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A NOTE ON NONLINEAR INTEGRAL EQUATIONS

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Abstract. This paper is devoted to the existence, uniqueness, and growth of the solutions of a system of triple nonlinear integral equations of the form:

$$x(t) + \int_0^t f(t, s, x(s), y(s), z(s))\, ds = p(t),$$

$$y(t) + \int_0^t g(t, s, x(s), y(s), z(s))\, ds = q(t),$$

$$z(t) + \int_0^t h(t, s, x(s), y(s), z(s))\, ds = r(t).$$

Key words. Volterra integral equations, existence, uniqueness.

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1. INTRODUCTION

The mathematical literature on this subject provided a good information concerning the existence, uniqueness, stability of various classes of nonlinear Volterra integral equations, see for example ([1–5]).

In this paper, we study the existence, uniqueness, and growth of solutions of a more general system of three nonlinear Volterra equations of the form:

$$\begin{aligned} x(t) &= p(t) - \int_0^t f(t, s, x(s), y(s), z(s))\, ds, \\ y(t) &= q(t) - \int_0^t g(t, s, x(s), y(s), z(s))\, ds, \qquad (1.1) \\ z(t) &= r(t) - \int_0^t h(t, s, x(s), y(s), z(s))\, ds, \end{aligned}$$

where $x(t) = (x^1(t), \dots, x^n(t)), y(t) = (y^1(t), \dots, y^n(t)),$
$z(t) = (z^1(t), \dots, z^n(t)), p(t) = (p^1(t), \dots, p^n(t)),$
$q(t) = (q^1(t), \dots, q^n(t)), r(t) = (r^1(t), \dots, r^n(t))$

are n-dimensional vector valued functions and continuous on $J = [0, T]$, $T > 0$ and

$$\begin{aligned}
f[t, s, x(s), y(s), z(s)] = {} & (f^1[t, s, x^1(s), \ldots, x^n(s), \\
& y^1(s), \ldots, y^n(s), z^1(s), \ldots, z^n(s)], \ldots, \\
& f^n[t, s, x'(s), \ldots, x^n(s), y^1(s), \ldots, y^n(s), \\
& z^1(s), \ldots, z^n(s)]), \\
g[t, s, x(s), y(s), z(s)] = {} & (g^1[t, s, x^1(s), \ldots, x^n(s), y^1(s), \ldots, \\
& y^n(s), z^1(s), \ldots, z^n(s)], \ldots, \\
& g^n[t, s, x^1(s), \ldots, x^n(s), \\
& y^1(s), \ldots, y^n(s), z^1(s), \ldots, z^n(s)])
\end{aligned}$$

and

$$\begin{aligned}
h[t, s, x(s), y(s), z(s)] = {} & (h^1[t, s, x^1(s), \ldots, x^n(s), y^1(s), \ldots \\
& y^n(s), z^1(s), \ldots, z^n(s)], \ldots, \\
& h^n t, s, x^1(s), \ldots, x^n(s), y^1(s), \ldots, \\
& y^n(s), z^1(s), \ldots, z^n(s)])
\end{aligned}$$

are n-dimensional vector valued functions defined and continuous on domain

$$D = \{0 \leqq s \leqq t \leqq T, \| x \|, \| y \|, \| z \| \leqq b;\ T, b < \infty\},$$

where $\| . \|$ denotes a convenient norm defined in R^n, the n-dimensional vector space.

2. MAIN RESULTS

In this paper, we shall employ the notation of upper and lower solutions to investigate the existence and uniqueness of the solutions of (1.1) and describe how these functions become upper and lower bounds of the solutions to the system (1.1). Throughout this paper, without further mention, we assume that all inequalities between vectors are componentwise.

Definition 1. *A triple of functions* $(\bar{u}, \bar{v}, \bar{w})$, $\bar{u}, \bar{v}, \bar{w} \in C[J, R^n]$ *is called an upper solution of* (1.1) *if*

$$\begin{aligned}
&\bar{u}(t) + \int_0^t f(t, s, \bar{u}(s), \bar{v}(s), \bar{w}(s))\, ds \geqq p(t), \\
(2.1) \qquad &\bar{v}(t) + \int_0^t g(t, s, \bar{u}(s), \bar{v}(s), \bar{w}(s))\, ds \geqq q(t), \\
&\bar{w}(t) + \int_0^t h(t, s, \bar{u}(s), \bar{v}(s), \bar{w}(s))\, ds \geqq r(t),
\end{aligned}$$

and similarly, a triple function $(\underline{u}, \underline{v}, \underline{w})$, $\underline{u}, \underline{v}, \underline{w} \in C[J, R^n]$ *is called a lower solution of* (1.1) *if the inequalities in* (2.1) *are reversed.*

