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ESTIMATION OF THE INDEX OF G^2

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Abstract. The present theses deal with the spectra of powers of graphs. A new estimation of the index of a graph is given and used for a description of the squares of the graphs having the index at most four.

Key words. Simple graph, spectrum of a graph, bound of the index of a graph.

MS Classification. 05 C 50.

1. INTRODUCTION AND NOTATION

There are many papers dealing with relations between a graph and the spectrum of its adjacency matrix. One such relationship proposed by A. J. Hoffman [4] asks the following question: If $\varrho(G)$ denotes the largest eigenvalue, or equivalently the index of a graph G and α is a real number, characterize those graphs that satisfy $\varrho(G) < \alpha$. For $\alpha = 2$ this problem was solved by J. H. Smith [5]. His result can be formulated as follows: $\alpha = 2$ is the largest real number such that for each $\varrho < 2$ there is a finite number of graphs G with the index $\varrho(G) = \varrho$.

In this paper we consider the following question: What is the similar α for the class of the second powers of graphs. During these considerations a new estimation of the index of a graph has been found.

All graphs if not stated otherwise will be undirected, without loops or multiple edges.

Let $G = (X, E)$ be a graph. The second power or equivalently the square of G is the graph $G^2 = (X, E')$ with the same vertex set X and in which diferent vertices are adjacent if and only if there is at least one path of the length 1 or 2 in G between them.

In this paper we use the following theorems from the matrix theory and their application to the spectrum of a graph:

1.1. Theorem [3]. *The maximal real eigenvalue r' of every principal submatrix (of order less than n) of a non-negative matrix A (of order n) does not exceed the* *maximal real eigenvalue r of A. If A is irreducible, then $r' < r$ always holds. If A is reducible, then $r' = r$ holds for at least one principal submatrix.*

1.2. Theorem [1]. *The increase of any element of a non-negative matrix A does not decrease the maximal real eigenvalue. The maximal real eigenvalue increases strictly if A is an irreducible matrix.*

1.3. Remark. Theorems 1.1. and 1.2. state that in a (strongly) connected multi-(di)graph G every subgraph (not equal G) has the index smaller than the index of G

2. THE ESTIMATION OF THE INDEX OF A GRAPH

2.1. Theorem. *For the index ϱ of a graph $G = (X, E)$, $X = \{v_1, v_2, \ldots, v_n\}$ it holds*

$$\sqrt{\frac{\sum_{i=1}^{n} d_G^2(v_i)}{n}} \leqq \varrho \leqq \min\left\{\max_i d_G(v_i); \sqrt{\sum_{i=1}^{n} d_G(v_i)}\right\}.$$

Proof. For an arbitrary vector $x = (x_1, \ldots, x_n)^T$ and the euclidian norm g_2 of the symmetric matrix A it holds

$$g_2(Ax) \leqq \varrho \cdot g_2(x).$$

Let $x = (1, \ldots, 1)^T$ so we obtain

$$g_2(\sum_{k=1}^{n} a_{1k}, \ldots, \sum_{k=1}^{n} a_{nk}) \leqq \varrho \sqrt{n}$$

and from

$$\sqrt{\sum_{i=1}^{n} d_G^2(v_i)} \leqq \varrho \sqrt{n}$$

we get the lower bound. The upper bound is obtained from the Schur's (or Frobenius's) norm of the matrix

$$N(A) = (\sum_{i=1}^{m} \sum_{k=1}^{n} | a_{ik} |^2)^{1/2},$$

the norm

$$g_3(x) = \max_i | x_i |$$

and from the inequalities

$$\varrho \leqq g_3(A),$$
$$\varrho = g_2(A) \leqq N(A)$$

(see, e.g. [2]).

2.2. Remark. For regular graphs the equality holds

$$\varrho = \sqrt{\frac{\sum_i d_G^2(v_i)}{n}}$$

since in that case the vector $x = (1, \ldots, 1)^T$ is a eigenvector for A belonging to ϱ.

