Josef Šlapal

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ON THE AXIOMS PRESERVED BY MODIFICATIONS OF TOPOLOGIES WITHOUT AXIOMS

JOSEF ŠLAPAL

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Abstract. Under a topology on a set P we understand a topology without axioms on P, i.e. a mapping u: $\exp P \to \exp P$. If u and v are two topologies on P, then u is called finer (coarser) than v if $uX \subseteq vX$ ($vX \subseteq uX$) holds for every subset $X \subseteq P$. Let f be a topological property. Then a topology possessing f is said to be an f-topology. The coarsest (finest) of all f-topologies on P which are finer (coarser) than a given topology u on P is called the lower (upper) f-modification of u. We consider the topological properties f given by means of the well-known axioms O, I, M, A, U, K, B^*, B, S. In the present paper the axioms are determined which are preserved by the ndividual f-modifications.

Key words. Topology (without axioms), O, I, M, A, U, K, B^*, B, S-axioms, lower modification of a topology, upper modification of a topology.

MS Classification. Primary 54 A 05, 54 A 10

INTRODUCTION

If P is a set and $u : \exp P \to \exp P$ a mapping fulfilling the following three axioms (1)$u\emptyset = \emptyset$, (2)$X \subseteq P \Rightarrow X \subseteq uX$, (3)$X \subseteq Y \subseteq P \Rightarrow uX \subseteq uY$, then u is called a Čech topology on P. Some modifications of Čech topologies are studied in [2], [3], [4], and the axioms pieserved by these modifications are fully dealt in [5]. We shall consider the topologies obtained by omitting all three axioms in the definition of Čech topologies. Some modifications of these generalized topologies are investigated in [6]. The piesent papei completes [6] by determining the axioms preserved by the modifications described in [6]. The results of both these papers generalize certain results of [3], [4], [5] and have many applications because generalized topologies occur in various branches of mathematics (see [6]).

A topology without axioms (briefly a topology) u on a set P is a mapping $u : \exp P \to \exp P$. A set P provided with a topology u on P is called a topological space and denoted by (P, u). We consider the following axioms for topologies u on a given set P:

1. $u\emptyset = \emptyset$	*O-axiom,*
2. $X \subseteq P \Rightarrow X \subseteq uX$	*I-axiom,*
3. $X \subseteq Y \subseteq P \Rightarrow uX \subseteq uY$	*M-axiom,*
4. $X, Y \subseteq P \Rightarrow u(X \cup Y) \subseteq uX \cup uY$	*A-axiom,*
5. $X \subseteq P \Rightarrow uuX \subseteq uX$	*U-axiom,*
6. $x, y \in P,\ x \in u\{y\},\ y \in u\{x\} \Rightarrow x = y$	*K-axiom,*
7. $x, y \in P,\ x \in u\{y\} \Rightarrow y \in u\{x\}$	*B*-axiom,*
8. $x \in P \Rightarrow u\{x\} \subseteq \{x\}$	*B-axiom,*
9. $\emptyset \neq X \subseteq P \Rightarrow uX \subseteq \bigcup_{x \in X} u\{x\}$	*S-axiom.*

Let $f \in \{O, I\ M, A, U, K, B^*, B, S\}$. If a topology u fulfils the f-axiom, then it is called an *f-topology*. We talk about an *fg-topology* u if u is an f-topology and g-topology simultaneously. Analogously *fgh-topology* etc. can be defined. For a topology u on a set P, the *lower f-modification u_f* of u (i.e. the coarsest f-topology on P which is finer than u) and the *upper f-modification u^f* of u (i.e. the finest f-topology on P which is coarser than u) are studied in [6]. Of course, for given topologies u and v on a set P we say that u is *finer* than v or that v is *coarser* than u when $X \subseteq P \Rightarrow \Rightarrow uX \subseteq vX$.

1. O-MODIFICATION

Theorem 1. *Let (P, u) be a topological space, let $f \in \{O, I, M, A, U, K, B^*, B, S\}$, and let u be an f-topology. Then it holds:*

a) u_0 is an f-topology.

b) If u^0 exists, then it is an f-topology.

Proof. *a)* According to [6], Theorem 1, *a)*, u_0 always exists and it is defined by:

$$\emptyset \neq X \subseteq P \Rightarrow u_0X = uX,$$
$$u_0\emptyset = \emptyset.$$

Let us suppose that u is an f-topology. For $f = 0$ the asseition is obvious, and for $f = I$ its proof is trivial.

$f = M$: Let $X \subseteq Y \subseteq P$ be subsets. If $X \neq \emptyset$, then $u_0X = uX \subseteq uY = u_0Y$. Otherwise, supposing $X = \emptyset$ we have $u_0X = \emptyset \subseteq u_0Y$. Thus, u_0 is an M-topology.

$f = A$: Let $X, Y \subseteq P$ be subsets. If $X \neq \emptyset \neq Y$, then $u_0(X \cup Y) = u(X \cup Y) \subseteq \subseteq uX \cup uY = u_0X \cup u_0Y$. If one of the sets X, Y is non-empty and the other is empty, then clearly $u_0(X \cup Y) \subseteq u_0X \cup u_0Y$. Finally, let $X = Y = \emptyset$. Then $u_0(X \cup Y) = u_0\emptyset = \emptyset \subseteq u_0X \cup u_0Y$. Thus u_0 is an A-topology.

