Archivum Mathematicum

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Archivum Mathematicum, Vol. 24 (1988), No. 3, 99--110

Persistent URL: http://dml.cz/dmlcz/107317

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 24, No. 3 (1988), 99–110

ON MODIFICATIONS OF TOPOLOGIES WITHOUT AXIOMS

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(Received May 14, 1985)

Abstract. A topology without axioms (briefly a topology) on a set P is a mapping u: $\exp P \to \exp P$. If u, v are two topologies on P, then u is called finer (coarser) than v if $uX \subseteq vX$($vX \subseteq uX$) holds for every subset $X \subseteq P$. Let f be a topological property. Then a topology possessing f is said to be an f-topology. The coarsest (finest) of all f-topologies on P which are finer (coarser) than a given topology u on P is called the lower (upper) f-modification of u. In the present paper those f-modifications are studied where f is one of the axioms O, I, M, A, U, K, B^*, B, S well-known from the literature. The results are illustrated by several examples.

Key words. Topology (without axioms), O, I, M, A, U, K, B^*, B, S-axioms, lower modification of a topology, upper modification of a topology.

MS Classification. Primary 54 A 05, 54 A 10

INTRODUCTION

In [12] and [14] the authors have found some modifications of a Čech topology. i.e. of such a mapping u: $\exp P \to \exp P$ (where P is a given set) for which the following three axioms are satisfied: (1) $u\emptyset = \emptyset$, (2) $X \subseteq P \Rightarrow X \subseteq uX$, (3) $X \subseteq Y \subseteq P \Rightarrow uX \subseteq uY$. These modifications are then studied in [15], [16] and [18]. In the present paper we shall find modifications of such topologies which we obtain by omitting all three axioms in the definition of Čech topologies. These topologies are sometimes called topologies without axioms or general topologies or Koutský topologies. From the point of view of topology they are studied in [13] and [19]. But, as a general mathematical structure, these topologies occur and are investigated in many other branches of mathematics, for example in abstract logics (see [3], [5] and [21]) and in the theory of games (see [11]).

Let P be a set. By a topology without axioms on P we mean a mapping u: $\exp P \to \exp P$. Briefly, we shall say a topology instead of a topology without axioms. If u, v are two topologies on P, we say that u is finer than v or that v is coarser than u when $X \subseteq P \Rightarrow uX \subseteq vX$. Then we write $u \leq v$. Clearly, $\leq$ is an order relation on the set of all topologies on P. Moreover, the set of all topologies on P ordered by $\leqq$ is a complete lattice as shown in [13] (and even a completely distributive complete Boolean algebra – see [19]).

For topologies u on a given set P the following axioms are considered:

1. $u\emptyset = \emptyset$	*O-axiom* ([8]),
2. $X \subseteq P \Rightarrow X \subseteq uX$	*I-axiom* ([8]),
3. $X \subseteq Y \subseteq P \Rightarrow uX \subseteq uY$	*M-axiom* ([8]),
4. $X, Y \subseteq P \Rightarrow u(X \cup Y) \subseteq uX \cup uY$	*A-axiom* ([7]),
5. $X \subseteq P \Rightarrow uuX \subseteq uX$	*U-axiom* ([12]),
6. $x, y \in P,\ x \in u\{y\},\ y \in u\{x\} \Rightarrow x = y$	*K-axiom* ([7]),
7. $x, y \in P,\ x \in u\{y\} \Rightarrow y \in u\{x\}$	*B*-axiom* ([14]),
8. $x \in P \Rightarrow u\{x\} \subseteq \{x\}$	*B-axiom* ([7]),
9. $\emptyset \neq X \subseteq P \Rightarrow uX \subseteq \bigcup_{x \in X} u\{x\}$	*S-axiom* ([17]).

Let $f, g \in \{O, I, M, A, U, K, B^*, B, S\}$. A topology u is called *f-topology* if it satisfies the f-axiom. If u satisfies both f-axiom and g-axiom, then it is called *fg-topology,* etc. It is easy to see that the notions B-topology and KB^*-topology are consistent, and that every MS-topology is an A-topology.

Many papers study topologies fulfilling some of the above listed axioms. More precisely:

IM-topology = *extended topology* in [10],

OIM-topology = *topology* in [7],

$OIMA$-topology = *closure operation* in [6],

$OIMAK$ or $OIMAB^*$ or $OIMAB$ or $OIMS$-topology respectively = *feebly semi-separated* or *semi-uniformizable* or *semi-separated* or *quasi-discrete closure operation* respectively in [6],

$OIMU$-topology = *topology* in [20],

$OIMAU$-topology = *topology* in [1], [2], [4],

$OIMAUK$-topology = T_0-*topology* in [2],

$OIMAUB^*$-topology = R_0-*topology* in [9],

$OIMAUB$-topology = T_1-*topology* in [2],

$OIMUS$-topology = *saturated topology* (or *S-topology*) in [17],

$OIMUKS$-topology = *discrete topology of Alexandroff* in [2].