Definition 2. *The pairs of functions* $(\bar{x}, \bar{y}, \bar{z})$ *and* $(\underline{x}, \underline{y}, \underline{z})$ *are called maximal and minimal solutions of* (1.1) *respectively, if every other solutions* (x, y, z) *of* (1.1) *satisfies the relations*:

$$\underline{x}(t) \leqq x(t) \leqq \bar{x}(t), \qquad \underline{y}(t) \leqq y(t) \leqq \bar{y}(t), \qquad \underline{z}(t) \leqq z(t) \leqq \bar{z}(t), \qquad t \in J.$$

Our main hypotheses are:

(i) The pairs of functions $(\underline{u}, \underline{v}, \underline{w})$ and $(\bar{u}, \bar{v}, \bar{w})$, $\underline{u}, \underline{v}, \underline{w}, \bar{u}, \bar{v}, \bar{w} \in C[J, R^n]$ with $\underline{u}(t) \leqq \bar{u}(t)$, $\underline{v}(t) \leqq \bar{v}(t)$, $\underline{w}(t) \leqq \bar{w}(t)$ for all $t \in J$ are lower and upper solutions of (1.1).

(ii) For each i, $f_i(t, s, x, y, z)$, $g_i(t, s, x, y, z)$, $r_i(t, s, x, y, z)$, $i \in \{1, \ldots, n\}$ are monotone decreasing in x, y and z for fixed $t, s \in J$, and

$$\underline{u}(t) \leqq x \leqq \bar{u}(t), \qquad \underline{v}(t) \leqq y \leqq \bar{v}(t), \qquad \underline{w}(t) \leqq z \leqq \bar{w}(t), \qquad \text{on } J.$$

(iii) For each $t, s \in J$, $x, y, z, k \in R^n$,

$$\forall \|x\|, \|y\|, \|z\|, \|k\| \leqq b < \infty,$$
$$\|f(t, s, x, y, z) - f(t, s, k, y, z)\| \leqq A \|x - k\|,$$
$$\|g(t, s, x, y, z) - g(t, s, x, k, z)\| \leqq B \|y - k\|,$$
$$\|h(t, s, x, y, z) - g(t, s, x, y, k)\| \leqq C \|z - k\|,$$

where A, B, C are non-negative constants.

We defined the following sequence which will be used in proving the existence of maximal solution of the system (1.1):

$$\begin{aligned} u_n(t) + \int_0^t f(t, s, u_{n-1}(s), v_{n-1}(s), w_{n-1}(s))\, ds &= p(t), \\ v_n(t) + \int_0^t g(t, s, u_{n-1}(s), v_{n-1}(s), w_{n-1}(s))\, ds &= q(t), \\ w_n(t) + \int_0^t h(t, s, u_{n-1}(s), v_{n-1}(s), w_{n-1}(s))\, ds &= r(t), \end{aligned} \tag{2.2}$$

with $u_0 = u$, $v_0 = v$, $w_0 = w$ on J, $\forall \|u\|, \|v\|, \|w\| \leqq b$.

Now, we will state and prove the existence of a maximal and minimal solution of (1.2).

Theorem 2.1. *Let the assumptions* (i) *and* (ii) *hold. Then, the sequence* $\{\bar{u}_n, \bar{v}_n, \bar{w}_n\}$ *given by* (2.2) *with* $\bar{u}_0 = \bar{u}, \bar{v}_0 = \bar{v}, \bar{w}_0 = \bar{w}$, *converges uniformly from above to the maximal solution* $(\bar{x}, \bar{y}, \bar{z})$ *of* (1.1) *while the sequence* $\{\underline{u}_n, \underline{v}_n, \underline{w}_n\}$ *given by* (2.2) *with* $\underline{u}_0 = \underline{u}$, $\underline{v}_0 = \underline{v}$, $\underline{w}_0 = \underline{w}$, *converges uniformly from below to a minimal solution*

