Stanislav Říha

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HAMILTONIAN LINES IN THE SQUARE OF GRAPHS

I. HAMILTONIAN CIRCUITS IN THE SQUARE OF CACTI

STANISLAV ŘÍHA

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Abstract. A graph is a cactus if each edge of G is in at most one cycle of G. Necessary and sufficient condition for the existence of the Hamiltonian circuit in the square of a cactus is given in this paper.

Key words. Graph, Hamiltonian circuit, cactus, square.

MS Classification. 05 C 45

In this paper we use the terminology and notation of Harary [2]. Now, we define some special notions.

Let G be any graph. For nonnegative integer i, $V_i(G)$ is the set of all vertices of the degree i in G. If H is a subgraph of G, then we define the graph $G-H$ as follows: $V(G-H) = V(G) - V_0(G - E(H))$, $E(G-H) = E(G) - E(H)$. A vertex u of G is free provided it is not a cut vertex. A block B of G is free provided at least $|V(B)| - 1$ its vertices are free in G (in the block cut vertex-tree of a graph G the end-vertices agree with all free blocks of G). Otherwise B is an inner block. We say that the subgraphs G_1, G_2 of G touch each other in a vertex v (in G) if $V(G_1) \cap V(G_2) = \{v\}$. A vertex z of G is of type X in G provided it is a cut vertex in which no two inner blocks touch each other in G. The set of all blocks and inner blocks of G is denoted by BL^G and $\overline{BL}^G$ respectively, the set of all blocks of G containing a common vertex w is denoted by $BL^G(w)$. For $BL \subseteq BL^G$, we define $BL^G(BL, w) = BL^G(w) - BL$. If a vertex w is of type X, then $BL^G(\overline{BL}^G, w) \neq \emptyset$. We say that a subgraph H of G is a BL-subgraph of G if and only if $BL^H \subseteq BL^G$.

Let v be a vertex of G. Then a v-fragment of G is any maximal connected subgraph of G in which v is not a cut vertex. If H is a BL-subgraph of G and v is a vertex of H, then an H, v-fragment of G is any v-fragment of G edge disjoint with H.

Let $y = y_0, \ldots, y_m$ and $x = x_1, \ldots, x_n$ be sequences of some vertices of G. We use the following notation and terminology: $F(y) = y_0$, $L(y) = y_m$, $V(y) = \{y_0, \ldots, y_m\}$. y^{-1} and (y), (x) indicates the sequence $y_m, \ldots, y_0$ and $y_0, \ldots, y_m, x_1, \ldots, x_n$ respectively. We say that y is a section in x if there are sequences a and c like that $x = (a), (y), (c)$ (either a or c or both may be the empty sequences). If $y_0 = y_m$, then a rotation of y is any sequence of the form $y_i, y_{i+1}, \ldots, y_m, y_1, \ldots, y_i$, where $i \in \{0, 1, \ldots, m-1\}$. A transform of y is any rotation of y or y^{-1}.

A connected graph $G = (V, E)$ is a cactus if and only if for each edge $e \in E(G)$ there is at most one subgraph H of G which is a cycle (i.e. a regular connected graph of degree 2) such that $e \in E(H)$. Then, in a cactus G, every block is either a cycle or a bridge. If C is a block of G, then the statement $C = c_1, \ldots, c_n$ indicates the following: $V(C) = \{c_1, \ldots, c_n\}$, c_i is adjacent to c_{i+1} in G for each $i \in \{1, \ldots, n-1\}$ (hence c_n is adjacent to c_1 in G, too).

Let v be a vertex of any graph G. We say that G is short (with respect to v) if there is a Hamiltonian path p in $G^2 - v$ such that $F(p)$ and $L(p)$ are both adjacent to v in G. We say that G is long (with respect to v) if G is not short (with respect to v) but there is a Hamiltonian path q in $G^2 - v$ such that $F(q)$ is adjacent to v in G and the vertices $L(q)$ and v have a distance 2 in G. If G is neither short nor long (with respect to v), it is unusable (with respect to v).

The following theorem was proved in [3].

Theorem. *Let G be a cactus with a block $C = c_1, \ldots, c_n$. Then G^2 is Hamiltonian if and only if*

(1) *no C, c_i-fragment of G is unusable for each $i \in \{1, \ldots, n\}$,*

(2) *no more than two C, c_i-fragments of G are long for each $i \in \{1, \ldots, n\}$,*

(3) *if two distinct C, c_i-fragments and two distinct C, c_j-fragments of G are long, then each nontrivial c_i, c_j-walk in G includes a vertex whose degree in G is 2 (c_i and c_j may be the same vertex).*

The condition given in this paper consists in describing the whole class of the „prohibited" graphs, i.e. such graphs a cactus G must not include as its BL-subgraph, in the case, its square is Hamiltonian. The condition is the generalization of the condition given in [5] for triangular cacti.