$f = U$: Let $X \subseteq P$ be a subset and suppose $X \neq \emptyset$. Consequently, $u_0u_0X = = u_0uX$. Now, if $uX \neq \emptyset$, then $u_0uX = uuX \subseteq uX = u_0X$, and if $uX = \emptyset$, then

$u_0uX = u_0\emptyset = \emptyset \subseteq u_0X$. Otherwise, supposing $X = \emptyset$ we have $u_0u_0X = \emptyset \subseteq u_0X$. Thus u_0 is a U-topology.

$f \in \{K, B^*, B\}$: In these cases the assertion follows from the fact that $u_0\{x\} = u\{x\}$ for every $x \in P$.

$f = S$: As evident, u_0 is an S-topology.

b) According to [6], Theorem 1, *b*), u^0 exists iff u is an O-topology, and then $u^0 = u$.

2. I-MODIFICATION

Theorem 2. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{O, I, M, A, U, K, B^*, B, S\}$ and u is na f-topology and if u_I exists, then u_I is an f-topology.

b) If $f \in \{O, I, M, A, AU, K, B^*, B, S\}$ and u is an f-topology, then u^I is an f-topology.

Proof. *a*) According to [6], Theorem 2, *a*), u_I exists iff u is an I-topology, and then $u_I = u$.

b) According to the same Theorem, *b*), u^I always exists and it is defined by:

$$X \subseteq P \Rightarrow u^IX = uX \cup X.$$

Let us suppose that u is an f-topology. For $f = O$ the proof is trivial, and for $f = I$ the assertion is obvious.

$f = M$: Let $X \subseteq Y \subseteq P$ be subsets. Then $u^IX = uX \cup X \subseteq uY \cup Y = u^IY$, and thus u^I is an M-topology.

$f = A$: Let $X, Y \subseteq P$ be subsets. Then $u^I(X \cup Y) = u(X \cup Y) \cup X \cup Y \subseteq uX \cup X \cup uY \cup Y = u^IX \cup u^IY$. Hence u^I is an A-topology.

$f = AU$: Let $X \subseteq P$ be a subset. Then $u^Iu^IX = u^I(uX \cup X) = uX \cup X \cup u(uX \cup X) \subseteq uX \cup X \cup uuX \cup uX = uX \cup X = u^IX$. Thus u^I is a U-topology and, according to the previous part of the proof, u^I is an AU-topology.

$f = K$: Let $x, y \in P$ be points, $x \in u^I\{y\}$, $y \in u^I\{x\}$. Then $x \in u\{y\}$ or $x = y$, and $y \in u\{x\}$ or $x = y$. Consequently, $x = y$ and u^I is a K-topology.

$f = B^*$: Let $x, y \in P$ be points, $x \in u^I\{y\}$. Then $x \in u\{y\}$ *or* $x = y$ and therefore $y \in u\{x\}$ or $y = x$, i.e. $y \in u^I\{x\}$. Hence, u^I is a B^*-topology.

$f = B$: In this case the assertion is proved already since u is a B-topology iff it is a KB^*-topology.

$f = S$: Let $\emptyset \neq X \subseteq P$ be a subset. Then $u^IX = uX \cup X = \bigcup_{x \in X} u\{x\} \cup \bigcup_{x \in X} \{x\} = \bigcup_{x \in X} (u\{x\} \cup \{x\}) = \bigcup_{x \in X} u^I\{x\}$, and thus u^I is an S-topology.

Remark 1. The upper I-modification of a U-topology need not be a U-topology in general — see the following example: Let $P = \{x, y, z\}$ and put $uX = X$ for any subset $X \subseteq P$ fulfilling $\{x\} \neq X \neq \{x, y\}$, $u\{x\} = \{y\}$, $u\{x, y\} = P$. Then clearly u is a U-topology (but not an A-topology) on P. It is $u^I u^I\{x\} = u(u\{x\} \cup \{x\}) \cup \cup u\{x\} \cup \{x\} = P$ while $u^I\{x\} = u\{x\} \cup \{x\} = \{x, y\}$. Therefore u^I is not a U-topology.

3. M-MODIFICATION

Theorem 3. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{O, I, M, A, U, K, B, S\}$ and u is an f-topology, then u_M is an f-topology.

b) If $f \in \{O, I, M, A, OK, IK, OB^*, IB^*, OB, IB, S\}$ and u is an f-topology, then u^M is an f-topology.

Proof. *a*) According to [6], Theorem 3, *a*), u_M always exists and it is defined by:

$$X \subseteq P \Rightarrow u_M X = \bigcap_{X \subseteq Z \subseteq P} uZ.$$

Let us suppose that u is an f-topology. For $f = O$ the proof is trivial.

$f = I$: Let $X \subseteq P$ be a subset. Since $X \subseteq Z \subseteq P \Rightarrow X \subseteq uZ$ for any set Z, we have $X \subseteq \bigcap_{X \subseteq Z \subseteq P} uZ = u_M X$. Thus, u_M is an I-topology.

$f = M$: Obviously, u_M is an M-topology.

$f = A$: Let $X, Y \subseteq P$ be subsets, $z \in u_M(X \cup Y)$ a point. Then $z \in uZ$ for any set Z with $X \cup Y \subseteq Z \subseteq P$. Suppose $z \notin u_M X \cup u_M Y$. Consequently, there exist sets X', Y' with $X \subseteq X' \subseteq P$, $Y \subseteq Y' \subseteq P$ such that $z \notin uX'$ and $z \notin uY'$. Thus, $z \notin uX' \cup uY'$, and we have $z \notin u(X' \cup Y')$ since $u(X' \cup Y') \subseteq uX' \cup uY'$. But this is a contradiction because $X \cup Y \subseteq X' \cup Y' \subseteq P$. Therefore $z \in u_M X \cup u_M Y$, and the inclusion $u_M(X \cup Y) \subseteq u_M X \cup u_M Y$ is proved. Hence, u_M is an A-topology.