Let (P, u) be a topological space and let $f \in \{O, I, M, A, U, K, B^*, B, S\}$. A topology v on P is said to be a *lower* (*upper*) *f-modification* of the topology u when v is the coarsest (finest) f-topology on P which is finer (coarser) than u. A lower (upper) f-modification of u will be noted as u_f (u^f).

1. O-MODIFICATION

Theorem 1. *Let* (P, u) *be a topological space. Then*

a) u_O *always exists and it is defined by*

$$\emptyset \neq X \subseteq P \Rightarrow u_O X = uX,$$

$$u_O \emptyset = \emptyset.$$

b) u^O *exists iff* u *is an O-topology and then* $u^O = u$. *If* u *is no O-topology, then there does not exist any O-topology on* P *coarser than* u.

Proof. Obvious.

2. I-MODIFICATION

Theorem 2. *Let* (P, u) *be a topological space. Then*

a) u_I *exists iff* u *is an I-topology and then* $u_I = u$. *If* u *is no I-topology, then there does not exist any I-topology on* P *finer than* u.

b) u^I *always exists and it is defined by*

$$X \subseteq P \Rightarrow u^I X = uX \cup X.$$

Proof. Obvious.

3. M-MODIFICATION

Theorem 3. *Let* (P, u) *be a topological space. Then*

a) u_M *always exists and it is defined by*

$$X \subseteq P \Rightarrow u_M X = \bigcap_{X \subseteq Z \subseteq P} uZ.$$

b) u^M *always exists and it is defined by*

$$X \subseteq P \Rightarrow u^M X = \bigcup_{Z \subseteq X} uZ.$$

Proof. *a)* For any subset $X \subseteq P$ put $vX = \bigcap_{X \subseteq Z \subseteq P} uZ$. Clearly, v is an M-topology on P, since for any subsets $X, Y \subseteq P, X \subseteq Y$, there holds $vX = \bigcap_{X \subseteq Z \subseteq P} uZ \subseteq \bigcap_{Y \subseteq Z \subseteq P} uZ = vY$. Obviously, $v \leqq u$ is valid. Let w be an M-topology on P such that $w \leqq u$. If $X \subseteq P$ is a subset, then $wX = \bigcap_{X \subseteq Z \subseteq P} wZ \subseteq \bigcap_{X \subseteq Z \subseteq P} uZ = vX$. Thus $w \leqq v$ and therefore v is the lower M-modification of u, i.e. $v = u_M$.

b) For any subset $X \subseteq P$ put $vX = \bigcup_{Z \subseteq X} uZ$. Clearly, v is an M-topology on P (since $X, Y \subseteq P$, $X \subseteq Y \Rightarrow vX = \bigcup_{Z \subseteq X} uZ \subseteq \bigcup_{Z \subseteq Y} uZ = vY$). Obviously, $u \leq v$. Let w be an M-topology on P such that $u \leq w$. If $X \subseteq P$ is a subset, then $vX = \bigcup_{Z \subseteq X} uZ \subseteq \bigcup_{Z \subseteq X} wZ = wX$. Thus $v \leq w$ and consequently v is the upper M-modification of u, i.e. $v = u^M$.

4. A-MODIFICATION

Lemma 1. *Let (P, u) be a topological space. Then*

$$u = \inf\{v \mid v \text{ is an } A\text{-topology on } P,\ u \leq v\}.$$

Proof. Put $\mathcal{G} = \{v \mid v \text{ is an } A\text{-topology on } P,\ u \leq v\}$. $\mathcal{G} \neq \emptyset$ since for the topology v on P defined by $X \subseteq P \Rightarrow vX = P$ we have $v \in \mathcal{G}$. Put $w = \inf \mathcal{G}$. Clearly, $u \leq w$. We shall show that conversely $w \leq u$. For every subset $Z \subseteq P$ let us define a topology $v_{(Z)}$ on P in the following way:

$$X \subseteq P \Rightarrow v_{(Z)}X = \begin{cases} uX \text{ for } X = Z, \\ P \text{ for } X \neq Z. \end{cases}$$

Evidently, $v_{(Z)} \geq u$ holds for every subset $Z \subseteq P$. Now, we shall show that $v_{(Z)}$ is an A-topology for every subset $Z \subseteq P$. On that account, let $X, Y, Z \subseteq P$ be subsets. Let $X \cup Y = Z$. If $X = Y = Z$, then $v_{(Z)}(X \cup Y) = v_{(Z)}X \cup v_{(Z)}Y$ is valid trivially. Otherwise, if at least one of the sets X, Y is different from Z, then $v_{(Z)}(X \cup Y) = v_{(Z)}Z = uZ \subseteq P = v_{(Z)}X \cup v_{(Z)}Y$. On the contrary, let $X \cup Y \neq Z$. Then at least one of the sets X, Y is different from Z. Thus $v_{(Z)}(X \cup Y) = P = v_{(Z)}X \cup v_{(Z)}Y$. Hence $v_{(Z)}$ is an A-topology. Consequently, $v_{(Z)} \in \mathcal{G}$ holds for every $Z \subseteq P$. Finally, let $X \subseteq P$ be an arbitrary subset. Then $wX \subseteq v_{(X)}X = uX$. From this $w \leq u$ follows and therefore $u = w$. The statement is proved.