The concepts established in the following four definitions are fundamental for a description of our results.

Definition 1. *Let G be a cactus and x a vertex of G. A C-generating sequence of G from the vertex x is any sequence of the cacti $G(1), \ldots, G(t) = G$ arising in the following manner.*

1. *$G(1) = \bigcup_{A \in BL^G(x)} A$. The set $BL^G(x)$ is called the first growth and we say that it is of the type (n), where $n = |BL^G(x)|$. The vertex x is a root.*

2. *Suppose, we have constructed a cactus $G(i-1)$ and $B = b_1, \ldots, b_r$ is an arbitrary free block from $G(i-1)$ such that the vertex b_r is either a cut vertex of $G(i-1)$ or $b_r = x$ and at least one of the vertices $b_1, \ldots, b_{r-1}$ is a cut vertex of G. Then $G(i) = G(i-1) \bigcup_{j=1}^{r-1} \bigcup_{A \in BL^G(B, b_j)} A$ and the set $\bigcup_{j=1}^{r-1} BL^G(B, b_j)$ is called an i-th growth. We add to the i-th growth an ordering sequence $(m_1, \ldots, m_{r-1})$, where $m_j = |BL^G(B, b_j)|$ for each $j \in \{1, \ldots, r-1\}$ and we say that the i-th growth starts from the block B and is of type $(m_1, \ldots, m_{r-1})$.*

If there is no block B of the mentioned properties, then evidently $G(i-1) = G$ and the construction of a C-generating sequence stops.

Definition 2. *Let G be a cactus, $G(1), \ldots, G(t) = G$ be any C-generating sequence of G from a vertex x. Suppose, the i-th growth of this sequence starts from a block $B = b_1, \ldots, b_r$, where either $b_r = x$ or b_r is a cut vertex of $G(i-1)$ and it is of type $(m_1, \ldots, m_{r-1})$. We say that the i-th growth is of*

1. *The first sort if $m_j = 1$ for each $j \in \{1, \ldots, r-1\}$ and all blocks of the i-th growth are free in G.*

2. *The second sort if either*

2a. *$m_1 = 0$ and there is an index $s \in \{2, \ldots, r-1\}$ such that $m_s = 2$, $m_j = 0$ for each $j \in \{2, \ldots, s-1\}$ and $m_j = 1$ for each $j \in \{s+1, \ldots, r-1\}$, or*

2b. *$m_{r-1} = 0$ and there is an index $s \in \{1, \ldots, r-2\}$ such that $m_s = 2$, $m_j = 0$ for each $j \in \{s+1, \ldots, r-1\}$ and $m_j = 1$ for each $j \in \{1, \ldots, s-1\}$, and all blocks of the i-th growth are free in G with the exception of the set of blocks $BL^G(B, b_s)$ which are the inner ones in G.*

Notes. It immediately follows from the preceeding definitions:

1. The cacti $G(1), \ldots, G(t)$ are the BL-subgraphs of G.

2. If x is a cut vertex of G, then $|\overline{BL}^G| = t-1$. Conversely, if $|\overline{BL}^G| = t'$ and $G(1), \ldots, G(t)$ is any C-generating sequence of G from any cut vertex of G, then $t' = t-1$.

3. The growth of the type (1) can never be of the second sort.

Definition 3. *A cactus G is a C-diad if there are a vertex x and a C-generating sequence $G(1), \ldots, G(t)$ of G from the vertex x such that*

1. *$t > 1$ and the first growth is of type (1).*

2. *An i-th growth is either of the first or the second sort for each $i \in \{2, \ldots, t\}$. The vertex x is called a root of G and the block $G(1)$ is called a root block of G. If $t = 2$ we say that a C-diad G is prime.*

Definition 4. *A cactus G is called a 3-C-diad if there are the BL-subgraphs G_1, G_2, G_3 of G such that*

1. G_1, G_2, G_3 *are the mutually edge-disjoint C-diads with a common root x,*
2. $\bigcup_{i=1}^{3} G_i = G.$

The vertex x is called a root of the 3-C-diad G.

Notes.

1. A 3-C-diad G can be also described in the following. There is a C-generating sequence $G(1), \dots, G(t)$ of G from a vertex x such that the first growth is of the type (3), every block of the first growth is the inner one of G and every further growth is either of the first or the second sort.