$f = U$: Let $X \subseteq P$ be a subset, $x \in u_M u_M X$ a point. Since $u_M u_M X = \bigcap_{\bigcap_{X \subseteq Z \subseteq P} uZ \subseteq Y \subseteq P} uY$, we have $x \in uY$ for each set Y fulfilling $\bigcap_{X \subseteq Z \subseteq P} uZ \subseteq Y \subseteq P$. Now, let V be a set with $X \subseteq V \subseteq P$. Then $\bigcap_{X \subseteq Y \subseteq P} uY \subseteq uV \subseteq P$, and thus $x \in uuV \subseteq uV$. Therefore $x \in \in \bigcap_{X \subseteq V \subseteq P} uV = u_M X$. Consequently, $u_M u_M X \subseteq u_M X$. Hence, u_M is a U-topology.

$f = K$: Let $x, y \in P$ be points, $x \in u_M\{y\}$, $y \in u_M\{x\}$. Then $x \in u\{y\}$, $y \in u\{x\}$, and therefore $x = y$. Thus u_M is a K-topology.

$f = B$: Let $x \in P$ be a point. Then $u_M\{x\} = \bigcap_{x \in Z \subseteq P} uZ \subseteq u\{x\} \subseteq \{x\}$, so that u_M is a B-topology.

$f = S$: Let $X \subseteq P$ be a subset, $y \in u_M X$ a point. Then $y \in uZ$ for any set Z with $X \subseteq Z \subseteq P$. Suppose $z \notin \bigcup_{x \in X} u_M\{x\}$. Consequently, for any point $x \in X$ there exists a set Y_x with $x \in Y_x \subseteq P$ such that $y \notin uY_x$. Thus, $y \notin \bigcup_{x \in X} uY_x$, and we have $y \notin$

$\notin u \bigcup_{x \in X} Y_x$, since $u \bigcup_{x \in X} Y_x \subseteq \bigcup_{x \in X} u Y_x$ follows immediately from the S-axiom. But this is a contradiction because $X \subseteq \bigcup_{x \in X} Y_x \subseteq P$. Therefore $z \in \bigcup_{x \in X} u_M\{x\}$, and the inclusion $u_M X \subseteq \bigcup_{x \in X} u_M\{x\}$ is proved. Hence, u_M is an A-topology.

b) Accoiding to [6], Theorem 3, *b*), u^M always exists and it is defined by:

$$X \subseteq P \Rightarrow u^M X = \bigcup_{Z \subseteq X} uZ.$$

Let us suppose that u is an f-topology. For $f = O$ the proof is trivial.

$f = I$: Let $X \subseteq P$ be a set. Then $X = \bigcup_{Z \subseteq X} Z \subseteq \bigcup_{Z \subseteq X} uZ = u^M X$. Therefore u^M is an I-topology.

$f = M$: Obviously, u^M is an M-topology.

$f = A$: Let $X, Y \subseteq P$ be subsets. Then $u^M(X \cup Y) = \bigcup_{Z \subseteq X \subseteq Y} uZ$. But for every subset $Z \subseteq X \cup Y$ some subsets $X' \subseteq X$ and $Y' \subseteq Y$ ($X' = X \cap Z$, $Y' = Y \cap Z$) exist fulfilling $Z = X' \cup Y'$ and consequently, $uZ \subseteq uX' \cup uY'$. This implies $\bigcup_{Z \subseteq X \cup Y} uZ \subseteq$ $\subseteq \bigcup_{X' \subseteq X} uX' \cup \bigcup_{Y' \subseteq Y} uY' = u^M X \cup u^M Y$. Thus u^M is an A-topology.

$f \in \{OK, IK\}$: Let $x, y \in P$ be points, $x \in u^M\{y\}$, $y \in u^M\{x\}$. Then $x \in u\emptyset \cup u\{y\} =$ $= u\{y\}$ and $y \in u\emptyset \cup u\{x\} = u\{x\}$. Therefore $x = y$, and u^M is a K-topology. Hence, u^M is an OK-topology for $f = OK$, and it is an IK-topology for $f = IK$.

$f \in \{OB^*, IB^*\}$: Let $x, y \in P$ be points, $x \in u^M\{y\}$. Then $x \in u\emptyset \cup u\{y\} = u\{y\} \Rightarrow$ $\Rightarrow y \in u\{x\} = u\emptyset \cup u\{x\} = u^M\{x\}$. Therefore, u^M is a B^*-topology. Thus, u^M is an OB^*-topology for $f = OB^*$, and it is an IB^*-topology for $f = IB^*$.

$f \in \{OB, IB\}$: As u is a B-topology iff it is a KB^*-topology, the assertion is proved in these cases already.

$f = S$: Let $\emptyset \neq X \subseteq P$ be a subset. Then $u^M X = \bigcup_{Z \subseteq X} uZ = u\emptyset \cup \bigcup_{\emptyset \neq Z \subseteq X} uZ \subseteq$ $\subseteq u\emptyset \cup \bigcup_{\emptyset \neq Z \subseteq X} \bigcup_{x \in Z} u\{x\} = u\emptyset \cup \bigcup_{x \in X} u\{x\} = \bigcup_{x \in X} (u\emptyset \cup u\{x\}) = \bigcup_{x \in X} u^M\{x\}$. Hence, u^M is an S-topology.