Theorem 4. *Let (P, u) be a topological space. Then*
a) *u_A always exists and it is defined by*

$$X \subseteq P \Rightarrow u_A X = \bigcap \{Z \subseteq P \mid Z = \bigcup_{i=1}^{m} uX_i,\ \bigcup_{i=1}^{m} X_i = X,\ m \in N\},$$

where N *is the set of all positive integers.*
b) *u^A exists iff u is* an *A-topology and then $u^A = u$.*

Proof. *a*) For any subset $X \subseteq P$ put $vX = \bigcap \{Z \subseteq P \mid Z = \bigcup_{i=1}^{m} uX_i,\ \bigcup_{i=1}^{m} X_i = X,\ m \in N\}$. At first, we prove that v is A-topology on P. On that account, let $X, Y \subseteq P$ be subsets, $x \in v(X \cup Y)$ a point. Then $x \in \bigcup_{i=1}^{p} uV_i$ for every system of sets $\{V_i \mid i = 1, \dots, p\}$, $p \in N$, fulfilling $\bigcup_{i=1}^{p} V_i = X \cup Y$. Let $\{X_i \mid i = 1, \dots, m\}$, $m \in N$, be a system of sets such that $\bigcup_{i=1}^{m} X_i = X$. Similarly, let $\{Y_i | \ i = 1, \ldots, n\}$, $n \in \mathbf{N}$, be a system of sets such that $\bigcup_{i=1}^{n} Y_i = Y$. Put $W_i = X_i$ for $i = 1, \ldots, m$, and $W_i = Y_{i-m}$ for $i = m + 1, \ldots, m + n$. Then $\bigcup_{i=1}^{m+n} W_i = X \cup Y$ holds and thus $x \in \bigcup_{i=1}^{m+n} uW_i$. From this $x \in \bigcup_{i=1}^{m} uW_i = \bigcup_{i=1}^{m} uX_i$ or $x \in \bigcup_{i=m+1}^{m+n} uW_i = \bigcup_{i=1}^{n} uY_i$. Therefore $x \in \bigcup_{i=1}^{m} uX_i$ holds for any system $\{X_i | \ i = 1, \ldots, m\}$ fulfilling $\bigcup_{i=1}^{m} X_i = X$, or $x \in \bigcup_{i=1}^{n} uY_i$ holds for any system $\{Y_i | \ i = 1, \ldots, n\}$ fulfilling $\bigcup_{i=1}^{n} Y_i = Y$. This implies $x \in vX$ or $x \in vY$, i.e. $x \in vX \cup vY$. Hence $v(X \cup Y) \subseteq vX \cup vY$ is valid.

Now, we shall show that $v \leqq u$. So, let $X \subseteq P$ be a subset, $x \in vX$ a point. Then $x \in \bigcup_{i=1}^{m} uX_i$ for every system $\{X_i | \ i = 1, \ldots, m\}$, $m \in \mathbf{N}$, for which $\bigcup_{i=1}^{m} X_i = X$. Particularly, for $m = 1$ and $X_1 = X$ we have $x \in uX$. Thus $v \leqq u$.

Finally, we prove that v is the coarsest of all A-topologies on P finer than u. Let w be an A-topology on P such that $w \leqq u$. Let $X \subseteq P$ be a subset and $x \in wX$ a point. Let $\{X_i | \ i = 1, \ldots, m\}$, $m \in \mathbf{N}$, be a system of sets fulfilling $\bigcup_{i=1}^{m} X_i = X$. Then we have $x \in w \bigcup_{i=1}^{m} X_i \subseteq \bigcup_{i=1}^{m} wX_i \subseteq \bigcup_{i=1}^{m} uX_i$. This yields $x \in vX$ and hence $wX \subseteq vX$. Consequently, $w \leqq v$. We have proved that v is the lower A-modification of u, i.e. $v = u_A$.

The assertion b) follows immediately from Lemma 1.

Remark 1. For Čech topologies the same theorem holds (see [12], 3.1. and 3.2.). The assertion b) of it is an immediate consequence of that contained in Theorem 4.

