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ON EQUALITY OF EDGE-CONNECTIVITY AND MINIMUM DEGREE OF A GRAPH

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Dedicated to the memory of Milan Sekanina

Abstract. Sufficient conditions for the equality of edge-connectivity and minimum degree of a graph or a bipartite graph are presented. Also previously known conditions are surveyed.

Key words. Graph, bipartite graph, edge-connectivity, minimum degree, distance.
MS Classification. 05 C 40, 05 C 38.

Our terminology is based on [1]. Given a graph G, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively; $n := V(G)$ is its order; $\lambda(G)$ is its edge-connectivity and $\delta(G)$ is the minimum degree of G. The distance between two vertices x and y is denoted $d(x, y)$ and diam (G) is the diameter of G. The vertex neighbourhood of a vertex x is denoted $V(x)$. For brevity, λ often stands for $\lambda(G)$ and δ for $\delta(G)$.

It is well known that $\lambda(G) \leqq \delta(G)$ and one may ask for conditions on G ensuring the equality $\lambda(G)$ and $\delta(G)$. In this paper we give first a survey of known sufficient conditions and then provide some new ones.

§ 1. A SURVEY OF KNOWN RESULTS

In this section we will give a series of known conditions ensuring $\lambda = \delta$ in terms of various parameters of a graph. Each of these conditions can be also referred to as a result, in which case it is meant the assertion that the condition yields $\lambda = \delta$.

The first such condition is due to Chartrand [3]:

(1) $$\delta(G) \geqq [n/2].$$

This was refined by Lesniak [6]:

(2) $$\deg (x) + \deg (y) \geqq n - 1$$

for any pair of nonadjacent vertices x, y.

The following result of Plesník [7] is based on the diameter and obviously implies results (1) and (2):

(3) $$\operatorname{diam}(G) \leqq 2.$$

Goldsmith and Entringer [5] observed: It is also sufficient that for each vertex x of minimum degree, the vertices in the neighbourhood $V(x)$ have large degree sums; more precisely:

(4) $$\sum_{w \in V(x)} \deg(w) \geqq \begin{cases} [n/2]^2 - [n/2] & \text{for all even } n \text{ and} \\ & \text{for odd } n \leqq 15, \\ [n/2]^2 - 7 & \text{for odd } n \geqq 15. \end{cases}$$

This result implies (1) but is independent of (2) and (3). Indeed, the graph in Fig. 1 fulfils (2) and (3) but not (4); on the other hand the graph from Fig. 2 fulfils (4) but not (3) or (2).

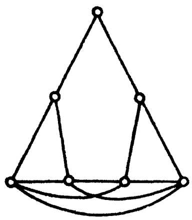

fig 1

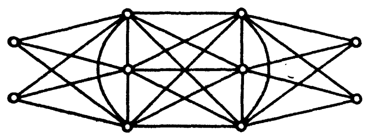

fig. 2

Bollobás [2] uses maximal graphs with $\delta > \lambda$ and derives several results. The following is a typical one and perhaps the most important of them: The degree sequence $d_1 \geqq d_2 \geqq \ldots \geqq d_n = \delta$ of G with $n \geqq 2$ fulfils

(5) $$\sum_{i=1}^{k} (d_i + d_{n-i}) \geqq kn - 1$$

for each k with $1 \leqq k \leqq \min\{[n/2] - 1, \delta\}$.

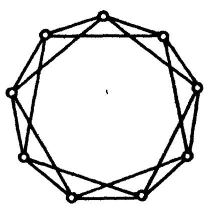

fig. 3

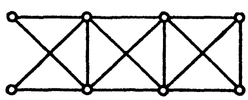

fig. 4

Although the result (5) implies (1) if n is even, in general (5) is independent of (1)–(4). This can be seen with aid of graphs in Figs. 3 and 4. The former fulfils (1)–(4) but not (5) and the latter works conversely.

Esfahanian [4] has given lower bounds on the edge-connectivity and, as a consequence, the following condition (Δ is the maximum degree of G and $D := \operatorname{diam}(G)$):

(6) $$n \geqq (\delta - 1)\frac{(\Delta - 1)^{D-1} + \Delta(\Delta - 2) - 1}{\Delta - 2} + 1.$$

The following similar condition is due to Soneoka, Nakada, Imase and Peyrat [8] and slightly improves (6):

(7) $$n > (\delta - 1)\frac{(\Delta - 1)^{D-1} + \Delta - 3}{\Delta - 2} + \Delta - 1.$$

As shown in [8] this bound is best possible (at least) for diameters $D = 3$ and 4. On the other hand, the graph of Fig. 3 does not fulfil (7) but fulfils (1)–(4).