Remark 2. *a*) The lower M-modification of a B^*-topology need not be a B^*-topology in general – sée the following example: Let $P = \{x, y\}$, $u\emptyset = \emptyset$, $u\{x\} = \{y\}$, $u\{y\} = \{x\}$, $uP = \{y\}$. Then u is a B^*-topology and $u_M\{x\} = \{y\}$ while $u_M\{y\} =$ $= \emptyset$. Thus, u_M is not a B^*-topology.

b) The next example showes that the upper M-modification of a U-topology need not be a U-topology in general: Let $P = \{x, y, z\}$, $u\emptyset = u\{y\} = u\{z\} =$ $= u\{y, z\} = uP = \emptyset$, $u\{x\} = \{x\}$, $u\{x, y\} = \{z\}$, $u\{x, z\} = \{y\}$. Then u is a U-topology, and $u^M\{x, y\} = \{x, z\}$ while $u^M u^M\{x, y\} = u^M\{x, z\} = \{x, y\}$. Therefore u^M is not a U-topology.

But, if a U-topology u on a set P fulfils $uZ \subseteq Z$ for any subset $Z \subseteq P$, then u^M

is a U-topology. Namely, $Z \subseteq P$, $uZ \subseteq Z \Rightarrow u^M u^M X = \bigcup_{Y \subseteq u^M X} uY = \bigcup_{\substack{Y \subseteq \bigcup uZ \\ Z \subseteq X}} uY \subseteq$
$\subseteq \bigcup_{\substack{Y \subseteq \bigcup Z \\ Z \subseteq X}} uY = \bigcup_{Y \subseteq X} uY = u^M X$ for any subset $X \subseteq P$. Especially, if a U-topology u is moreover a BS-topology, then u^M is a U-topology.

Now we shall show that the assertion $b)$ of Theorem 3 does not hold for any $f \in \{K, B^*, B\}$ in general. On that account, let $P = \{x, y, z\}$, $u\emptyset = \{y, z\}$, $u\{x\} = \{x\}$, $u\{y\} = u\{z\} = \emptyset$, $u\{x, y\} = u\{x, z\} = u\{y, z\} = uP = P$. Then u is a B-topology, i.e. a KB^*-topology. Since $u^M\{x\} = P$ and $u^M\{y\} = u^M\{z\} = \{y, z\}$, u^M is neither K-topology nor B^*-topology. Consequently, u^M is not a B-topology.

4. A-MODIFICATION

Theorem 4. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{O, I, M, A, MU, K, B^*, B, S\}$ and u is an f-topology, then u_A is an f-topology.

b) If $f \in \{O, I, M, A, U, K, B^*, B, S\}$ and u is an f-topology and if u^A exists, then u^A is an f-topology.

Proof. $a)$ According to [6], Theorem 4, $a)$, u_A always exists and it is defined by:

$$X \subseteq P \Rightarrow u_A X = \bigcap \{Z \subseteq P \mid Z = \bigcup_{i=1}^{m} uX_i,\ \bigcup_{i=1}^{m} X_i = X,\ \mathrm{m} \in \mathbf{N}\},$$

where $\mathbf{N}$ denotes the set of all positive integers. Let us suppose that u is an f-topology. For $f = O$ the proof is trivial.

$f = I$: Let $X \subseteq P$ be a subset, $x \in X$ a point. Then for any system of sets $\{X_i \mid i = 1, \dots, \mathrm{m}\}$ ($\mathrm{m} \in \mathbf{N}$) fulfilling $\bigcup_{i=1}^{m} X_i = X$, there exists an index $i_0 \in \{1, \dots, \mathrm{m}\}$ such that $x \in X_{i_0}$. This implies $x \in uX_{i_0}$ and consequently, $x \in \bigcup_{i=1}^{m} uX_i$. Therefore $x \in u_A X$ which yields $X \subseteq u_A X$. Thus u_A is an I-topology.

$f = M$: Let $X \subseteq Y \subseteq P$ be subsets, $x \in u_A X$ a point. Let $\{Y_i \mid i = 1, \dots, \mathrm{m}\}$ ($\mathrm{m} \in \mathbf{N}$) be a system of sets such that $\bigcup_{i=1}^{m} Y_i = Y$. Put $X_i = Y_i \cap X$ for each $i \in \{1, \dots, \mathrm{m}\}$. Then $\bigcup_{i=1}^{m} X_i = X$ and thus $x \in \bigcup_{i=1}^{m} uX_i \subseteq \bigcup_{i=1}^{m} uY_i$. Consequently, $x \in u_A Y$. Therefore $u_A X \subseteq u_A Y$, i.e. u_A is an M-topology.

$f = A$: The assertion is obvious.

$f = MU$: Let $X \subseteq P$ be a subset, $x \in u_A u_A X$ a point. Then $x \in \bigcup_{i=1}^{n} uY_i$ for any system of sets $\{Y_i | i = 1, \dots, n\}$ $(n \in N)$ fulfilling $\bigcup_{i=1}^{n} Y_i = u_A X$. Let $\{X_i | i = 1, \dots, m\}$ $(m \in N)$ be a system of sets such that $\bigcup_{i=1}^{m} X_i = X$. Put $Y_i = u_A X_i$ for each $i \in \{1, \dots, m\}$. Since u_A is an MA-topology, we have $\bigcup_{i=1}^{m} Y_i = \bigcup_{i=1}^{m} u_A X_i = u_A \bigcup_{i=1}^{m} X_i = u_A X$. Therefore $x \in \bigcup_{i=1}^{m} u Y_i = \bigcup_{i=1}^{m} u u_A X_i \subseteq \bigcup_{i=1}^{m} u u X_i \subseteq \bigcup_{i=1}^{m} u X_i$. Consequently $x \in u_A X$ which implies $u_A u_A X \subseteq u_A X$. Thus u_A is a U-topology and hence an MU-topology.