2. There are more roots in a 3-C-diad G. All roots of the 3-C-diad in the fig. 1 are indicated.

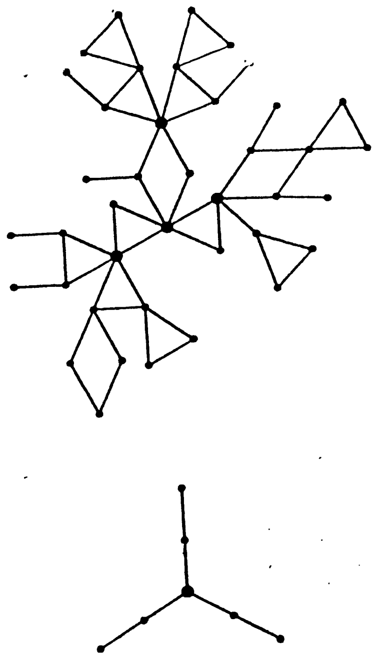

fig. 1

Theorem 1. *Let G be a cactus with at least three vertices such that*

1. *No 3-C-diad is included in G as a BL-subgraph.*

2. *All vertices of every inner block of G are the cut vertices. If* $Z = \{z_j, j \in J\}$ *is the set of all vertices of type X in G and, for each* $j \in J$, $A_j = a_{j1}, \dots, a_{j,n_j}, z_j$ *is an arbitrary block of* $BL^G(\overline{BL}^G, z_j)$, *then there is a Hamiltonian circuit h in* G^2 *having the following properties.*

a) *For each* $j \in J$, *there is a transform of* h *of the form*

$$(x_j), a_{j1}, \ldots, a_{j,n_j}, z_j, (y_j).$$

b) *For every sequence* $a_1, \ldots, a_m$ *of mutually different free vertices of* G *in which* a_t *is adjacent to* a_{t+1} *for each* $t \in \{1, \ldots, m-1\}$ *there is a transform of* h *of the form* $(x), a_1, \ldots, a_m, (y)$.

Proof. If $|\overline{BL}^G| = 0$, then G is either a cycle and theorem holds or there is just a single vertex z of type X in G that is the common vertex of all blocks of G, i.e. $BL^G = BL^G(z) = BL^G(\emptyset, z)$. Let $A_1, \ldots, A_s$, $A_r = a_{r1}, \ldots, a_{r,n_r}, z$ for each $r \in \{1, \ldots, s\}$, be all blocks of G. Then even for two arbitrary blocks of $BL^G(\emptyset, z)$, say A_i and A_j $(i < j)$, there is a Hamiltonian circuit h in G^2, $h = z, (w_j^{-1}), (w_{j+1}), \ldots, \ldots, (w_s), (w_1), \ldots, (w_{i-1}), (w_{i+1}), \ldots, (w_{j-1}), (w_i), z$ where $w_r = a_{r1}, \ldots, a_{r,n_r}$ for each $r \in \{1, \ldots, s\}$, which complies with a) and b), too.

Suppose $|\overline{BL}^G| = n \geqq 1$ and the theorem holds for every cactus with less then n inner blocks. As G is not a cycle, there is a vertex v of G which is of type X. Let $G(1), \ldots, G(n+1)$ be a C-generating sequence of G from the vertex v. Assume that $(n+1)$-st growth starts from a block $C = c_1, \ldots, c_k, z$, where z is a cut vertex of G, and is of type $(m_1, \ldots, m_k)$. Then $m_j \geqq 1$ for each $j \in \{1, \ldots, k\}$ and z is of type X in $G(n)$. If z is not of type X in $G(n)$, there are two inner blocks in $G(n)$ that touch each other in the vertex z. As all vertices of every inner block are the cut vertices of G, these two blocks together with B are the root blocks of three edge disjoint prime C-diad with common root z, that are the BL-subgraphs of G, which is not the case.

Now, for each $i \in \{1, \ldots, k\}$, let $C_{ij} = c^j_{i1}, \ldots, c^j_{i,p_{ij}}, c_i$ where $j \in \{1, \ldots, m_i\}$, be all blocks that touch the block C in a vertex c_i. As $\overline{BL}^G \cap BL^G(C, c_i) = \emptyset$ for each $i \in \{1, \ldots, k\}$, all vertices $c_1, \ldots, c_k$ are of type X in G. Let us differentiate two cases.