Soneoka et al. [8] have established also the following generalization of (3) with g standing for the girth of G:

(8) $$D \leqq \begin{cases} g - 1 & \text{for } g \text{ odd}, \\ g - 2 & \text{for } g \text{ even}. \end{cases}$$

They show that this condition is best possible for an infinite number of values of δ when g is 4 or g is odd.

Figs. 2 and 4 provide examples of graphs fulfilling (4) and (5), respectively, and not fulfilling (8). Also there are examples in [8] where (7) works but (8) does not.

We conclude the survey by a result of Volkmann [9]:

(9) $$G \text{ is bipartite and } \delta \geqq \frac{n + 1}{4}.$$

Two disjoint copies of complete bipartite graph $K(n/4, n/4)$ provide an example demonstrating that this result is best possible. Moreover, it is not a corollary of (8), because there is a bipartite graph with $g = 4$ and $D > 2$ fulfilling (9) (e.g. with $n = 7$, $\delta = 2$). A generalization of (9) for p-partite graphs is given in [10].

§ 2. A NEW DISTANCE CONDITION

Here we show that the condition (3) can be slightly relaxed in sense that some distances can be greater than 2.

2.1. Theorem. *If in a connected graph no four vertices u_1, v_1, u_2, v_2 with*

(10) $$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) \geqq 3$$

exist, then $\lambda = \delta$.

Proof. For a contradiction consider a graph G fulfilling the distance condition with $\lambda < \delta$. Let E_0 be an edge cut of cardinality λ and let A and $\bar{A}$ be the vertex sets of the components arising after deleting E_0 from G. Further, let $A_1 \subseteq A$ and $\bar{A}_1 \subseteq \bar{A}$ be the sets of vertices incident with edges of E_0 and put $A_0 := A - A_1$ and $\bar{A}_0 := \bar{A} - \bar{A}_1$ (see Fig. 5). Denote the cardinalities of $A_0, A_1, \bar{A}_1$ and $\bar{A}_0$ by $a_0, a_1, \bar{a}_1$ and $\bar{a}_0$, respectively. Clearly $\lambda \geqq a_1$ and $\lambda \geqq \bar{a}_1$.

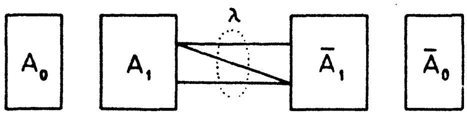

fig 5

The distance condition in our theorem implies that $a_0 \geqq 2$ and $\bar{a}_0 \geqq 2$ cannot hold simultaneously (otherwise there are $u_1, v_1 \in A_0$ and $u_2, v_2 \in \bar{A}_0$ fulfilling (10)). Thus owing to the reason of symmetry we can assume that $a_0 \leqq 1$. Each edge going from a vertex x of A ends in $A_0 \cup A_1$ or belongs to E_0. Since G has no loops or multiple edges, we have

$$\sum_{x \in A} \deg(x) \leqq \begin{cases} a_1(a_1 - 1) + \lambda \leqq \lambda(a_1 - 1) + \lambda = \lambda a_1 & \text{if} \quad a_0 = 0, \\ (a_1 + 1) a_1 + \lambda \leqq \lambda a_1 + a_1 + \lambda & \text{if} \quad a_0 = 1. \end{cases}$$

On the other hand

$$\sum_{x \in A} \deg(x) \geqq \begin{cases} a_1 \delta \geqq a_1(\lambda + 1) = \lambda a_1 + a_1 & \text{if} \quad a_0 = 0, \\ (a_1 + 1)\delta \geqq (a_1 + 1)(\lambda + 1) = \lambda a_1 + a_1 + \lambda + 1 & \text{if} \quad a_0 = 1. \end{cases}$$

Being compared these inequalities give a contradiction in either case. ∎

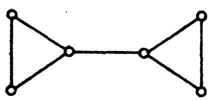

fig. 6

Fig. 6 shows that Theorem 2.1 is in a sense a best possible result. We have immediately:

2.2. Corollary. *If a connected graph G contains such a vertex v_0 that $d(x, y) \leqq 2$ for all $x, y \in V(G) - \{v_0\}$, then $\lambda = \delta$.*

§3. DISTANCE CONDITION FOR BIPARTITE GRAPHS

Now we will give an analog of Theorem 2.1 for bipartite graphs and show that it yields the result (9).