5. *U*-MODIFICATION

Lemma 2. *Let (P, u) be a topological space. If u is an M-topology, then*

$$u = \sup \{v | \ v \text{ is a } U\text{-topology on } P, v \leqq u\}.$$

Proof. Put $\mathcal{G} = \{v | \ v \text{ is a } U\text{-topology on } P, \ v \leqq u\}$. Then $\mathcal{G} \neq \emptyset$ since for the topology v on P defined by $X \subseteq P \Rightarrow vX = \emptyset$ we have $v \in \mathcal{G}$. Put $w = \sup \mathcal{G}$. Clearly, $w \leqq u$. We shall show that conversely $u \leqq w$. For every subset $Z \subseteq P$ let us define a topology $v_{(Z)}$ on P in the following way:

$$X \subseteq P \Rightarrow v_{(Z)}X = \begin{cases} \emptyset & \text{for } Z \nsubseteq X, \\ uZ & \text{for } Z \subseteq X. \end{cases}$$

Obviously, $v_{(Z)} \leq u$ holds for every subset $Z \subseteq P$. At first, we shall show that $v_{(Z)}$ is an M-topology for any subset $Z \subseteq P$. Let $X, Y, Z \subseteq P$ be subsets, $X \subseteq Y$. If $Z \nsubseteq Y$, then $Z \nsubseteq X$ and therefore $v_{(Z)}X = \emptyset = v_{(Z)}Y$. Otherwise, let $Z \subseteq Y$. Now, if $Z \nsubseteq X$, then $v_{(Z)}X = \emptyset \subseteq v_{(Z)}Y$, and if $Z \subseteq X$, then $Z \subseteq Y$ and thus $v_{(Z)}X = uZ = v_{(Z)}Y$. Hence, in every case we have $v_{(Z)}X \subseteq v_{(Z)}Y$. Consequently, $v_{(Z)}$ is an M-topology for any subset $Z \subseteq P$.

Next, we shall show that $v_{(Z)}$ is a U-topology for every subset $Z \subseteq P$. On that account, let $X, Z \subseteq P$ be subsets. If $Z \nsubseteq X$, then $v_{(Z)}X = \emptyset$ and $v_{(Z)}v_{(Z)}X = v_{(Z)}\emptyset \subseteq v_{(Z)}X$ since $v_{(Z)}$ is an M-topology. Otherwise, let $Z \subseteq X$. Then $v_{(Z)}v_{(Z)}X = v_{(Z)}uZ$. Now, if $Z \nsubseteq uZ$, then $v_{(Z)}uZ = \emptyset$ so that $v_{(Z)}v_{(Z)}X \subseteq v_{(Z)}X$, and if $Z \subseteq uZ$, then $v_{(Z)}uZ = uZ = v_{(Z)}X$. Thus, in every case we have $v_{(Z)}v_{(Z)}X \subseteq v_{(Z)}X$. Hence $v_{(Z)}$ is a U-topology. Consequently, $v_{(Z)} \in \mathcal{G}$ for every subset $Z \subseteq P$.

Finally, let $X \subseteq P$ be an arbitrary subset. Then $uX = v_{(X)}X \subseteq wX$. From this $u \leq w$ follows and therefore $u = w$. The statement is proved.

Theorem 5. *Let (P, u) be a topological space. If u is an M-topology, then*

a) u_U exists iff u is a U-topology and then $u_U = u$.

b) u^U always exists and it is defined by

$$X \subseteq P \Rightarrow u^U X = \bigcap \{Y \subseteq P \mid uX \subseteq Y, uY \subseteq Y\}.$$

Proof. The assertion *a)* follows immediately from Lemma 2.

b) For any subset $X \subseteq P$ put $vX = \bigcap \{Y \subseteq P \mid uX \subseteq Y, uY \subseteq Y\}$. Let $X \subseteq P$ be a subset and $x \in vvX$ a point. Then $x \in Y$ for every subset $Y \subseteq P$ fulfilling $uvX \subseteq Y$ and $uY \subseteq Y$. Since u is an M-topology, there holds $uvX = u \bigcap \{Y \subseteq P \mid uX \subseteq Y, uY \subseteq Y\} \subseteq \bigcap \{uY \subseteq P \mid uX \subseteq Y, uY \subseteq Y\} \subseteq \bigcap \{Y \subseteq P \mid uX \subseteq Y, uY \subseteq Y\} = vX$. Thus, if we put $Y = vX$, then $Y \subseteq P$, $uvX \subseteq Y$ and $uY \subseteq Y$. Therefore $x \in Y = vX$ and we have proved the inclusion $vvX \subseteq vX$, so that v is a U-topology on P. Obviously, $u \leq v$. Let w be a U-topology on P such that $u \leq w$. Let $X \subseteq P$ be a subset and $x \in vX$ a point. Then $x \in Y$ for every subset $Y \subseteq P$ fulfilling $uX \subseteq Y$ and $uY \subseteq Y$. From $u \leq w$ the implication $wY \subseteq Y \Rightarrow uY \subseteq Y$ follows. Thus $x \in Y$ for every subset $Y \subseteq P$ fulfilling $uX \subseteq Y$ and $wY \subseteq Y$. Now, if we put $Y = wX$, then $Y \subseteq P$, $uX \subseteq Y$ and $wY \subseteq Y$. Therefore $x \in Y = wX$ and we have proved the inclusion $vX \subseteq wX$. Consequently, $v \leq w$. This yields that v is the upper U-modification of u, i.e. $v = u^U$.