2.3. Remark. Recall that a multigraph G is called semiregular of degrees r_1, r_2 if it is bipartite having a representation $G = (X_1, X_2, U)$ with $| X_1 | = n_1, | X_2 | = n_2, n_1 + n_2 = n$, where each vertex $x_i \in X_i$ has valency r_i $(i = 1, 2)$. As according to Theorem 2.1.

$$\varrho \geqq \sqrt{\frac{n_1 r_1^2 + n_2 r_2^2}{n_1 + n_2}} = \sqrt{r_1 r_2 \frac{\frac{n_1 r_1}{r_2} + \frac{n_2 r_2}{r_1}}{n_1 + n_2}} = \sqrt{r_1 r_2},$$

$(n_1 r_1 = n_2 r_2)$ and $x = (\underbrace{\sqrt{r_1}, \ldots, \sqrt{r_1}}_{r_1}, \underbrace{\sqrt{r_2}, \ldots, \sqrt{r_2}}_{r_2})$ is an eigenvector belonging to $\sqrt{r_1 r_2}$ the equality holds for semiregular graphs too.

2.4. Remark. Substituting $d_G^+(v_i) = d_G^-(v_i)$ for $d_G(v_i)$ we obtain the estimation of the index of the multi-digraph with the symmetric adjacency matrix.

3. THE INDEX OF G^2

3.1. Lemma. *Let G contain a vertex x so that $d_G(x) \geqq 4$ or a circuit C_m of the length $m \geqq 5$. Then $\varrho(G^2) \geqq 4$.*

Proof. As in those cases G^2 contains either K_5 or C_m^2 and $\varrho(K_5) = 4 = \varrho(C_m^2)$ the assertion follows for Remark 1.3.

3.2. Lemma. *If G contains as a subgraph a tree with at least 5 pendant vertices, then*

$$\varrho(G^2) > 4.$$

Proof. That graph has either a vertex of the degree at least 4 and the assertion follows from Lemma 3.1. or it has one of the following graphs as its subgraph (see fig. 1).

As from Theorem 2.1. we get

$$\varrho(G_{1,1}^2) \geqq \sqrt{18}, \qquad \varrho(G_{1,2}^2) \geqq \sqrt{\frac{158}{9}},$$

$$\varrho(G^2_{1,3+k}) \geqq \sqrt{16 + \frac{12}{k+10}}, \qquad \varrho(G^2_{2,1}) \geqq \sqrt{\frac{174}{10}},$$

$$\varrho(G^2_{2,2+k}) \geqq \sqrt{16 + \frac{12}{k+11}}, \qquad \varrho(G^2_{3+i,1+j}) \geqq \sqrt{16 + \frac{10}{i+j+12}},$$

the assertion follows from Remark 1.3.