3.1. Theorem. *Let G be a bipartite graph with bipartition $[A, B]$. Then $\lambda = \delta$ whenever at least one of the following two conditions holds:*

(i) diam $(G) \leqq 4$ *and neither part contains four vertices u_1, v_1, u_2, v_2 such that*

(11) $$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) = 4.$$

(ii) *There exists a part P with $d(x, y) \leqq 2$ for all $x, y \in P$.*

Proof. Suppose for a contradiction that there is an edge cut E_0 with cardinality $\lambda < \delta$. Clearly $\lambda > 0$. After deleting the edges of E_0 from G, we obtain two components with vertex sets S and $\bar{S} := V(G) - S$. In accordance with Fig. 7, $A_1, \bar{A}_1$,

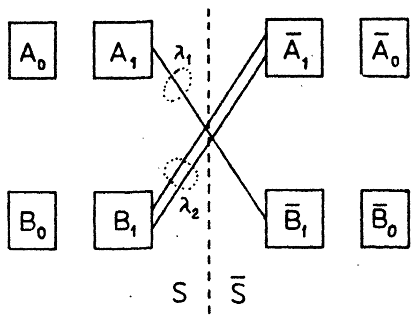

fig. 7

$B_1, \bar{B}_1$ denote the sets of vertices incident with some edge of the cut E_0 and lying in $A \cap S$, $A \cap \bar{S}$, $B \cap S$ and $B \cap \bar{S}$, respectively. The remaining vertices form the sets $A_0, \bar{A}_0, B_0$ and $\bar{B}_0$, i.e. $A_0 = A \cap S - A_1$, etc. Let the number of edges between A_1 and $\bar{B}_1$ be λ_1 and that between $\bar{A}_1$ and B_1 be λ_2. Thus $\lambda = \lambda_1 + \lambda_2$. Finally, let the cardinalities of the sets $A_0, \bar{A}_0, \ldots, \bar{B}_1$ be denoted by the corresponding small letters, i.e. $a_0, \bar{a}_0, \ldots, \bar{b}_1$. Clearly we have

$$a_1 \leqq \lambda_1, \; \bar{b}_1 \leqq \lambda_1, \; \bar{a}_1 \leqq \lambda_2, \; b_1 \leqq \lambda_2.$$

(i) First suppose that the condition (i) holds. We have to distinguish several cases, but owing to the reason of symmetry we can confine to the following:
Case 1: $a_0 \geqq 2$ and $\bar{a}_0 \geqq 2$. Then we can find $u_1, v_1 \in A_0$ and $u_2, v_2 \in \bar{A}_0$ fulfilling (11).

Thus without loss of generality in what follows we can suppose $a_0 \leqq 1$.
Case 2: $a_0 = b_0 = 0$. Then $a_1 + b_1 > 0$ and we can suppose that $A_1 \neq \emptyset$. For any $x \in A_1$ we have deg $(x) \leqq \lambda_1 + b_1$. On the other hand deg $(x) \geqq \delta \geqq \lambda + 1 = \lambda_1 + \lambda_2 + 1 \geqq \lambda_1 + b_1 + 1$, a contradiction.
Case 3: $a_0 = 0$, $b_0 \geqq 1$. Then for every $x \in B_0$ we have deg $(x) \leqq a_1 \leqq \lambda_1 < \delta$, what is impossible.
Case 4: $a_0 = 1$, $b_0 = 1$. Then for $x \in A_0$ we have deg $(x) \leqq b_1 + 1 \leqq \lambda_2 + 1$ and for $y \in B_0$ analogously deg $(y) \leqq a_1 + 1 \leqq \lambda_1 + 1$. Thus we can write $2\delta \leqq$

$\leq \deg(x) + \deg(y) \leq \lambda_1 + \lambda_2 + 2 = \lambda + 2 \leq \delta + 1$, which yields $\delta \leq 1$, i.e. $\lambda = 0$, a contradiction.