(1) $n \geqq 2$. Then $z \neq v$ and z is not of type X in G. Let $z_1, \ldots, z_p, c_1, \ldots, c_k$ be all vertices of type X in G, $A_s = a_{s1}, \ldots, a_{s,n_s}, z_s$ be an arbitrary block of $BL^G(\overline{BL}^G, z_s)$ for each $s \in \{1, \ldots, p\}$ and C_{i,r_i} be an arbitrary block of $BL^G(\overline{BL}^G, c_i)$ for each $i \in \{1, \ldots, k\}$. Then $z_1, \ldots, z_p, z$ are all vertices of type X in $G(n)$. As $|\overline{BL}^{G(n)}| < n$ and a cactus $G(n)$ fulfils all assumptions of the theorem, then for the same choice of the blocks A_s of $BL^{G(n)}(\overline{BL}^{G(n)}, z_s)$ $(= BL^G(\overline{BL}^G, z_s))$ for each $s \in \{1, \ldots, p\}$ like in G and for the choice C from $BL^{G(n)}(\overline{BL}^{G(n)}, z)$ there is, assumed by the induction, a Hamiltonian circuit h in $G(n)^2$ having properties a) and b). Especially, there is a transform of $\bar{h}$ of the form $z, (w), c_1, \ldots, c_k, z$ (obviously $a_{s1}, \ldots, a_{s,n_s}, z_s$ is a section either in w or in w^{-1}, for each $s \in \{1, \ldots, p\}$). Let us denote $w_{ij} = c^j_{i1}, \ldots, c^j_{i,p_{ij}}$ for each $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, m_i\}$. Then $h = z$, $(w_{11}), \ldots, (w_{1,r_1-1}), (w_{1,r_1+1}), \ldots, (w_{1,m_1}), (w_{1,r_1}), c_1, \ldots, (w_{k1}), \ldots, (w_{k,r_k-1}), (w_{k,r_k+1}), \ldots, (w_{k,m_k}), (w_{k,r_k}), c_k, (w^{-1}), z$ is a Hamiltonian circuit in G^2. Next, it

immediately follows from the induction assumption and from the form of the extending of $\bar{h}$ on h that h has both properties a) and b).

(2) $n = 1$. Then $z = v$ and z is the only vertex of type X in $G(1)$. As $| BL^{G(1)} | = 0$, the theorem (part a)) holds in $G(1)$ for a choice of two blocks of $BL^{G(1)}(\emptyset, z)$. In the same way like in (1) we can extend a Hamiltonian circuit from $G(1)^2$ on G^2 so that the theorem holds.

Theorem 2. *Let G be a cactus with at least three vertices which includes no 3-C-diad as its BL-subgraph and let $Z = \{z_j, j \in J\}$ be the set of all vertices of type X in G. Now, if $A_j = a_{j1}, \ldots, a_{j,n_j}, z_j$ is an arbitrary block from $BL^G(\overline{BL}^G, z_j)$, for each $j \in J$, then there is a Hamiltonian circuit h in G^2 having properties a) and b) from Theorem 1.*

Proof. If $| BL^G | = 1$, then G is a cycle and the theorem holds. Suppose, $| BL^G | = n \geqq 2$ and the theorem holds for every cactus with less then n blocks. If all vertices of every inner block of G are the cut vertices, the theorem follows from Theorem 1. Otherwise, there is an inner block with at least one free vertex in G. Let us consider the following possibilities (1) and (2) (we shall prove later that there are no other possibilities).

(1) There is an inner block $B = b_1, \ldots, b_k, b$ of G, where b is a cut vertex, that $| \overline{BL}^G \cap BL^G(B, b) | \leqq 1$ and at least one of the vertices b_1 and b_k is free in G. Let us say b_1 is such vertex.

(2) There is an inner block $B = b_1, \ldots, b_k, b$, where b is a cut vertex such that $| \overline{BL}^G \cap BL^G(B, b) | \geqq 2$ and both vertices b_1 and b_k are free in G.

Let G_0 be the component of $G - B$ containing a vertex b. The cacti G_1 and G_2, are defined in the following way: $G_1 = G_0 \cup B$, $G_2 = G - G_0$. Then $| BL^{G_1} | < n$, $| BL^{G_2} | < n$ and both G_1 and G_2 correspond to the assumptions of the theorem. Let us denote Z, Z_1, Z_2 the sets of all vertices of type X in G, G_1, G_2 respectively. Then $Z_1 \cap Z_2 = \emptyset$, $Z \subseteqq Z_1 \cup Z_2$ and for each $z \in Z$ either $BL^G(\overline{BL}^G, z) \subseteq BL^{G_1}(\overline{BL}^{G_1}, z)$ (if $z \in G_1$) or $BL^G(\overline{BL}^G, z) \subseteq BL^{G_2}(\overline{BL}^{G_2}, z)$ (if $z \in G_2$). Let C_z be a chosen block of $BL^G(\overline{BL}^G, z)$ for each $z \in Z$. Now, let us choose a block C_z^1 of $BL^{G_1}(\overline{BL}^{G_1}, z)$ for each $z \in Z_1$ and C_z^2 of $BL^{G_2}(\overline{BL}^{G_2}, z)$ for each $z \in Z_2$ so that we put $C_z^1 = C_z$ or $C_z^2 = C_z$ if $z \in Z$.