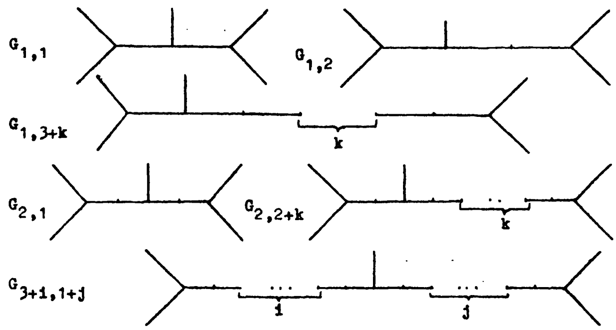

Fig. 1

3.3. Lemma. $\varrho(G^2) > 4$, *if* G *contains one of the following graphs as a subgraph*

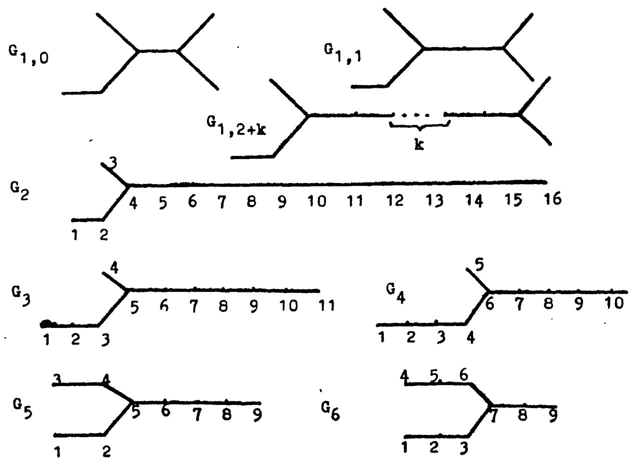

Fig. 2

Proof: The indices of $G^2_{1,0}, \ldots, G^2_{1,9}, G^2_2, \ldots, G^2_6$ (see Table) are greater than 4, so the assertion follows from Remark 1.3.

3.4. Lemma. *Let G contain a block (not necessarily block of G) on at least 5 vertices or two blocks connected by a path, one of these blocks contains at least 4 vertices, then*

$$\varrho(G^2) > 4.$$

Proof. If G contains a block on 5 vertices, it has either C_m $m > 5$ or $K_{2,3}$ as its subgraph. As $\varrho(C_m^2) = \varrho(K_{2,3}^2) = 4$ the assertion follows from Remark 1.3. In the second case G has either a vertex of degree at least 4 and the assertion follows from Lemma 3.1. or it has one of the following graphs as its subgraph

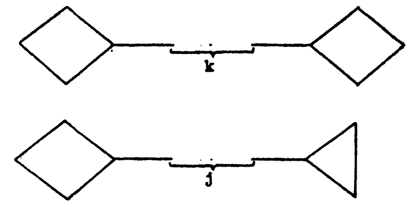

Fig. 3

and the assertion follows from Lemma 3.3.

3.5. Lemma. *Let a connected graph G contain at least three blocks, then*

$$\varrho(G^2) > 4.$$

Proof. With regard to Lemmas 3.1. and 3.4. there remain the cases, when G contains as its subgraph one of the following graphs

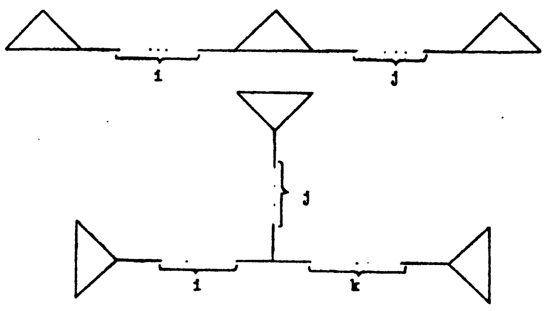

Fig. 4

and the assertion follows from Lemma 3.2.

3.6. Lemma. *Let a connected graph G contains a block on 4 vertices and has at least two pendant vertices. Then*

$$\varrho(G^2) > 4.$$

Proof. That G contains as a subgraph one of the following graphs

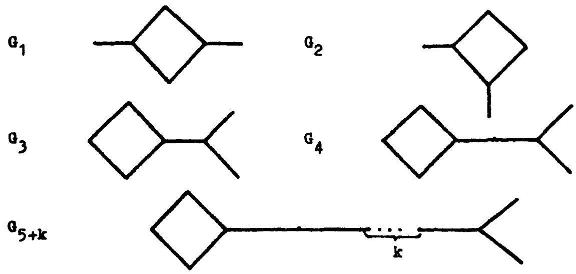

Fig. 5

According to Theorem 2.1. we obtain

$$\varrho(G_1^2) \geqq \sqrt{\frac{100}{6}}, \quad \varrho(G_2^2) \geqq \sqrt{\frac{100}{6}}, \quad \varrho(G_3^2) \geqq \sqrt{\frac{120}{7}}, \quad \varrho(G_4^2) \geqq \sqrt{17},$$

$$\varrho(G_{5+k}^2) \geqq \sqrt{16 + \frac{6}{k+9}},$$

so the assertion follows from Remark 1.3.