Case 5: $a_0 = 1$ and $\bar{b}_0 = 1$. Then because of Cases 3 and 4 we have $b_0 \geq 2$ and $\bar{a}_0 \geq 2$ (use the symmetry). For any $x \in B_0$ we get $\deg(x) \leq a_1 + 1 \leq \lambda_1 + 1$ and for any $y \in \bar{A}_0$ we have $\deg(y) \leq \bar{b}_1 + 1 \leq \lambda_1 + 1$. Hence $2(\lambda_1 + \lambda_2 + 1) = 2(\lambda + 1) \leq 2\delta \leq \deg(x) + \deg(y) \leq 2\lambda_1 + 2$, i.e. $\lambda_2 = 0$ and thus $\bar{a}_1 = b_1 = 0$. But for $u \in A_0$, $v \in \bar{B}_0$ we have $d(u, v) \geq 5$, which contradicts our assumption (i).

Case 6: $a_0 = 1$, $b_0 \geq 2$, $\bar{b}_0 \geq 2$. This is excluded by Case 1 (use the symmetry).

Having covered all possibilities the proof is completed if (i) is assumed to hold.

(ii) Now let the condition (ii) hold. We can assume that $P = A$, i.e. $d(x, y) \leq 2$ for all $x, y \in A$. This yields $d(u, v) \leq 4$ for all $u, v \in B$ and $d(x, u) \leq 3$ for all $x \in A$, $u \in B$. Hence diam $(G) \leq 4$. However, $d(x, y) = 4$ for any $x \in A_0, y \in \bar{A}_0$ (see Fig. 7). Therefore $a_0 \,.\, \bar{a}_0 = 0$ and we can assume that $a_0 = 0$. Then the considerations of above mentioned Cases 2 and 3 will work. ■

Fig. 8 shows that the assumption diam $(G) \leq 4$ cannot be dropped; on the other hand this condition is not sufficient if the rest of (i) does not hold (see Fig. 9).

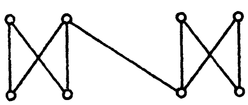

fig 8

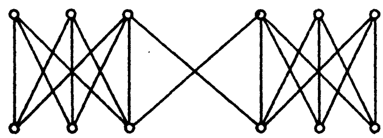

fig. 9

3.2. Corollary. *Let G be a bipartite graph with* diam $(G) \leq 4$. *If in either part P there exists such a vertex* v_0 *that* $d(x, y) \leq 2$ *for all* $x, y \in P - \{v_0\}$, *then* $\lambda = \delta$.

Proof. Immediately, since (i) is fulfilled. ■

3.3. Corollary. *If a bipartite graph G has* diam $(G) \leq 3$, *then* $\lambda = \delta$.

Proof. Now the condition (ii) is fulfilled because the distances in the same part are even. ■

Our theorem implies also the above mentioned result (9) of Volkmann [9]:

3.4. Corollary. *If G is a bipartite graph with* $\delta \geq (n + 1)/4$, *then* $\lambda = \delta$.

Proof. We will prove that the condition (ii) of Theorem 3.1 holds. Indeed, if it is not the case, then there exist vertices $x, y \in A$ with $d(x, y) > 2$ and so $V(x) \cap V(y) = \emptyset$. Consequently, B has at least $(n + 1)/4 + (n + 1)/4 = (n + 1)/2$ vertices. Symmetrically, A has at least $(n + 1)/2$ vertices too, what is impossible. ■

Examples from Figs. 10 and 11 show that there are no other relations between the conditions (i) and (ii) of Theorem 3.1 and (9). The graphs have $n = 11$, $\delta = 2$.

In Fig. 10 we have $d(1,4) = d(1,5) = 4$ and $d(x,y) \leqq 2$ for all $x, y \in A - \{1\}$. Also $d(6,10) = d(6,11) = 4$ and $d(x,y) \leqq 2$ for all $x, y \in B - \{6\}$. Thus (i) is fulfilled but neither (ii) nor (9) hold.

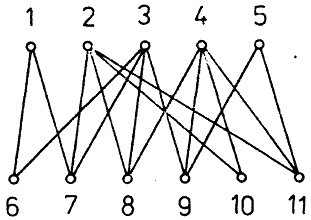

fig. 10

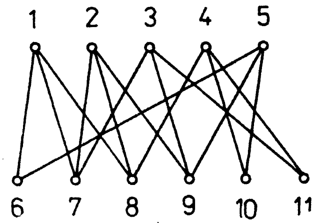

fig. 11

In Fig. 11 we see that $d(x,y) \leqq 2$ for all $x, y \in A$. Further $d(6,11) = d(7,10) = 4$. Thus (ii) holds but (i) and (9) do not.

Moreover, both these graphs have $g = 4$ and thus not even (8) is fulfilled.

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