$f \in \{K, B^*, B, S\}$: Clearly, for any point $x \in P$ it holds $u_A\{x\} = u\{x\}$, and from this the assertion follows in these cases.

b) According to [6], Theorem 4, *b*, u^A exists iff u is an A-topology, and then $u^A = u$.

Remark 3. The lower A-modification of a U-topology need not be a U-topology in general — see the following example: Let $P = \{x, y, z\}$, $u\emptyset = \{y\}$, $u\{x, y\} = \{z\}$, $uX = X$ for any subset $X \subseteq P$ fulfilling $\emptyset \neq X \neq \{x, y\}$. Then u is a U-topology, and $u_A\{x, y\} = \{y, z\} \cap \{z\} \cap \{x, y\} = \emptyset$ while $u_A u_A\{x, y\} = u_A\emptyset = \{y\}$. Thus u_A is not a U-topology.

5. U-MODIFICATION

Theorem 5. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{M, OM, IM, MA, MU, MK, MB^*, MB, MS\}$ and u is an f-topology and if u_U exists, then u_U is an f-topology.

b) If $f \in \{M, OM, IM, MA, MU, MB, MS\}$ and u is an f-topology, then u^U is an f-topology.

Proof. *a*) According to [6], Theorem 5, *a*), if u is an M-topology, then u_U exists iff u is an U-topology, and then $u_U = u$.

b) According to the same Theorem, *b*), if u is an M-topology, then u^U exists and it is defined by:

$$X \subseteq P \Rightarrow u^U X = \bigcap \{Z \subseteq P | uX \subseteq Z, uZ \subseteq Z\}.$$

Let us suppose that u is an f-topology.

$f = M$: Let $X \subseteq Y \subseteq P$ be subsets, $x \in u^U X$ a point. Then $x \in Z$ for every set $Z \subseteq P$ fulfilling $uX \subseteq Z$ and $uZ \subseteq Z$. Let $T \subseteq P$ be a set with $uY \subseteq T$ and $uT \subseteq T$. Then $uX \subseteq uY \subseteq T$, and thus $x \in T$. Therefore $x \in u^U Y$, and we have proved the inclusion $u^U X \subseteq u^U Y$. So that u^U is an M-topology.

$f \in \{OM, IM\}$: The proof is trivial in these cases.

$f = MA$: Let $X, Y \subseteq P$ be subsets, $x \in u^U(X \cup Y)$ a point. Then $x \in Z$ for every set $Z \subseteq P$ fulfilling $u(X \cup Y) \subseteq Z$ and $uZ \subseteq Z$. Thus, $x \in Z$ for each set $Z \subseteq P$ with $uX \cup uY \subseteq Z$ and $uZ \subseteq Z$. Let $T, U \subseteq P$ be subsets fulfilling $uX \subseteq T$, $uT \subseteq T$, $uY \subseteq U$, $uU \subseteq U$. Then $uX \cup uY \subseteq T \cup U$, and $u(T \cup U) \subseteq uT \cup uU \subseteq$ $\subseteq T \cup U$. Therefore, $x \in T \cup U$, i.e. $x \in T$ or $x \in U$. Consequently, $x \in u^U X$ or $x \in u^U Y$. From this $x \in u^U X \cup u^U Y$, and we have proved the inclusion $u^U(X \cup Y) \subseteq$ $\subseteq u^U X \cup u^U Y$. So that u^U is an A-topology and hence an MA-topology.

$f \in \{MU, MB\}$: The proof is trivial in these cases.

$f = MS$: Let $\emptyset \neq X \subseteq P$ be a subset, $z \in u^U X$ a point. Then $z \in Z$ for every set $Z \subseteq P$ fulfilling $uX \subseteq Z$ and $uZ \subseteq Z$. Thus, $z \in Z$ for any set $Z \subseteq P$ with $\bigcup_{x \in X} u\{x\} \subseteq Z$ and $uZ \subseteq Z$. Let $\{Y_x \subseteq P | x \in X\}$ be an arbitrary system of sets such that $u\{x\} \subseteq Y_x$ and $uY_x \subseteq Y_x$ for each $x \in X$. Then $\bigcup_{x \in X} u\{x\} \subseteq \bigcup_{x \in X} Y_x$ and since u is an S-topology, we have $u \bigcup_{x \in X} Y_x \subseteq \bigcup_{x \in X} uY_x \subseteq \bigcup_{x \in X} Y_x$. Therefore $z \in \bigcup_{x \in X} Y_x$ and consequently, a point $x_0 \in X$ exists such that $z \in Y_{x_0}$ for any above defined system of sets. Hence $z \in u^U\{x_0\} \subseteq \bigcup_{x \in X} u^U\{x\}$, and the inclusion $u^U X \subseteq \bigcup_{x \in X} u^U\{x\}$ is proved. So, u^U is an S-topology, and thus an MS-topology.