3.7. Lemma. *Let G contain as a subgraph one of the following graphs* then

$$\varrho(G^2) > 4.$$

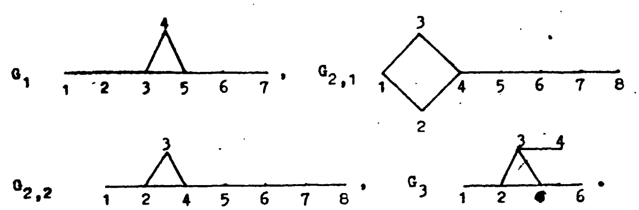

Fig. 6

Proof.

$$A(G_1^2) = \begin{bmatrix} 0&1&1&0&0&0&0 \\ 1&0&1&1&1&0&0 \\ 1&1&0&1&1&1&0 \\ 0&1&1&0&1&1&0 \\ 0&1&1&1&0&1&1 \\ 0&0&1&1&1&0&1 \\ 0&0&0&0&1&1&0 \end{bmatrix} \quad \begin{matrix} A(G_{2,i}^2) = \\ (i = 1, 2) \end{matrix} \begin{bmatrix} 0&1&1&1&0&0&0&0 \\ 1&0&1&1&1&0&0&0 \\ 1&1&0&1&1&0&0&0 \\ 1&1&1&0&1&1&0&0 \\ 0&1&1&1&0&1&1&0 \\ 0&0&0&1&1&0&1&1 \\ 0&0&0&0&1&1&0&1 \\ 0&0&0&0&0&1&1&0 \end{bmatrix}.$$

As the edge labeled digraphs with the matrices

$$A(H_1^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{20}{21} & 0 & 1 & \frac{20}{21} & 1 & 0 & 0 \\ \frac{20}{21} & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & \frac{20}{21} \\ 0 & 0 & 1 & \frac{20}{21} & 1 & 0 & \frac{20}{21} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}; \quad \begin{array}{c} A(H_{2,i}^2) = \\ (i = 1, 2) \end{array} \begin{bmatrix} 0 & 1 & 1 & \frac{22}{23} & & 0 & 0 & 0 \\ \frac{30}{31} & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \frac{30}{31} & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \frac{30}{31} & 1 & 1 & 0 & 1 & \frac{30}{31} & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \frac{30}{31} & \frac{10}{11} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & \frac{12}{13} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{30}{31} & 1 & 0 \end{bmatrix}$$

have the eigenvalue 4 and the corresponding eigenvectors

$$x_1 = (21, 38, 46, 42, 46, 38, 21)^T,$$
$$x_2 = (31, 40, 40, 46, 44, 31, 22, 13)^T$$

and with respect to $\varrho(G_3^2) \geq \sqrt{17}$ (see Theorem 2.1.) the assertion follows from Theorem 1.2. and Remark 1.3.

3.8. Theorem. *Let $\mathscr{G}^2$ denote the class of graphs that are the second power of some graph. In this class the graphs with the index $\varrho < 4$ are characterized by the spectrum.*

Proof. The following edge labeled graphs have the eigenvalue $\varrho = 4$, for the corresponding eigenvectoı x_ϱ (for the indexing of vertices given in Figures). The unlabeled edges have the weight 1.