(1) Next, we put $C_b^1 = B$ if $b \notin Z$ (in this case b is always of type X in G_1). Now, there are Hamiltonian circuits h_1^1 and h_1^2 in G_1^2 and G_2^2 respectively, if induction is assumed, following theorem. Then there are especially a transform of h_1^1 of the form $b_1, \ldots, b_k, b, (w_1), b_1$ and a transform of h_1^2 of the form $(w_2), b, b_1, (w_3)$. Then $h_1 = (w_2), b, (w_1), b_1, (w_3)$ is a Hamiltonian circuit in G^2.

If $b \in Z$, then $| \overline{BL}^{G_1} | = 0$ and the theorem (part a)) holds for a choice of two blocks of $BL^{G_1}(\emptyset, b)$, concretely C_b and B (according to the proof of Theorem 1). Also in this case there is a Hamiltonian circuit h_1^1 in G_1^2 such that some transform of it is of the form $b_1, \ldots, b_k, b, (w_1), b_1$. Then $h_1 = (w_2), b, (w_1), b_1, (w_3)$ is a Hamiltonian circuit in G^2.

(2) There are Hamiltonian circuits h_2^1 and h_2^2 in G_1^2 and G_2^2 respectively, if induction is assumed, following theorem. There are especially a transform of h_2^1 of the form $b_1, \ldots, b_k, (x_1), b_1$ and a transform of h_2^2 of the form $(x_2), b_k, b, b_1, (x_3)$. Hence $h_2 = (x_2), b_k, (x_1), b_1, (x_3)$ is a Hamiltonian circuit in G^2.

As in both cases (1) and (2) every path of the free vertices of G is a path of the free vertices either in G_1 or G_2, then under the induction assumption and from the form of the connection of the circuits h_1^1, h_1^2 and h_2^1, h_2^2 it follows that the Hamiltonian circuits h_1 and h_2 prove the validity of the theorem 2.

If neither (1) nor (2) occurs, then for every cut vertex a of every inner block A of G there holds either $| BL^G(A, a) \cap \overline{BL}^G | \leqq 1$ and the vertices which are adjacent to the vertex a in A are both the cut vertices in G or $| BL^G(A, a) \cap \overline{BL}^G | \geqq 2$ and at least one of the vertices which are adjacent to the vertex a in A is a cut vertex in G. As there is at least one inner block D_1 containing a free vertex in G (otherwise Theorem 1 holds) so for at least one cut vertex d of D_1 it holds $| BL^G(D_1, d) \cap \cap \overline{BL}^G | \geqq 2$. Hence, in G there are three different inner blocks D_1, D_2, D_3 having the common vertex d. Let us consider the block $D_1 = d_1, \ldots, d_k, d$. At least one from the vertices d_1 and d_k is a cut vertex in G (possibility (2) does not occur). Let d_1 be a cut vertex. If all vertices $d_2, \ldots, d_k$ are cut vertices, the block D_1 is a root block of a prime C-diad which is a BL-subgraph of G. If one of the vertices $d_2, \ldots, d_k$ is free in G, there is an index $j \in \{2, \ldots, k-1\}$ such that $| BL^G(D_1, d_j) \cap \cap \overline{BL}^G | \geqq 2$ and the vertices d_i are the cut vertices for each $i \in \{1, \ldots, j-1\}$. The blocks from $BL^G(D_1, d_j) \cap \overline{BL}^G$ and the ones D_2, D_3 can be discussed in the same way like D_1. From the definition of a C-diad it follows immediatelly that all blocks D_1, D_2, D_3 are the root blocks of three mutually edge disjoint C-diads with the common vertex d, which are the BL-subgraphs of G. Hence G includes a 3-C-diad as its BL-subgraph. It is not possible, therefore either (1) or (2) must occur.

Suppose v is a cut vertex of a graph G and suppose a Hamiltonian circuit in G^2 is $x_1, e_1, x_2, e_2, \ldots, x_{n-1}, e_{n-1}, x_n$ with vertices $x_1, \ldots, x_n$, edges $e_1, \ldots, e_{n-1}$ and $v = x_1 = x_n$. If we erase such edges from h which are not incident with v and which join a vertex of one v-fragment to a vertex of a different v-fragment, we obtain paths $p_1, \ldots, p_s$ in G^2 which are disjoint sections of h with the following properties.

1. $s \geqq 2$.
2. Both vertices $F(p_i)$, $L(p_i)$ are adjacent to the vertex v in G for each $i \in \in \{2, \ldots, s-1\}$.
3. Both vertices $F(p_s)$, $L(p_1)$ are adjacent to the vertex v in G.
4. $\bigcup_{i=1}^{s} V(p_i) = V(G)$.

