path in G_1^2 joining z with a G-neighbour of z, and h_2 is a Hamiltonian path in G_2^2 joining z with a G-neighbour of z.

Definition. $(h_1, G_1) \mapsto (h_2, G_2)$ iff property (12) is satisfied; $(h_1, G_1) \leftrightarrow (h_2, G_2)$ iff property (13) is satisfied.

Representing the paths h_1, h_2 by vertex-sequences we see immediately

Corollary 3. *If* $(h_1, G_1) \mapsto (h_2, G_2)$ *then*

$$h_1 + h_2 := (h_1, h_2, z'),$$

where z' *is the initial vertex of* h_1, *is a Hamiltonian cycle in* G^2. *If* $(h_1, G_1) \leftrightarrow (h_2, G_2)$ *then*

$$h_1 \cup h_2 := (h_1, h_2^{-1})$$

is a Hamiltonian cycle in G^2. □

(Of course, $(h_1, G_1) \leftrightarrow (h_2, G_2)$ holds iff $(h_2^{-1}, G_2) \leftrightarrow (h_1^{-1}, G_1)$; however, $(h_1, G_1) \mapsto (h_2, G_2)$ does not imply $(h_2, G_2) \mapsto (h_1, G_1)$.)

Corollary 4. If G_1, G_2 *form a non-trivial separation of a connected graph* G *with* $V(G_1) \cap V(G_2) = \{z\}$ *for some* $z \in V(G)$, *and if there exists a Hamiltonian cycle* h *in* G^2, *then there are paths* h_1 *and* h_2 *in* G_1 *and* G_2, *respectively, satisfying* $(h_1, G_1) \mapsto (h_2, G_2)$ *or* $(h_2, G_2) \mapsto (h_1, G_1)$ or $(h_1, G_1) \leftrightarrow (h_2, G_2)$.

Corollary 4 can be easily proved by considering the maximal G_1-sections and the maximal G_2-sections of h. □

Note that the *block-cutpoint-graph* $bc(G)$ of a connected graph G with $|V(G)| \geqq 2$ is a tree and that its endvertices (i.e. vertices of valency $\leqq 1$) in every case are representing some (maximal) blocks of G. (If G is a block then $bc(G)$ is a one-vertex-tree, and this vertex is also considered to be an endvertex of $bc(G)$.) We define $bc(G) := \varnothing$ if $|V(G)| \leqq 1$.

Theorem 3. *Let* G *be a connected graph with* $|V(G)| \geqq 3$ *satisfying the property that* G^2 *is Hamiltonian. Suppose that* $bc(G)$ *has at least one endvertex representing a non-trivial (maximal) block of* G. *Then there is a Hamiltonian cycle in* G^2 *containing some edge* $l \in E(G)$.

Proof: If G is a block then we only need apply Theorem 2 to G.

If G is not a block consider an endvertex of $bc(G)$ representing a non-trivial block G_1 of G, and let z be the cutpoint of G belonging to G_1. Then G_1 and $G_2 := G - (V(G_1) - \{z\}) = G((V(G) - V(G_1)) \cup \{z\})$ form a non-trivial separation of G with $V(G_1) \cap V(G_2) = \{z\}$, and Corollary 4 implies the existence of some h_1, h_2 such that $(h_1, G_1) \mapsto (h_2, G_2) \vee (h_2, G_2) \mapsto (h_1, G_1) \vee (h_1, G_1) \leftrightarrow (h_2, G_2)$ holds Because G_1 is a non-trivial block according to Theorem 2 there is a Hamiltonian path h_1' in $G_1^2 - z$ joining two G_1-neighbours (i.e. G-neighbours) of z and containing an edge $l \in E(G_1)$.

If $(h_1, G_1) \mapsto (h_2, G_2)$ then $((z, h_1'), G_1) \leftrightarrow ((z, h_2), G_2)$, and $(z, h_1') \cup (z, h_2)$ is a Hamiltonian cycle in G^2 containing $l \in E(G)$. If $(h_2, G_2) \mapsto (h_1, G_1)$ then $(h_2, G_2) \mapsto (h_1', G_1)$, and $h_2 + h_1'$ is a Hamiltonian cycle in G^2 containing $l \in E(G)$.

If $(h_1, G_1) \leftrightarrow (h_2, G_2)$ then $((z, h_1'), G_1 \leftrightarrow (h_2, G_2)$, and $(z, h_1') \cup h_2$ is a Hamiltonian cycle in G^2 containing $l \in E(g)$. □

For a connected graph G with $|V(G)| \geqq 3$ we form $G^{(1)} := G - V_1(G)$, where $V_1(G) := \{x \in V(G): v(x : G) = 1\}$. Then it is easy to show

Corollary 5. *Let G be a connected graph with $|V(G)| \geqq 3$ satisfying the property that G^2 is Hamiltonian. Suppose that all endvertices of $bc(G)$ are representing trivial (maximal) blocks of G. If $bc(G^{(1)}) = \emptyset$, or if $bc(G^{(1)})$ has an endvertex representing a trivial (maximal) block of $G^{(1)}$ then there is a Hamiltonian cycle in G^2 containin an edge $l \in E(G)$.* □

Now it remains the case that all endvertices of $bc(G)$ are representing trivial (maximal) blocks of G and all endvertices of $bc(G^{(1)})$ are representing non-tiivial (maximal) blocks of $G^{(1)}$. It is rather obvious that this problem could be solved if the following statement were true.

Conjecture: For every connected graph G with $|V(G)| \geqq 3$ fulfilling (14) and every vertex $x \in V(G^{(1)})$ with $\iota(x : G^{(1)}) = v(x : G)$ the existence of a Hamiltonian path in $G^2 - x$ joining two G-neighbours of x implies the existence of a Hamiltonian path in $G^2 - x$ joining two suitable G-neighbours of x and containing some edge of G.

(14) $G^{(1)}$ is a non-trivial block $\wedge$ for any different veitices $x, y \in V_1(G)$ their G-neighbours are different (i.e. $N(x : G) \neq N(y : G)$).

We remark that this Conjecture holds in case that $|V_1(G)| \leqq 1$ because of Theorem 2.

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