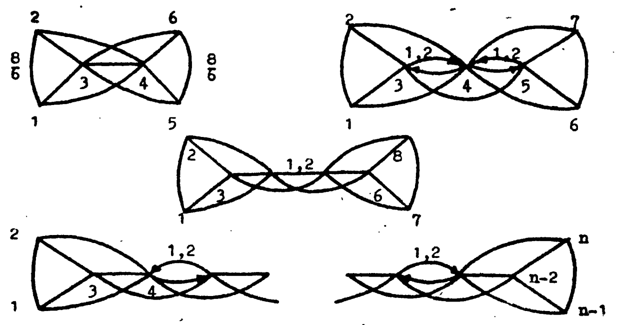

Fig. 7

$$x_\varrho = (6, 6, 8, 10, \ldots, 10, 8, 6, 6)^T.$$

With regard to Theorem 1.2., Remark 1.3. and with respect to Table we get, that second powers only the graphs P_n ($n \geqq 1$), $G_{1,n}$ ($n \geqq 3$), $G_{2,n}$ ($n \geqq 6$), $G_{3,n}$ ($6 \leqq \leqq n \leqq 15$), $G_{4,n}$ ($7 \leqq n \leqq 8$), $G_{5,n}$ ($8 \leqq n \leqq 10$), $G_{6,n}$ ($4 \leqq n \leqq 7$) from Table have the index $\varrho < 4$.

3.8.' Theorem. *The number* 4 *is the greatest real number so that for each* $\alpha < 4$ *there is only a finite number of graphs G so that*

$$\varrho(G^2) \leqq \alpha$$

From the assertions 3.1.—3.7. it follows for circuits:

3.9. Theorem. *The* C_m^2 *is characterized by its spectrum in* $\mathscr{G}^2$.

3.10. Corollary. *In Theorem* 3.8. $\varrho < 4$ *can be replaced by* $\varrho \leqq 4$.

TABLE

Coefficients of characteristic polynomials

$$P_G(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$$

and maximal eigenvalues of the second powers of the following graphs:
($a_1 = 0$, since G^2 contains no loops)