$(\underline{x}, \underline{y}, \underline{z})$ *of* (1.1). *Furthermore, if any solutions of* (1.1) *such that*

$$\underline{u} \leqq x \leqq \bar{u}, \qquad \underline{v} \leqq y \leqq \bar{v}, \qquad \underline{w} \leqq z \leqq \bar{w},$$

then,

$$\underline{u} \leqq \underline{u}_1 \leqq \underline{u}_2 \leqq \ldots \leqq \underline{u}_n \leqq \ldots \leqq \underline{x} \leqq x \leqq \bar{x} \leqq \ldots \leqq \bar{u}_n \leqq \ldots \leqq \bar{u}_2 \leqq \bar{u}_1 \leqq \bar{u},$$
$$\underline{v} \leqq \underline{v}_1 \leqq \underline{v}_2 \leqq \ldots \leqq \underline{v}_n \leqq \ldots \leqq \underline{y} \leqq y \leqq \bar{y} \leqq \ldots \leqq \bar{v}_n \leqq \ldots \leqq \bar{v}_2 \leqq \bar{v}_1 \leqq \bar{v},$$
$$\underline{w} \leqq \underline{w}_1 \leqq \underline{w}_2 \leqq \ldots \leqq \underline{w}_n \leqq \ldots \leqq \underline{z} \leqq z \leqq \bar{z} \leqq \ldots \leqq \bar{w}_n \leqq \ldots \tag{2.3}$$
$$\ldots \leqq \bar{w}_2 \leqq \bar{w}_1 \leqq \bar{w} \qquad \text{on } J.$$

Proof. Define $R_i(t) = \underline{u}_{i1}(t) - \underline{u}_i(t)$, f^i monotone nondecreasing in $\underline{u}, \underline{v}, \underline{w}$, for fixed $t, s \in J$ and p^i, $i = 1, \ldots, n$ vector valued functions and continuous on $J = [0, T]$, $T > 0$. Hence, we have

$$R_i(t) \geqq p^i(t) - \int_0^t f^i(t, s, \underline{u}(s), \underline{v}(s), \underline{w}(s))\, ds - p^i(t) +$$
$$+ \int_0^t f^i(t, s, \underline{u}(s), \underline{v}(s), \underline{w}(s))\, ds = 0,$$

which implies $\underline{u}(t) \leqq \underline{u}_{i1}(t)$ on J. In the same way, we can show that

$$\underline{v}(t) \leqq \underline{v}_{i1}(t) \quad \text{and} \quad \underline{w}(t) \leqq w_{i1}(t) \quad \text{on } J.$$

By following on induction argument, we have:

$$\underline{u}_{n-1}(t) \leqq \underline{u}_n(t), \qquad \underline{v}_{n-1}(t) \leqq \underline{v}_n(t), \qquad \underline{w}_{n-1}(t) \leqq \underline{w}_n(t),$$

for all n on J. By the same technique, we can show that:

$$\bar{u}_n(t) \leqq \bar{u}_{n-1}(t), \qquad \bar{v}_n(t) \leqq \bar{v}_{n-1}(t), \qquad \bar{w}_n(t) \leqq \bar{w}_{n-1}(t)$$

for all n on J.

Also, now define $R_i(t) = \underline{u}_i(t) - \bar{u}_i(t)$, and using the fact that:

$$\underline{u}(t) \leqq \bar{u}(t), \qquad \underline{v}(t) \leqq \bar{v}(t), \qquad \underline{w}(t) \leqq \bar{w}(t), \qquad \forall\, t \in J,$$

we have:

$$R_i(t) \leqq p^i(t) - \int_0^t f^i(t, s, \bar{u}(s), \bar{v}(s), \bar{w}(s))\, ds - p^i(t) +$$
$$+ \int_0^t f^i(t, s, \bar{u}(s), \bar{v}(s), \bar{w}(s))\, ds = 0,$$

where p^i, $i = 1, \ldots, n$ vector valued functions, continuous on $J = [0, T]$, $T > 0$, f^i monotone nondecreasing in $\bar{u}, \bar{v}, \bar{w}$ for fixed $t, s \in J$, which implies $\underline{u}_i \leqq \bar{u}_i(t)$ on J.