Remark 2. For Čech topologies the same theorem holds (see [12], 3.7. and 3.8.). The assertion *a)* of it is an immediate consequence of that contained in Theorem 5.

6. *K*-MODIFICATION

Lemma 3. *Let* (P, u) *be a topological space. Then*

$$u = \sup \{v \mid v \text{ is a } K\text{-topology on } P, v \leqq u\}.$$

Proof. Put $\mathcal{G} = \{v \mid v \text{ is a } K\text{-topology on } P, v \leqq u\}$. Then $\mathcal{G} \neq \emptyset$ since for the topology v on P defined by

$$X \subseteq P, X \text{ is not one-point} \Rightarrow vX = uX,$$

$$x \in P \Rightarrow v\{x\} = \begin{cases} \emptyset & \text{for } x \notin u\{x\}, \\ \{x\} & \text{for } x \in u\{x\}, \end{cases}$$

we have $v \in \mathcal{G}$. Put $w = \sup \mathcal{G}$. Obviously, $w \leqq u$. We shall show that conversely $u \leqq w$. For every point $z \in P$ let us define a topology $v_{(z)}$ on P in the following way:

$$X \subseteq P, X \text{ is not one-point} \Rightarrow v_{(z)}X = uX,$$

$$x \in P \Rightarrow v_{(z)}\{x\} = \begin{cases} \emptyset & \text{for } x \neq z, \\ u\{z\} & \text{for } x = z. \end{cases}$$

Clearly, $v_{(z)} \leqq u$ holds for every point $z \in P$. We shall show that $v_{(z)}$ is a K-topology for any $z \in P$. On that account, let $x, y, z \in P$ be points, $x \in v_{(z)}\{y\}$, $y \in v_{(z)}\{x\}$. Then $y = z$ and $x = z$, so that $x = y$ and $v_{(z)}$ is a K-topology. Consequently, $v_{(z)} \in \mathcal{G}$ for any point $z \in P$. Next, let $X \subseteq P$ be a set. If X is not one-point, then let $z \in P$ be an arbitrary point, and we have $uX = v_{(z)}X \subseteq wX$. Otherwise, if $X = \{x\}$, $x \in P$, then $uX = u\{x\} = v_{(x)}\{x\} \subseteq w\{x\} = wX$. Thus, in both cases we have $uX \subseteq wX$. From this $u \leqq w$ follows and therefore $u = w$. The statement is proved.

Theorem 6. *Let* (P, u) *be a topological space. Then*

a) u_K *exists iff* u *is a* K*-topology and then* $u_K = u$.

b) u^K *exists iff* u *is a* K*-topology and then* $u^K = u$. *If* u *is no* K*-topology, then there does not exist any* K*-topology on* P *coarser than* u.

Proof. The assertion *a)* follows immediately from Lemma 3 and b) is evident.

Remark 3. For Čech topologies the same theorem holds (see [14], 4.3. and 4.4.). It is an immediate consequence of Theorem 6.

7. *B**-MODIFICATION

Theorem 7. *Let* (P, u) *be a topological space. Then*

a) u_{B*} *always exists and it is defined by*

$$X \subseteq P, X \text{ is not one-point} \Rightarrow u_{B*}X = uX,$$

$$x \in P \Rightarrow u_{B*}\{x\} = u\{x\} \cap \{z \in P \mid x \in u\{z\}\}.$$

b) u^{B*} *always exists and it is defined by*

$$X \subsetneq P, X \text{ is not one-point} \Rightarrow u^{B*}X = uX,$$
$$x \in P \Rightarrow u^{B*}\{x\} = u\{x\} \cup \{z \in P \mid x \in u\{z\}\}.$$

Proof. *a)* For any not one-point set $X \subsetneq P$ put $vX = uX$ and for any point $x \in P$ put $v\{x\} = u\{x\} \cap \{z \in P \mid x \in u\{z\}\}$. Clearly, v is a B^*-topology on P and $v \leqq u$. Let w be a B^*-topology on P such that $w \leqq u$. Let $X \subsetneq P$ be a subset. If X is not one-point, then $wX \subsetneq uX = vX$. Let $X = \{x\}$, $x \in P$, and let $y \in wX = w\{x\}$ be a point. Then $y \in u\{x\}$, and from $y \in w\{x\}$ it follows $x \in w\{y\}$, and thus $x \in u\{y\}$. Hence, $y \in u\{x\} \cap \{z \in P \mid x \in u\{z\}\} = v\{x\} = vX$. Therefore $wX \subsetneq vX$ holds, again. Consequently, $w \leqq v$. This implies that v is the lower B^*-modification of u, i.e. $v = u_{B*}$.

b) For any not one-point subset $X \subsetneq P$ put $vX = uX$ and for any point $x \in P$ put $v\{x\} = u\{x\} \cup \{z \in P \mid x \in u\{z\}\}$. Clearly, v is a B^*-topology and $u \leqq v$. Let w be a B^*-topology on P such that $u \leqq w$. Let $X \subsetneq P$ be a subset. If X is not one-point, then $vX = uX \subsetneq wX$. Let $X = \{x\}$, $x \in P$. Then $vX = v\{x\} = u\{x\} \cup \{z \in P \mid x \in u\{z\}\} \subsetneq w\{x\} \cup \{z \in P \mid x \in w\{z\}\} = w\{x\} \cup \{z \in P \mid z \in w\{x\}\} = w\{x\} = wX$. Therefore $vX \subsetneq wX$ holds, again. Consequently, $v \leqq w$. This implies that v is the upper B^*-modification of u, i.e. $v = u^{B*}$.