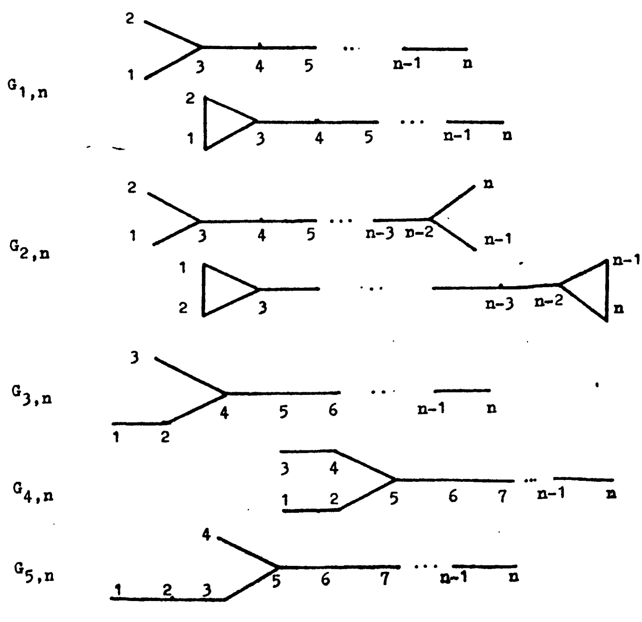

Fig. 8

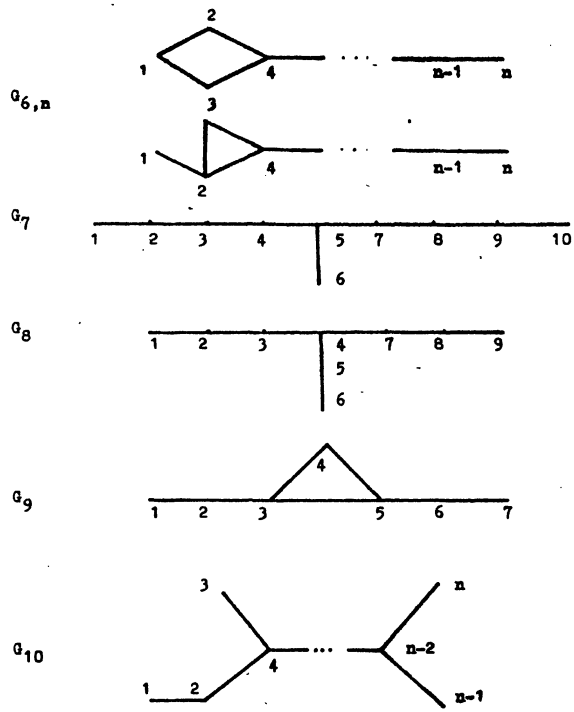

Fig. 9

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Table

	a_2 a_{10}	a_3 a_{11}	a_4 a_{12}	a_5 a_{13}	a_6 a_{14}	a_7 a_{15}	a_8 a_{16}	a_9 ϱ
$G_{1,4}^2$	−6	−8	−3					 3
$G_{1,5}^2$	−8	−10	−1	2				 3,323404
$G_{1,6}^2$	−10	−12	7	14	4			 3,534422
$G_{1,7}^2$	−12	−14	20	42	23	4		 3,651862
$G_{1,8}^2$	−14	−16	37	78	41	0	−3	 3,730611
$G_{1,9}^2$	−16	−18	58	122	44	−50	−39	−6 3,783904
$G_{1,10}^2$	−18 −5	−20	83	174	23	−184	−164	−50 3,822365
$G_{1,11}^2$	−20 6	−22 4	112	234	−30	−426	−387	−108 3,850884
$G_{1,12}^2$	−22 200	−24 84	145 8	302	−123	−800	−689	−42 3,872724
$G_{1,13}^2$	−24 917	−26 470	182 92	378 6	−264	−1330	−1034	372 3,889831
$G_{1,14}^2$	−26 2550	−28 1332	223 216	462 −20	−461 −5	−2040	−1370	−1422 3,903510
$G_{1,15}^2$	−28 5493	−30 2488	268 −308	554 −588	−722 −155	−2954 −10	−1629	3460 3,914636
$G_{1,16}^2$	−30 10108	−32 3212	317 −3296	654 −3286	−1055 −1109	−4096 −152	−1727 −7	6902 3,923821

Continuation

	a_2 / a_{10}	a_3 / a_{11}	a_4 / a_{12}	a_5 / a_{13}	a_6 / a_{14}	a_7 / a_{15}	a_8 / a_{16}	a_9 / ϱ
$G^2_{2,6}$	−11	−16	3	16	1			3,828427
$G^2_{2,7}$	−13	−18	19	52	33	6		3,917286
$G^2_{2,8}$	−15	−20	39	108	91	32	4	3,917285
$G^2_{2,9}$	−17	−22	62	162	111	−10	−37	−10 3,917285
$G^2_{2,10}$	−19 −9	−24	89	224	98	−172	−216	−84 3,942820
$G^2_{2,11}$	−21 −59	−26 −4	120	294	48	−482	−601	−294 3,950439
$G^2_{2,12}$	−23 65	−28 80	155 13	372	−46	−948	−1117	−428 3,955788
$G^2_{2,13}$	−25 913	−30 660	194 170	458 12	−192	−1594	−1709	−154 3,960342
$G^2_{2,14}$	−27 3151	−32 2332	237 748	552 96	−398 4	−2444	−2316	896 3,964023
$G^2_{2,15}$	−29 7317	−34 5088	284 1147	654 −268	−672 −154	−3522 −16	−2862	3130 3,967140
$G^2_{2,16}$	−31 13882	−36 8108	335 −1070	764 −3536	−1022 −1657	−4852 −300	−3255 −15	7020 3,969785