As before, by using an induction argument, we have $\underline{u}_n(t) \leqq \bar{u}_n(t)$ for all n on J. Similarly, we can show that $\underline{v}_n(t) \leqq \bar{v}_n(t)$, $\underline{w}_n(t) \leqq \bar{w}_n(t)$ foı all n on J. Thus, the sequence $\{(\underline{u}_n, \underline{v}_n, \underline{w}_n)\}$, $\{(\bar{u}_n, \bar{v}_n, \bar{w}_n)\}$ are monotone nondecreasing respectively and

$$\underline{u} \leqq \underline{u}_n \leqq \bar{u}_n \leqq \bar{u}, \qquad \underline{v} \leqq \underline{v}_n \leqq \bar{v}_n \leqq \bar{v}, \qquad \underline{w} \leqq \underline{w}_n \leqq \bar{w}_n \leqq \bar{w} \qquad \text{on } J.$$

Furthermore, using standard arguments used by Candre and Davis [1], it follows that these sequence converge unifoımly and monotonically to the solutions $(\underline{x}, \underline{y}, \underline{z})$ and $(\bar{x}, \bar{y}, \bar{z})$ of (1.1).

Let the triple (x, y, z) be any solution of (1.1) such that $\underline{u} \leqq x \leqq \bar{u}$, $\underline{v} \leqq y \leqq \bar{v}$, $\underline{w} \leqq z \leqq \bar{w}$. Then, by the induction argument, it is easily seen that $x \leqq \bar{u}_n$, $y \leqq \bar{v}_n$, $z \leqq \underline{w}_n$, for every $n = 0, 1, 2, 3, \ldots$

Thus, we have:

$$\underline{x} \leqq x \leqq \bar{x}, \qquad \underline{y} \leqq y \leqq \bar{y} \qquad \text{and} \qquad \underline{z} \leqq z \leqq \bar{z}.$$

This shows that the triple $(\bar{x}, \bar{y}, \bar{z})$ is a maximal solution and the triple $(\underline{x}, \underline{y}, \underline{z})$ is a minimal solution of (1.1). This completes the proof of the theorem.

Remark 1. We remark that the maximal solutions established in the above theorem are not necessarily the same. If f, g and h in the system (1.1) satisfy the conditions in our hypothesis (iii), then we have the following uniqueness result.

Theorem 2.2. *If the hypotheses* (i), (ii) *and* (iii) *hold, then the maximal solution* $(\bar{x}, \bar{y}, \bar{z})$ *and the minimal* ˎ*olution* $(\underline{x}, \underline{y}, \underline{z})$ *obtained in theorem* 2.1 *coincıde on* J, *that is*

$$\bar{x}(t) = \underline{x}(t), \qquad \bar{y}(t) = \underline{y}(t), \qquad \bar{z}(t) = \underline{z}(t) \qquad \text{foı } t \in J.$$

Proof. Let $(\bar{x}, \bar{y}, \bar{z})$ and $(\underline{x}, \underline{y}, \underline{z})$ be the maximal and minimal solutions of the systems (1.1) respectively. Then, we have:

$$\bar{x}(t) + \int_0^t f(t, s, \bar{x}(s), \bar{y}(s), \bar{z}(s))\, ds = p(t), \tag{2.4}$$

$$\bar{y}(t) + \int_0^t g(t, s, \bar{x}(s), \bar{y}(s), \bar{z}(s))\, ds = q(t), \tag{2.5}$$

$$\bar{z}(t) + \int_0^t h(t, s, \bar{x}(s), \bar{y}(s), \bar{z}(s))\, ds = r(t), \tag{2.6}$$

and

$$\underline{x}(t) + \int_0^t f(t, s, \underline{x}(s), \underline{y}(s), \underline{z}(s))\, ds = p(t), \tag{2.7}$$

$$\underline{y}(t) + \int_0^t g(t, s, \underline{x}(s), \underline{y}(s), \underline{z}(s))\, ds = q(t), \tag{2.8}$$

$$\underline{z}(t) + \int_)^t h(t, s, \underline{x}(s), \underline{y}(s), \underline{z}(s))\, ds = r(t), \tag{2.9}$$

Define $R(t) = \| \underline{x}(t) - \bar{x}(t) \|$. Then from (2.4), (2.7), (2.3), and hypotheses (ii) and (iii), we obtain

$$R(t) \leqq A \int_0^t R(s)\,ds.$$

Note that $R(0) = 0$. This implies $\bar{x}(t) = \underline{x}(t)$ foı $t \in J$. Similarly, we can prove that $\bar{y}(t) = \underline{y}(t)$ and $\bar{z}(t) = \underline{z}(t)$ for all $t \in J$. This completes the proof of the theorem.

Remark 2. The above technique can be easily applied to similar system of n-integral equations.

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