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THE SUFFICIENT CONDITION OF THE ASYMPTOTIC STABILITY OF TWO-DIMENSIONAL LINEAR SYSTEMS

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Abstract. The differential system of second-order with variable coefficients is studied, and a sufficient condition of the asymptotic stability for solutions is given.

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1. INTRODUCTION

In the present paper we consider a system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)\,x_1 + a_{12}(t)\,x_2, \\ \frac{dx_2}{dt} &= a_{21}(t)\,x_1 + a_{22}(t)\,x_2, \end{aligned} \tag{1.1}$$

where $a_{ik}\colon R^+ \to R$ $(i, k = 1, 2)$ are functions summable on every finite segment.

It will be assumed throughout that

$$\sigma a_{12}(t) > 0, \qquad \sigma a_{21}(t) < 0 \tag{1.2}$$

if $t \in R^+$, where $\sigma \in \{-1, 1\}$ and the function $\frac{a_{21}}{a_{12}}$ is summable on every finite segment.

Let

$$c(t) = \sigma(|\,a_{12}(t)\,a_{21}(t)\,|)^{-1/2}\left[\frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{1}{4}\left(\ln\left|\frac{a_{21}(t)}{a_{12}(t)}\right|\right)'\right], \tag{1.3}$$

$$l_i(t) = \left(\left|\frac{a_{ij}(t)}{a_{ji}(t)}\right|\right)^{1/4} \exp\left[\frac{1}{2}\int_0^t (a_{11}(\tau) + a_{22}(\tau))\,d\tau\right] \qquad (i \neq j;\ i, j = 1, 2), \tag{1.4}$$

$$\psi(t) = \int_0^t \sqrt{|\,a_{12}(\tau)\,a_{21}(\tau)\,|}\,d\tau. \tag{1.5}$$

Lemma 1. *By means of the transformations*

$$x_i(t) = l_i(t)\, y_i(s) \qquad (i = 1, 2),\ s = \psi(t) \tag{1.6}$$

the system (1.1) will take the form

$$\frac{dy_1}{ds} = \sigma[\alpha(s)\, y_1 + y_2], \qquad \frac{dy_2}{ds} = -\sigma[y_1 + \alpha(s)\, y_2], \tag{1.7}$$

where

$$\alpha(s) = c(\psi^{-1}(s)) \qquad \text{if } 0 \leqq s < s_0,\ s_0 = \lim_{t\to\infty} \psi(t) \tag{1.8}$$

and ψ^{-1} *is the inverse to* ψ.

Proof. Let (x_1, x_2) be an arbitrary solution of the system (1.1). In view of (1.4), (1.5) and (1.6)

$$l_i'(t) = \left[\frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{1}{4}\left(\ln\left|\frac{a_{ij}(t)}{a_{ji}(t)}\right|\right)'\right] l_i(t), \tag{1.9}$$

where $i \neq j$, $i, j = 1, 2$ and

$$x_i'(t) = l_i'(t)\, y_i(s) + l_i(t)\, (|\, a_{12}(t)\, a_{21}(t)\,|)^{1/2}\, y_i'(s) \tag{1.10}$$

$(i = 1, 2)$. By substitung (1.6), (1.9) and (1.10) into (1.1) we obtain (1.7). The lemma is proved.

Lemma 2. *Let the function* α *in* (1.8) *be absolutely continous on every finite segment and let there exist* $s_1 \in (0, s_0)$ *and* $\delta \in (0, 1)$ *such that* $|\,\alpha(s)\,| < \delta$ *for* $s_1 \leqq s < s_0$. *Then every solution* (y_1, y_2) *of* (1.7) *satisfies the estimate*

$$y_1^2(s) + y_2^2(s) \leqq \frac{1 + \delta}{1 - \delta}\,[y_1^2(s_1) + y_2^2(s_1)] \exp\left[\frac{1}{1 - \delta} \int_{s_1}^{s} |\,\alpha'(\xi)\,|\, d\xi\right] \tag{1.11}$$

for $s_1 \leqq s < s_0$.

Proof. Let (y_1, y_2) be an arbitrary solution of the system (1.7). From (1.7) we have

$$-y_1(s)\, y_1'(s) - \alpha(s)\, y_2(s)\, y_1'(s) = \alpha(s)\, y_1(s)\, y_2'(s) + y_2(s)\, y_2'(s).$$

Therefore

$$(y_1^2(s) + y_2^2(s))' = -2\alpha(s)\, (y_1(s)\, y_2(s))'.$$

Integrating of this equality from s_1 to s yields

$$y_1^2(s) + y_2^2(s) = y_1^2(s_1) + y_2^2(s_1) + 2\alpha(s_1)\, y_1(s_1)\, y_2(s_1) - {} - 2\alpha(s)\, y_1(s)\, y_2(s) + 2 \int_{s_1}^{s} \alpha'(\xi)\, y_1(\xi)\, y_2(\xi)\, d\xi. \tag{1.12}$$

Let $u(s) = y_1^2(s) + y_2^2(s)$. Then $2\,|\, y_1(s)\, y_2(s)\,| \leqq u(s)$ and from (1.12) we get

$$u(s) \leqq (1 + \delta)\, u(s_1) + \delta u(s) + \int_{s_1}^{s} |\, \alpha'(\xi) \,|\, u(\xi)\, d\xi,$$

$$u(s) \leqq \frac{1 + \delta}{1 - \delta} u(s_1) + \frac{1}{1 - \delta} \int_{s_1}^{s} |\, \alpha'(\xi) \,|\, u(\xi)\, d\xi$$

for $s_1 \leqq s < s_0$. Hence according to the Gronwall–Bellman lemma

$$u(s) \leqq \frac{1 + \delta}{1 - \delta} u(s_1) \exp\left[\frac{1}{1 - \delta} \int_{s_1}^{s} |\, \alpha'(\xi) \,|\, d\xi\right]$$

for $s_1 \leqq s < s_0$. The lemma is proved.

2. THE ASYMPTOTIC STABILITY OF THE SYSTEM (1.1)

Theorem 1. *Let for large t the inequality*

$$\delta_1 < \left| \frac{a_{21}(t)}{a_{12}(t)} \right| < \delta_2, \tag{2.1}$$

where δ_1 and δ_2 are positive constans, hold. Moreover, let the function c be absolutely continuous on every finite segment,

$$\limsup_{t \to \infty} |\, c(t) \,| < 1, \qquad \int_0^{\infty} |\, c'(t) \,|\, dt < \infty \tag{2.2}$$

and

$$\lim_{t \to \infty} \int_0^t (a_{11}(\tau) + a_{22}(\tau))\, d\tau = -\infty. \tag{2.3}$$

Then the system (1.1) is asymptotically stable.

Proof. According to (2.2) the conditions of Lemma 2 hold. Therefore, from (1.6) by means of (2.1) and (2.3) we conclude that (1.1) is asymptotically stable. This completes the proof.

Corollary 1. *Let $a_{11}(t) = 0$, $a_{12}(t) = -a_{21}(t) > 0$ for $t \in R^+$,*

$$\limsup_{t \to \infty} \left(-\frac{a_{22}(t)}{2a_{12}(t)} \right) < 1, \qquad \int_0^{\infty} \left| d\left(-\frac{a_{22}(t)}{2a_{12}(t)} \right) \right| < \infty$$

and

$$\lim_{t \to \infty} \int_0^t a_{22}(\tau)\, d\tau = -\infty.$$

Then the system (1.1) is asymptotically stable.

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