Suppose $r_1, \ldots, r_k$ are all of the sections $p_1, \ldots, p_s$ in a particular v-fragment F of G and suppose p_1 and p_s are not among $r_1, \ldots, r_k$. Then there are edges in G^2 by which the sections $r_1, \ldots, r_k$ can be joined together into a single path p_F in G^2 which includes all of the vertices of F except v and both vertices $x = F(p_F)$, $y = L(p_F)$ are adjacent in G to v ($x = y$ will occur if F has just two vertices). If p_1 is one of the sections $r_1, \ldots, r_k$ and p_s is not or p_s is one of $r_1, \ldots, r_k$ and p_1 is not we can proceed similarly. In these two cases, the resulting path p_F includes all of the vertices of F and $F(p_F) = v$ and $L(p_F)$ are adjacent in G to v or $L(p_F) = v$ and $F(p_F)$ are adjacent in G to v. Finally, if p_1 and p_s are both among $r_1, \ldots, r_k$ we can join p_1 and p_s in the order p_s, p_1 at the vertex v and then join the remaining sections in $r_1, \ldots, r_k$ by edges of G^2 as before. We obtain a path p_F in G^2 which includes all of the vertices of F and both vertices $F(p_F)$, $L(p_F)$ are adjacent in G to v. Now, these paths can be joined together end to end by edges from G^2 (except that if each of two paths have v on one of its ends, the two paths of this sort are joined at v). The result is a Hamiltonian circuit in G^2 which passes through all of the vertices other then v in each v-fragment before going on the next v-fragment. Hence, if k is any Hamiltonian circuit in G^2, there is a Hamiltonian circuit l in G^2 and there is an ordering $F_1, \ldots, F_t$ of all v-fragments of G such that some transform of l is of the form $v, (w_1), \ldots, (w_t), (w_{t+1}), v$, where $V(w_i) \subseteq V(F_i)$ for each $i \in \{1, \ldots, t\}$ and $V(w_{t+1}) \subseteq V(F_1)$ if $V(w_{t+1}) \neq \emptyset$. We call such a circuit l a simplification of k at v.

The notion of a simplification of a Hamiltonian circuit was used for the first time in [1] and in [4] it was used, too. In this paper it is used in a proof of the following theorem which enables to prove the necessity of the condition from Theorem 2.

Theorem 3. *Let G be a cactus with at least three vertices which includes no 3-C-diad as its BL-subgraph and let b be a free vertex in G. Then the following assertions are equivalent.*

(1) *There is a Hamiltonian circuit h in G^2 some transform of which is of the form $(x), a, b, c, (y)$, where the vertices a and c are both adjacent to b in G.*

(2) *There is no C-diad with a root b which is a BL-subgraph of G.*

Proof. (2) $\Rightarrow$ (1). If $|BL^G| = 1$, i.e. G is a cycle, there is nothing to prove. Suppose $|BL^G| = n > 1$ and the implication holds for every cactus with less then n blocks. Let $B = b_1, \ldots, b_k, b$ be a block of G containing a free vertex b. If both b_1 and b_k, $b_1 \neq b_k$ (otherwise b is a root of a prime C-diad), are free, then (1) follows from Theorem 2. Suppose b_k is a cut vertex of G and $G_1, \ldots, G_m$ are all the B, b_k-fragments.

a) $m = 1$. If $k = 2$, i.e. B is a triangle, b_1 is a free vertex and b_2 is of type X in G. Then (1) follows from Theorem 2. If $k > 2$, a cactus G_* is defined as follows: $V(G_*) = V(G) - V(G_1)$, $E(G_*) = (E(G) - (E(G_1) \cup \{bb_k, b_kb_{k-1}\})) \cup \{bb_{k-1}\}$.

In G_* no BL-subgraph can be a C-diad with a root b. As $| BL^{G*} | < n$, there exists a Hamiltonian circuit h_* in G_*^2 under the induction assumption such that some transform of h_* is of the form $(x), b_{k-1}, b, b_1, (y)$. A vertex b_k is of type X in $G_{**} = G_1 \cup B$ and according to Theorem 2 there is a Hamiltonian circuit h_{**} in G_{**}^2 such that some transform of h_{**} is of the form $b_k, b, b_1, \ldots, b_{k-1}, (z), b_k$. Then $h = (x), b_{k-1}, (z), b_k, b, b_1, (y)$ is a Hamiltonian circuit in G^2 such that (1) holds.