Remark 4. From [5], 2.10, it follows, that the assertion *b*) of Theorem 5 does not hold for any $f \in \{MK, MB^*\}$ in general.

6. K-MODIFICATION

Theorem 6. *Let (P, u) be a topological space. Then it holds: If $f \in \{O, I, M, A, U, K, B^*, B, S\}$ and u is an f-topology and if u_K or u^K respectively exists, then u_K or u^K respectively is an f-topology.*

Proof. According to [6], Theorem 6, u_K or u^K respectively exists iff u is a K-topology, and then $u_K = u$ or $u^K = u$ respectively.

7. B^*-MODIFICATION

Theorem 7. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{O, I, M, K, B^*, B\}$ and u is an f-topology, then u_{B^*} is an f-topology.

b) If $f \in \{O, I, A, B^*, B, S\}$ and u is an f-topology, then u^{B^*} is an f-topology.

Proof. *a*) According to [6], Theorem 7, *a*), u_{B^*} always exists and it is defined by:

$$X \subseteq P \text{ is no one-point set} \Rightarrow u_{B^*}X = uX,$$

$$x \in P \Rightarrow u_{B^*}\{x\} = u\{x\} \cap \{z \in P | x \in u\{z\}\}.$$

Let us suppose that u is an f-topology. For $f \in \{O, I\}$ the proof is trivial.

$f = M$: Let $X \subseteq Y \subseteq P$ be subsets. If both X and Y are no one-point sets, then $u_{B*}X \subseteq u_{B*}Y$ holds trivially. Let $X = \{x\}$. If Y is a one-point set, then $X = Y$ so that $u_{B*}X \subseteq u_{B*}Y$ is true. Otherwise, let Y be no one-point set. Let $y \in u_{B*}X = u_{B*}\{x\}$ be a point. Then $y \in u\{x\} = uX$ and consequently, $y \in uY = u_{B*}Y$. Thus, in each case the inclusion $u_{B*}X \subseteq u_{B*}Y$ is valid. Therefore, u_{B*} is an M-topology.

$f = K$: Let $x, y \in P$ be points, $x \in u_{B*}\{y\}$, $y \in u_{B*}\{x\}$. Then $x \in u\{y\}$, $y \in u\{x\}$, and this yields $x = y$. Hence, u_{B*} is a K-topology.

For $f = B^*$ and, since B-axiom $\Rightarrow$ B^*-axiom, for $f = B$ as well, the assertion is obvious.

b) According to [6], Theorem 7, *b*), u^{B^*} always exists and it is defined by:

$$X \subseteq P \text{ is no one-point set} \Rightarrow u^{B^*}X = uX,$$
$$x \in P \Rightarrow u^{B^*}\{x\} = u\{x\} \cup \{z \in P \mid x \in u\{z\}\}.$$

Let us suppose that u is an f-topology. For $f \in \{O, I\}$ the proof is trivial.

$f = A$: Let $X, Y \subseteq P$ be subsets. If $X = Y$ or if at least one of the sets X, Y is empty, then $u^{B^*}(X \cup Y) \subseteq u^{B^*}X \cup u^{B^*}Y$ holds trivially. Otherwise, let $X \neq Y$, $X \neq \emptyset \neq Y$. Then $X \cup Y$ is no one-point set, and therefore $u^{B^*}(X \cup Y) = u(X \cup Y) \subseteq uX \cup uY \subseteq u^{B^*}X \cup u^{B^*}Y$. Thus, u^{B^*} is an A-topology.

$f \in \{B^*, B\}$: The assertion is obvious.

$f = S$: Let $\emptyset \neq X \subseteq P$ be a subset. Supposing X is a one-point set, we have $u^{B^*}X \subseteq \bigcup_{x \in X} u^{B^*}\{x\}$ trivially. Let us admit that X is no one-point set. Then $u^{B^*}X = uX \subseteq \bigcup_{x \in X} u\{x\} \subseteq \bigcup_{x \in X} u^{B^*}\{x\}$. Therefore, u^{B^*} is an S-topology.

Remark 5. *a*) The assertion *a*) of Theorem 7 does not hold for any $f \in \{A, U, S\}$ in general—for $f \in \{A, S\}$ see the example (26) and for $f = A$ the example (27) of [5].

b) We shall give some examples showing that the upper B^*-modification of an f-topology u need not be an f-topology for any $f \in \{M, U, K\}$ in general.

$f = M$: Let $P = \{x, y, z\}$, $u\emptyset = \emptyset$, $u\{x\} = \{x\}$, $u\{y\} = \{x, y\}$, $u\{z\} = P$, $u\{x, y\} = \{x, y\}$, $u\{x, z\} = u\{y, z\} = uP = P$. Then u is an OIM-topology (i.e. a Čech topology), and $u^{B^*}\{x\} = \{x\} \cup \{y, z\} = P$ while $u^{B^*}\{x, y\} = u\{x, y\} = \{x, y\}$. Hence u^{B^*} is not an M-topology. This example confirms the following fact introduced in Remark 4 of [6]: The upper B^*-modification of a Čech topology defined in [4], 2.5., is generally different from the upper B^*-modification of a topology (without axioms) defined in Theorem 7, *b*), of [6].

$f = U$: Let $P = \{x, y, z\}$, $u\{x\} = \{x\}$, $u\{y\} = \{y\}$, $uX = P$ for any subset $X \subseteq P$ with $\{x\} \neq X \neq \{y\}$. Then u is a U-topology, and $u^{B^*}\{x\} = \{x, z\}$ while $u^{B^*}u^{B^*}\{x\} = u^{B^*}\{x, z\} = P$. Hence u^{B^*} is not a U-topology.