Remark 4. For Čech topologies the assertion *a)* of Theorem 7 holds, only. The upper B^*-modification of a Čech topology always exists, but it is defined in another way. (See [14], 2.4. and 2.5.)

8. *B*-MODIFICATION

Theorem 8. *Let* (P, u) *be a topological space. Then*

a) u_B always exists and it is defined by

$$X \subsetneq P, X \text{ is not one-point} \Rightarrow u_B X = uX,$$
$$x \in P \Rightarrow u_B\{x\} = \begin{cases} \{x\} & \text{for } x \in u\{x\}, \\ \emptyset & \text{for } x \notin u\{x\}. \end{cases}$$

b) u^B *exists iff* u *is a B-topology and then* $u^B = u$. *If* u *is no B-topology, then there does not exist any B-topology on P coarser than* u.

Proof. *a)* For any not one-point subset $X \subsetneq P$ put $vX = uX$ and for any point $x \in P$ put $v\{x\} = \{x\}$ if $x \in u\{x\}$ and $v\{x\} = \emptyset$ if $x \notin u\{x\}$. Clearly, v is a B-topology on P and $v \leqq u$. Let w be a B-topology on P such that $w \leqq u$. Let $X \subsetneq P$ be a subset. If X is not one-point, then $wX \subsetneq uX = vX$. Otherwise, let $X = \{x\}$, $x \in P$. Now, if $x \in u\{x\}$, then $wX = w\{x\} \subsetneq \{x\} = v\{x\} = vX$, and if $x \notin u\{x\}$, then $wX = w\{x\} = \emptyset = vX$ (since in the other case $w\{x\} \neq \emptyset$ we have $w\{x\} = \{x\}$ which implies $x \in u\{x\}$ and this is a contradiction). Therefore $wX \subsetneq vX$ holds for any $X \subsetneq P$ and consequently $w \leqq v$. This yields that v is the lower B-modification of u, i.e. $v = u_B$.

The assertion *b*) is evident.

Remark 5. For Čech topologies the same theorem holds (see [12], 3.3. and 3.4.). The assertion *b*) of it is an immediate consequence of that contained in Theorem 8.

9. *S*-MODIFICATION

Lemma 4. *Let (P, u) be a topological space. If u is an M-topology, then*

$$u = \inf\{v \mid v \text{ is an } S\text{-topology on } P,\ u \leqq v\}.$$

Proof. Put $\mathscr{G} = \{v \mid v$ *is an S-topology on* $P,\ u \leqq v\}$. Then $\mathscr{G} \neq \emptyset$ since for the topology v on P defined by $X \subseteqq P \Rightarrow vX = P$ we have $v \in \mathscr{G}$. Put $w = \inf \mathscr{G}$. Obviously, $u \leqq w$. We shall show that conversely $w \leqq u$. For any subset $Y \subseteqq P$ let us define a topology $v_{(Y)}$ on P in the following way:

$$X \subseteqq P \Rightarrow v_{(Y)}X = \begin{cases} uY & \text{for } X \subseteqq Y, \\ P & \text{for } X \nsubseteq Y. \end{cases}$$

Since u is an M-topology, it is $u \leqq v_{(Y)}$ for every $Y \subseteqq P$. We shall show that $v_{(Y)}$ is an S-topology for any subset $Y \subseteqq P$. On that account, let $Y \subseteqq P$, $\emptyset \neq X \subseteqq P$ be subsets. Let $X \subseteqq Y$. Then the implication $x \in X \Rightarrow x \in Y$ holds and therefore $v_{(Y)}\{x\} = uY$ is valid for every point $x \in X$. Consequently, $v_{(Y)}X = uY = \bigcup_{x \in X} v_{(Y)}\{x\}$. Otherwise, let $X \nsubseteq Y$. Then there exists a point $x_0 \in X$ such that $x_0 \notin Y$, and therefore it is $v_{(Y)}\{x_0\} = P$. Consequently, $v_{(Y)}X = P = \bigcup_{x \in X} v_{(Y)}\{x\}$. Thus $v_{(Y)}$ is an S-topology and $v_{(Y)} \in \mathscr{G}$ holds for every subset $Y \subseteqq P$. Finally, let $X \subseteqq P$ be an arbitrary subset. Then $wX \subseteqq v_{(X)}X = uX$. Hence $w \leqq u$, and therefore $u = w$. The statement is proved.