b) $m \geqq 2$. There is at least one index $i \in \{1, \ldots, m\}$ such that no C-diad which is a BL-subgraph of G_i has a root in b_k. Suppose $i = m$. Let us define a cactus $\bar{G} = G - G_m$. As $| BL^{\bar{G}} | < n$, $| BL^{G_m} | < n$ there are (according to the induction assumption) Hamiltonian circuits $\bar{h}$ and h_m in $\bar{G}^2$ and G_m^2 respectively such that some transform of them are of the form $b_k, b, b_1, (x), b_k$ and $b_k, (y), b_k$ respectively and where both $F(y)$ and $L(y)$ are adjacent to b_k in G_m ($F(y) = L(y)$ if G_m includes just two vertices). Let k be a simplification of $\bar{h}$ at b_k. Then there is an ordering $(i_1, \ldots, i_{m-1})$ of the set $\{1, \ldots, m - 1\}$ such that some transform of k is of the form $b_k, b, b_1, (w), (w_1), \ldots, (w_{m-1}), (\bar{w}), b_k$, where $V(w_j) \subseteq V(G_{i_j})$ for each $j \in \{1, \ldots, m - 1\}$ and the vertices from $V(w)$ and $V(\bar{w})$ belong to the b_k-fragment which includes the block B. As the vertices $L(b_1, (w))$ and $F(w_1)$ are adjacent to b_k in $\bar{G}$ and hence in G, too, $b_k, b, b_1, (w), (y), (w_1), \ldots, (w_{m-1}), (\bar{w}), b_k$ is a Hamiltonian circuit in G^2 that holds (1).

(1) $\Rightarrow$ (2). If $| BL^G | = 1$, then (2) holds. Suppose (2) holds for every cactus with less then n blocks, $n > 1$, and there is a cactus G such that $| BL^G | = n$, a block $B = b_1, \ldots, b_k, b$ in G with a free vertex b, a Hamiltonian circuit h in G^2 such that some transform of it is of the form $a, b, c, (w), a$, where both a and c are adjacent to b in G (then $k \geqq 2$ and we can suppose $a = b_1$, $c = b_k$) and a C-diad with a root b, which is a BL-subgraph of G.

For each $i \in \{1, \ldots, k\}$, if b_i is a cut vertex let G_i be the union of all the B, b_i-fragments.

Suppose b_k is a cut vertex. If $b_k, (w), b_1$ is of the form $b_k, (w_k^1), (x), (w_k^2), (y), b_1$, where $V(w_k^1) \subseteq V(G_k), \emptyset \neq V(w_k^2) \subseteq V(G_k), V(x) \neq \emptyset, V(x) \cap V(G_k) = \emptyset, F((y), b_1) \notin$ $\notin V(G_k)$, the vertices $L(x) \neq b$, $F((y), b_1) \neq b$ are different and both are adjacent to b_k in $G - G_k$. As b_k is adjacent in $G - G_k$ just to b and b_{k-1}, it is not possible and $b_k, (w), b_1$ must be of the form $(w_k), (\bar{w}_{k-1}), b_1$, where $V(w_k) = V(G_k)$, $V(\bar{w}_{k-1}) \cap V(G_k) = \emptyset$ and $F(w_k) = b_k$, $F((\bar{w}_{k-1}), b_1) = b_{k-1}$.

Suppose all vertices $b_2, \ldots, b_k$ are cut vertices. In the same way it can be successively proved that $b_k, (w), b_1$ is of the form $(w_k), \ldots, (w_2), (\bar{w}_1), b_1$, where $V(w_i) =$ $= V(G_i)$ for each $i \in \{2, \ldots, k\}$ and $F((\bar{w}_1), b_1) = b_1$. Then necessarily $V(\bar{w}_1) = \emptyset$ and b_1 is free in G. From this it follows that no prime C-diad can posses b as its root and at least one vertex from $\{b_1, \ldots, b_k\}$ is free in G. Therefore, according to the definition of a C-diad there is an index $i \in \{1, \ldots, k\}$ such that just two B, b_i-fragments include some C-diad with a root b_i as its BL-subgraph and b_j is a cut vertex for each $j \in \{i+1, \ldots, k\}$ or $j \in \{1, \ldots, i-1\}$. Suppose the first possibility occurs. In the same way as earlier there can be successively proved that $b_k, (w), b_1$ is of the form $(w_k), \ldots, (w_i), (\bar{w}_{i-1}), b_1$, where $V(w_j) = V(G_j)$ and $F(w_j) = b_j$ for each $j \in \{i+1, \ldots, k\}$ and the vertices $L(w_i)$ and $F(\bar{w}_{i-1}) = b_{i-1}$ are adjacent to b_i in G. Then $k_i = (w_i), b_i$ is a Hamiltonian circuit in G_i^2. Suppose $\bar{k}_i$ is a simplification of k_i at b_i and $G_1^i, \ldots, G_r^i$ are all the b_i-fragments of G_i (these are just all B, b_i-fragments of G). Then there is such ordering $(i_1, \ldots, i_r)$ of the set $\{1, \ldots, r\}$ that a transform of $\bar{k}_i$ is of the form $b_i, (d_1), \ldots, (d_r), (d_{r+1}), b_i$, where $V(d_j) \subseteq V(G_{i_j})$ for each $j \in \{1, \ldots, r\}$ and $V(d_{r+1}) \subseteq V(G_{i_1})$ if $V(d_{r+1}) \neq \emptyset$. As $L(w_i)$ is adjacent to b_i in G_i then for at least $r-1$ indices $s \in \{1, \ldots, r\}$, both vertices $F(d_s)$ and $L(d_s)$ are adjacent to b_i in G_{i_s}. Hence there is an index t such that b_i is a root of a C-diad which is a BL-subgraph of G_{i_t}, $(d_t), b_i, F(d_t)$ is a Hamiltonian circuit in $G_{i_t}^2$ and the vertices $F(d_t)$ and $L(d_t)$ are adjacent to b_i in G_{i_t}. But this is not possible under the induction assumption as $|BL^{G_t}| < n$. Therefore (1) implies (2).