$f = K$: Let $P = \{x, y\}$, $u\{x\} = \{x\}$, $u\emptyset = u\{y\} = uP = P$. Then u is a K-topology, and $u^{B^*}\{x\} = \{x, y\} = u^{B^*}\{y\}$. Therefore u^{B^*} is not a K-topology.

8. B-MODIFICATION

Theorem 8. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{O, I, OM, OU, K, B^*, B\}$ and u is an f-topology, then u_B is an f-topology.

b) If $f \in \{O, I, M, A, U, K, B^*, B, S\}$ and u is an f-topology and if u^B exists, then u^B is an f-topology.

Proof. *a)* According to [6], Theorem 8, *a)*, u_B always exists and it is defined by:

$$X \subseteq P \text{ is no one-point set} \Rightarrow u_B X = uX,$$

$$x \in P \Rightarrow u_B\{x\} = \begin{cases} \{x\} & \text{for } x \in u\{x\}, \\ \emptyset & \text{for } x \notin u\{x\}. \end{cases}$$

Let us suppose that u is an f-topology. For $f \in \{O, I\}$ the proof is trivial.

$f = OM$: Let $X \subseteq Y \subseteq P$ be subsets. If both X and Y are no one-point sets, then $u_B X \subseteq u_B Y$ holds trivially. Thus, *let* $X = \{x\}$ and let us admit that Y is no one-point set. If $x \in u\{x\}$, then $u_B X = u_B\{x\} = \{x\} \subseteq u\{x\} = uX \subseteq uY = u_B Y$. Otherwise, if $x \notin u\{x\}$, then $u_B X = u_B\{x\} = \emptyset \subseteq u_B Y$. Next, let us suppose that X is no one-point set and let Y be a one-point set. Then $X = \emptyset$ and consequently $u_B X = u_B \emptyset = u\emptyset = \emptyset \subseteq u_B Y$. Finally, suppose both X and Y are one-point sets. Then $X = Y$ so that $u_B X \subseteq u_B Y$ is true. Thus the inclusion $u_B X \subseteq u_B Y$ holds in every case. Therefore u_B is an M-topology, and hence an OM-topology.

$f = OU$: Let $X \subseteq P$ be a subset. Let us admit that X is no one-point set. Then $u_B u_B X = u_B uX$. If uX is not a one-point set, then $u_B uX = uuX \subseteq uX = u_B X$. Otherwise, let $uX = \{x\}$. Now, if $x \in u\{x\}$, then $u_B uX = u_B\{x\} = \{x\} = uX = u_B X$, and if $x \notin u\{x\}$, then $u_B uX = u_B\{x\} = \emptyset \subseteq u_B X$. Thus, for any not-one-point subset $X \subseteq P$ we have $u_B u_B X \subseteq u_B X$. Next, suppose X is a one-point set, $X = \{y\}$. If $y \in u\{y\}$, then $u_B u_B X = u_B u_B\{y\} = u_B\{y\} = u_B X$. Otherwise, if $y \notin u\{y\}$, then $u_B u_B X = u_B u_B\{y\} = u_B \emptyset = u\emptyset = \emptyset \subseteq u_B X$. Thus, for every subset $X \subseteq P$ the inclusion $u_B u_B X \subseteq u_B X$ holds. Therefore u_B is a U-topology, and hence an OU-topology.

$f \in \{K, B^*\}$: The proof is trivial in these cases.

$f = B$: The assertion is obvious.

b) According to [6], Theorem 8, *b)*, u^B exists iff u is a B-topology, and then $u^B = u$.

Remark 6. The following examples show that the assertion *a)* of Theorem 8 does not hold for any $f \in \{M, A, U, S\}$ in general.

$f = M$: Let $P = \{x, y\}$, $u\emptyset = u\{x\} = \{x\}$, $u\{y\} = uP = P$. Then u is an M-topology, and $u_B \emptyset = \{x\}$ while $u_B\{y\} = \{y\}$. Thus u_B is not an M-topology.

$f \in \{A, S\}$: Let $P = \{x, y, z\}$, $u\{x\} = \{x, y\}$, $u\{x, z\} = P$, $uX = X$ for any subset $X \subseteq P$ for which $\{x\} \neq X \neq \{x, z\}$. Then u is an AS-topology, and

$u_B\{x, z\} = P$ while $u_B\{x\} \cup u_B\{z\} = \{x, z\}$. Hence, u_B is neither A-topology nor S-topology.

$f = U$: Let $P = \{x, y\}$, $uX = \{x\}$ for any subset $X \subseteq P$. Then u is a U-topology, and $u_B\{y\} = \emptyset$ while $u_B u_B\{y\} = \{x\}$. Therefore u_B is not a U-topology.

9. S-MODIFICATION

Theorem 9. *Let (P, u) be a topological space. Then it holds:*

a) If $f \in \{M, OM, IM, MA, MU, MK, MB^*, MB, MS\}$ and u is an f-topology, then u_s is an f-topology.

b) If $f \in \{M, OM, IM, MA, MU, MK, MB^*, MB, MS\}$ and u is an f-topology and if u^s exists, then u^s is an f-topology.

Proof. *a)* According to [6], Theorem 9, *a)*, if u is an M-topology, then u_s exists and it is defined by:

$$u_s\emptyset = u\emptyset,$$
$$\emptyset \neq X \subseteq P \Rightarrow u_s X = \bigcup_{x \in X} u\{x\}.$$

Let us suppose that u is an f-topology.