Theorem 9. *Let (P, u) be a topological space. If u is an M-topology, then*
a) u_S always exists and it is defined by

$$u_S\emptyset = \emptyset,$$
$$\emptyset \neq X \subseteqq P \Rightarrow u_S X = \bigcup_{x \in X} u\{x\}.$$

b) u^S exists iff u is an S-topology and then $u^S = u$.

Proof. *a*) Put $v\emptyset = u\emptyset$ and $vX = \bigcup_{x \in X} u\{x\}$ whenever $\emptyset \neq X \subseteqq P$. Obviously v is an S-topology on P and since u is an M-topology, there holds $v \leqq u$. Let w be an S-topology on P such that $w \leqq u$. Let $X \subseteqq P$ be a subset. If $X = \emptyset$, then $wX = w\emptyset \subseteqq u\emptyset = v\emptyset = vX$. Let $X \neq \emptyset$. Then $wX \subseteqq \bigcup_{x \in X} w\{x\} \subseteqq \bigcup_{x \in X} u\{x\} = vX$. Therefore $wX \subseteqq vX$ is valid for any subset $X \subseteqq P$. Consequently, $w \leqq v$. This implies that v is the lower S-modification of u, i.e. $v = u_S$.

The assertion *b*) follows immediately from Lemma 4.

Remark 6. For closure operations (i.e. *OIMA*-topologies) the same theorem holds. As for the assertion *a*) of it see [6], 26A.4. The assertion *b*) of it is an immediate consequence of that contained in Theorem 9. The lower *S*-modification of a Bourbaki topology (i.e. of an *OIMAU*-topology) *u* is exactly the *A-complement* of *u* defined and studied in [1].

10. EXAMPLES

Example 1. Let G be a set and ϱ a binary relation on G. Let us define a topology u on G by

$$X \subseteq G \Rightarrow uX = \{y \in G \mid \exists x \in X: x \varrho y\}.$$

The topology u is called *associated* with the binary relation ϱ.

a) The topology u is not I-topology in general and u^I is the topology associated with the relation $\varrho \cup \varepsilon$ where ε is the diagonal relation on G.

b) The topology u is not U-topology in general, but it is an M-topology, and u^U is the topology associated with the transitive closure $\hat{\varrho}$ of ϱ (of course, $\hat{\varrho} = \bigcup_{n=1}^{\infty} \varrho^n$).

c) The topology u is not B^*-topology in general and u_{B*} is associated with the relation $\varrho \cap \varrho^{-1}$ while u^{B^*} is associated with the relation $\varrho \cup \varrho^{-1}$ (ϱ^{-1} is the inverse relation to ϱ).

d) The topology u is not B-topology in general and u_B is associated with the relation $\varrho \cap \varepsilon$ (again, ε is the diagonal relation on G).

Example 2. Let $(G, .)$ be a groupoid with a unit element e. Let us define a topology u on G by

$$X \subseteq G \Rightarrow uX = \{y \in G \mid \exists x \in X: x \, . \, y = e\}.$$

Thus, uX is the set of all right inverse elements to the elements of X. Let u be called a topology *induced* by the operation.. The topology u is not I-topology in general and according to Theorem 2 u^I is defined by $X \subseteq G \Rightarrow u^I X = uX \cup X$. Let $(G, \circ)$ be another groupoid with the unit element e, where the operation $\circ$ is defined as follows:

$$x, y \in G \Rightarrow x \circ y = \begin{cases} x \, . \, y & \text{for } x \neq y, \\ e & \text{for } x = y. \end{cases}$$

We can easily show that u^I is induced by the operation $\circ$.

Example 3. Let P be a set and u an *OIMAU*-topology (i.e. a Bourbaki topology) on P. Let us define a topology v on P by

$$X \subseteq P \Rightarrow vX = uX \cap u(P - X).$$

Then vX is called the *boundary* of the set X in the topology u, and it is well known that $uX = X \cup vX$ holds for any subset $X \subseteq P$.

a) The topology v is not I-topology in general and according to Theorem 2 we have $v^I X = vX \cup X = uX$ for any subset $X \subseteq P$. Thus, $v^I = u$ is valid.

b) The topology v is not M-topology in general and according to Theorem 3 we have $v_M X = \bigcap_{X \subseteq Z \subseteq P} vZ = \bigcap_{X \subseteq Z \subseteq P} [uZ \cap u(P - Z)] = \emptyset$ for any subset $X \subseteq P$.

Example 4. Let P be a set and u an $OIMAUB$-topology (i.e. T_1-topology) on P.

a) Let u be *connected*, i.e. let the implication

$$X, Y \subseteq P, X \cap Y = \emptyset, P = X \cup Y \Rightarrow uX \neq X \quad \text{or } uY \neq Y$$

hold. Same as in Example 3, denote by v the boundary in the topology u (i.e. $X \subseteq P \Rightarrow$ $\Rightarrow vX = uX \cap u(P - X)$). Then v is not M-topology in general. We shall show that $v^M = u$ holds.