Continuation

	a_2 / a_{10}	a_3 / a_{11}	a_4 / a_{12}	a_5 / a_{13}	a_6 / a_{14}	a_7 / a_{15}	a_8 / a_{16}	a_9 / ϱ
$G^2_{3,6}$	−10	−12	4	6	1			3,592615
$G^2_{3,7}$	−12	−14	16	26	4	−2		3,753006
$G^2_{3,8}$	−14	−16	33	62	22	−6	−3	3,839262
$G^2_{3,9}$	−16	−18	54	106	32	−30	−11	2 3,894023
$G^2_{3,10}$	−18 4	−20	79	158	19	−124	−67	4 3,929279
$G^2_{3,11}$	−20 22	−22 [illegible]	108	218	−26	−326	−223	−12 3,953424
$G^2_{3,12}$	−22 104	−24 14	141 −3	286	−111	−660	−473	22 3,970400
$G^2_{3,13}$	−24 492	−26 108	178 −26	362 −6	−244	−1150	−782	308 3,982717
$G^2_{3,14}$	−26 1589	−28 486	219 −79	446 −50	−433 −5	−1820	−1098	1134 3,991847
$G^2_{3,15}$	−28 3820	−30 1158	264 −430	538 −252	−686 −12	−2694 4	−1353	2852 3,998750
$G^2_{3,16}$	−30 7579	−32 1622	313 −2272	638 −1340	−1011 −87	−3796 68	−1463 8	5878 4,004054

Contiuuation

	a_2 a_{10}	a_3 c_{11}	a_4 a_{12}	a_5 a_{13}	a_6 a_{14}	a_7 a_{15}	a_8 a_{16}	a_9 ϱ
$G^2_{4,7}$	−12	−14	12	12	−7	0		$1 + \sqrt{8}$
$G^2_{4,8}$	−14	−16	28	40	−7	−14		3,956310
$G^2_{4,9}$	−16	−18	49	84	4	−36	−7	4,021879
$G^2_{5,8}$	−14	−16	32	54	4	−18	−3	3,883260
$G^2_{5,9}$	−16	−18	53	98	8	−64	−34	−4 3,951670
$G^2_{5,10}$	−18 3	−20	78	150	−4	−154	−87	−4 3,994069
$G^2_{5,11}$	−20 46	−22 4	107	210	−47	−338	−189	44 4,020463
$G^2_{6,5}$	−9	−14	−6	0				3,645751
$G^2_{6,6}$	−11	−16	1	10	3			3,858951
$G^2_{6,7}$	−13	−18	15	40	23	4		3,980637
$G^2_{6,8}$	−15	−20	34	86	58	12	0	4,041789

Continuation

	a_2 a_{10}	a_3 a_{11}	a_4 a_{12}	a_5 a_{13}	a_6 a_{14}	a_7 a_{15}	a_8 a_{16}	a_9 ϱ
G_7^2	−18 −4	−20	78	150	−12	−194	−153	−44 4,010636
G_8^2	−16	−18	48	76	−20	−66	−21	0 4,065691
G_9^2	−13	−18	12	28	9	0		4,032934
$G_{10,7}^2$	−13	−18	13	30	7	−4		4,023336
$G_{10,8}^2$	−15	−20	34	82	45	0	−3	4,058568
$G_{10,9}^2$	−17	−22	58	146	96	0	−18	−4 4,040037
$G_{10,10}^2$	−19 7	−24	85	208	97	−96	−89	−6 4,040576
$G_{10,11}^2$	−21 23	−26 6	116	278	56	−350	−343	−76 4,036625
$G_{10,12}^2$	−23 75	−28 40	151 4	356	−30	−776	−802	−216 4,035110
$G_{10,13}^2$	−25 477	−30 200	190 −8	442 −10	−168	−1382	−1365	−118 4,033446
$G_{10,14}^2$	−27 1925	−32 956	233 35	536 −72	−366 −9	−2192	−1960	636 4,032300
$G_{10,15}^2$	−29 5120	−34 2750	280 132	638 −308	−632 −72	−3230 −4	−2510	2478 4,031320
$G_{10,16}^2$	−31 10585	−36 5158	331 −1026	748 −1642	−974 −344	−4520 44	−2923 13	5880 4,030541