$f = M$: Let $X \subseteq Y \subseteq P$ be subsets. For $X = \emptyset = Y$ the inclusion $u_s X \subseteq u_s Y$ holds trivially. Suppose $X = \emptyset$, $Y \neq \emptyset$. Then $u_s X = uX = u\emptyset \subseteq \bigcup_{x \in Y} u\{x\} = u_s Y$, since $u\emptyset \subseteq u\{x\}$ holds for any point $x \in Y$. Finally, suppose $X \neq \emptyset \neq Y$. Then $u_s X = \bigcup_{x \in X} u\{x\} \subseteq \bigcup_{x \in Y} u\{x\} = u_s Y$. Therefore u_s is an M-topology.

$f = OM$: The proof is trivial.

$f = IM$: Let $X \subseteq P$ be a subset. If $X = \emptyset$, then $X \subseteq u_s X$ holds trivially. Suppose $X \neq \emptyset$. Then $X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} u\{x\} = u_s X$. Therefore u_s is an I-topology, and hence an IM-topology.

$f = MA$: Let $X, Y \subseteq P$ be subsets. If at least one of the sets X and Y is empty, then $u_s(X \cup Y) \subseteq u_s X \cup u_s Y$ holds trivially. Suppose $X \neq \emptyset \neq Y$. Then $u_s(X \cup Y) = \bigcup_{x \in X \cup Y} u\{x\} = \bigcup_{x \in X} u\{x\} \cup \bigcup_{x \in Y} u\{x\} = u_s X \cup u_s Y$. Thus, u_s is an A-topology, and hence an MA-topology.

$f = MU$: Let $X \subseteq P$ be a subset. Suppose $X = \emptyset$. Then $u_s u_s X = u_s uX$. If $uX = \emptyset$, then $u_s uX = uuX \subseteq uX = u_s X$. Otherwise, let $uX \neq \emptyset$. Then $u_s uX = \bigcup_{x \in uX} u\{x\} \subseteq uX = u_s X$, since $x \in uX \Rightarrow u\{x\} \subseteq uuX \subseteq uX$. Thus, for the empty set X we have $u_s u_s X \subseteq u_s X$. Next, suppose $X \neq \emptyset$. If $u_s X = \emptyset$, then we have $u_s u_s X = u_s \emptyset \subseteq u_s X$ (since u_s is an M-topology according to the first part of the proof). Otherwise, let $u_s X \neq \emptyset$. Then $u_s u_s X = \bigcup_{x \in u_s X} u\{x\} = \bigcup_{x \in \bigcup_{y \in X} u\{y\}} u\{x\} = \bigcup_{\substack{y \in X \\ u\{y\} \neq \emptyset}} \bigcup_{x \in u\{y\}} u\{x\} =$

$= \bigcup_{\substack{y \in X \\ u\{y\} \neq \emptyset}} u_s u\{y\} \subseteq \bigcup_{\substack{y \in X \\ u\{y\} \neq \emptyset}} uu\{y\} \subseteq \bigcup_{\substack{y \in X \\ u\{y\} \neq \emptyset}} u\{y\} = \bigcup_{y \in X} u\{y\} = u_s X$. Thus, for every subset $X \subseteq P$ the inclusion $u_s u_s X \subseteq u_s X$ holds. Therefore, u_s is a U-topology, and hence an MU-topology.

$f \in \{MK, MB^*, MB\}$: In these cases the assertion follows from the fact that $u_s\{x\} = u\{x\}$ for each point $x \in P$.

$f = S$: The assertion is obvious.

b) According to [6], Theorem 9, *b*), if u is an M-topology, then u^s exists iff u is an S-topology, and then $u^s = u$.

Remark 7. From the Theorems proved in the present paper and the corresponding ones proved in [6] some results attained in [1], [3] and [4] follow. So, Theorems 4 imply 3.1. and 3.2. of [3], Theorems 5 imply 3.7. and 3.8. of [3], Theorems 6 imply 4.3. and 4.4. of [4], Theorems 7, *a*), imply 2.4. of [4], Theorems 8 imply 3.3. and 3.4. of [3], Theorems 9, *a*), imply 26.A.4. of [1]. Many results of [5] follow from those of this note, too.

Remark 8. Let $f, g \in \{O, I, M, A, U, K, B^*, B, S\}$. According to [5], we say that the lower f-modifying and the lower g-modifying are commutative if for each topology u it holds $(u_f)_g = (u_g)_f$, provided that u_f, u_g, $(u_f)_g$, $(u_g)_f$ exist. The commutativity of the upper modifying is defined analogically. Although the modifying of Čech topologies only is dealt in [5], all considerations contained in the first paragraph of [5] are valid for the modifying of topologies without axioms, too. Particularly, the statement (7) of [5] implies:

A sufficient (necessary) condition for the lower f-modifying and the lower g-modifying to be commutative is that for each topology u for which u_f, u_g (u_f, u_g, $(u_f)_g$, $(u_g)_f$) exist, the following two conditions hold:

(1) If u is a g-topology, then u_f is a g-topology.

(2) If u is an f-topology, then u_g is an f-topology.

For the upper modifying the analogical assertion is valid.

Thus, the results attained in the present article solve the problem, under which conditions the lower (upper) modifying of topologies without axioms is commutative.

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Josef Šlapal
Department of Mathematics
Technical University of Brno
Technická 2
616 69 Brno, Czechoslovakia