Since $v \leq u$ and u is an M-topology, we have $v^M \leq u$. Let X be a set and $x \in uX$ a point. Then $x \in X$ or $x \in vX$. At first, suppose that $x \in X$. Then $u\{x\} \neq \{x\}$ or $u(P - \{x\}) \neq P - \{x\}$ since u is connected. But u is an IB-topology and therefore $u\{x\} = \{x\}$. Hence $u(P - \{x\}) \neq P - \{x\}$ and this yields $u(P - \{x\}) = P$ since u is an I-topology. From this $v\{x\} = u\{x\} \cap u(P - \{x\}) = \{x\}$. Consequently, $x \in \bigcup_{Z \subseteq X} vZ$. At second, suppose that $x \in vX$. But then $x \in \bigcup_{Z \subseteq X} vZ$ holds trivially. We have proved the implication $x \in uX \Rightarrow x \in \bigcup_{Z \subseteq X} vZ$. As $\bigcup_{Z \subseteq X} vZ = v^M X$ (according to Theorem 3), it is $uX \subseteq v^M X$. Hence $u \leq v^M$. Indeed, $v^M = u$ is valid.

b) The topology u is not S-topology in general and according to Theorem 9 we have $u_S \emptyset = u\emptyset = \emptyset$ and $\emptyset \neq X \subseteq P \Rightarrow u_S X = \bigcup_{x \in X} u\{x\} = \bigcup_{x \in X} \{x\} = X$. So that u_S is the so called *discrete* topology on P.

Appendix. For the modifications of Čech topologies (found in [12] and [14]) the axioms preserved by these individual modifications are determined in [16]. To complete our investigations of the modifications of topologies without axioms, it is necessary to do an analogous determination for them. It will be done in the forthcoming paper.

Acknowledgment. The autor would like to express his gratitude to Professor J. Chvalina for his helpful comments and references.

REFERENCES

[1] D. Adnadževič, *A*-complements of topological spaces (Russian), Math. Balcanica 4 (1974), 9–12.

[2] C. E. Aull and W. J. Thron, *Separation axioms between T_0 and T_1*, Indag. Math. 24 (1962), 26–37.
[3] S. L. Bloom and D. J. Brown, *Classical abstract logics*, Dissert. Math. CII (1973), 43–52.
[4] N. Bourbaki, *Topologie générale*, p. 1., II. ed., Paris, 1951.
[5] D. J. Brown and R. Suszko, *Abstract logics*, Dissert. Math. CII (1973), 5–41.
[6] E. Čech, *Topological spaces* (Revised by Z. Frolík and M. Katětov), Academia, Prague, 1966.
[7] E. Čech, *Topological papers of Eduard Čech*, ch. 28, Academia, Prague, 1968, 436–472.
[8] J. Chvalina, *On the number of general topologies on a finite set*, Scripta Fac. Sci. Nat. UJEP Brno, Math. I, 3 (1973), 7–22.
[9] A. S. Davis, *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly 68 (1961), 886–893.
[10] P. C. Hammer, *Extended topology: set-valued set-functions*, Nieuw Arch. Wisk. X (1962), 55–77.
[11] J. Hanák, *Game-theoretical approach to some modifications of generalized topologies*, Gen. Topology and its Relations to Modern Analysis and Algebra, Prague, 1971, 173–179.
[12] K. Koutský, *O některých modifikacích dané topologie*, Rozpravy II. třídy České akademie 48 (1938), n. 22, 1–13.
[13] K. Koutský, *Určenost topologických prostorů pomocí úplných systémů okolí bodů*, Spisy přír. fak. MU Brno č. 374 (1956), 1–11.
[14] K. Koutský and M. Sekanina, *On the R-modification and several other modifications of a topology*, Publ. Fac. Sci. Univ. Brno, No. 410 (1960), 45–64.
[15] K. Koutský and M. Sekanina, *On the systems of topologies with a given modification*, Publ. Fac. Sci. Univ. Brno 418 (1960), 425–464.
[16] K. Koutský, V. Polák and M. Sekanina, *On the commutativity of the modifying*, Publ. Fac. Sci. Univ. Brno 454, ser. A26 (1964), 275–292.
[17] F. Lorrain, *Notes on topological spaces with minimum neighborhoods*, Amer. Math. Monthly 76 (1969), 616–627.
[18] V. Polák, N. Poláková and M. Sekanina, *Remarks to inner constellation in topological spaces*, Publ. Fac. Sci. Univ. Brno 507 (1969), 317–332.
[19] M. Sekanina, *Systems of topologies on a given set* (Russian), Czech. Math. Journ. 15 (1965), 9–29.
[20] W. Sierpiński, *Introduction to general topology*, Toronto, 1934.
[21] R. Wojcicki, *Some remarks on the consequence operation in sentential logics*, Fund. Math. LXVIII (1970), 269–279.

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