Theorem 4. *Let G be a cactus with at least three vertices. Then G^2 is Hamiltonian if and only if G includes no 3-C-diad as its BL-subgraph.*

Proof. Suppose h is a Hamiltonian circuit in G^2 and G includes some 3-C-diad as its BL-subgraph. Suppose b is a root of the 3-C-diad and k is a simplification of h at b. Then there is an ordering $G_1, \ldots, G_n$ of all b-fragments such that a transform of k is of the form $b, (d_1), \ldots, (d_n), (d_{n+1}), b$, where $n \geqq 3$, $V(d_i) \subseteq V(G_i)$ for each $i \in \{1, \ldots, n\}$, $V(d_{n+1}) \subseteq V(G_1)$ if $V(d_{n+1}) \neq \emptyset$. As at least three b-fragments include some C-diads with a root b as their BL-subgraphs and at least for $n-2$ indices $t \in \{1, \ldots, n\}$ both vertices $F(d_t)$ and $L(d_t)$ are adjacent to b in G_t, there is an index $t \in \{1, \ldots, n\}$ such that b is a root of a C-diad which is a BL-subgraph of G_t, $(d_t), b, F(d_t)$ is a Hamiltonian circuit in G_t^2 and the vertices $L(d_t)$ and $F(d_t)$ are adjacent to b in G_t. This is not possible according to Theorem 3. Hence no 3-C-diad can be includes in G as a BL-subgraph.

The converse implication was proved by Theorem 2.

Suppose G is any graph, b is a vertex of G and k is a positive integer. We define a graph $G(k, b)$ in the following way: $V(G(k, b)) = \{V(G) - \{b\}\} \times \{1, \ldots, k\} \cup \cup \{b\}$, $\{xy\} \in E(G(k, b))$ if and only if either $x = (u, i)$, $y = (v, j)$, $i = j$, u is adjacent to v in G or $x = (u, i)$, $y = b$ and u is adjacent to b in G ($G(k, b)$ is constructed from k copies of G by connecting at b).

Corollary 1. *Let G be a cactus with at least three vertices which includes no 3-C-diad as its BL-subgraph and let $B = b_1, \ldots, b_k, b$ be an arbitrary cycle such that $V(G) \cap V(B) = \{b\}$. Then*

1. G is short with respect to b if and only if no b-fragment of G includes a C-diad with a root b as its BL-subgraph.

2. *G is long with respect to b if and only if $G \cup B$ includes no 3-C-diad as its BL-subgraph and just one b-fragment of G includes a C-diad with a root b as its BL-subgraph.*

Proof. 1. Suppose G is short with respect to b and at least one b-fragment of G includes a C-diad with a root b as its BL-subgraph. Then $G(3, b)$ includes a 3-C-diad as its BL-subgraph and there is a Hamiltonian circuit in $G(3, b)^2$. This is not possible according to Theorem 4.

The converse implication immediately follows from Theorem 3.

2. Suppose G is long with respect to b and p is a Hamiltonian path in $G^2 - b$ such that $F(p)$ is adjacent to b in G and the vertices $L(p)$ and b have the distance 2 in G. Then $b, (p^{-1}), b_1, \ldots, b_k, b$ is a Hamiltonian circuit in $(G \cup B)^2$ and according to Theorem 4, $G \cup B$ includes no 3-C-diad as its BL-subgraph. Suppose just k of all b-fragments of G include a C-diad with a root b. If $k = 0$, G is short with respect to b. If $k \geqq 2$, $G(2, b)$ includes a 3-C-diad as its BL-subgraph and there is a Hamiltonian circuit in $G(2, b)^2$. It is not possible, hence $k = 1$.

Conversely. Suppose $G_1, \ldots, G_k$ are all b-fragments of G and suppose just a cactus G_1 includes a C-diad with a root b as its BL-subgraph. As $G_1 \cup B$ includes no 3-C-diad as its BL-subgraph and b is of type X in $G_1 \cup B$, theie is (according to Theorem 2) a Hamiltonian circuit in $(G_1 \cup B)^2$ a transform of which is of the form $b_1, \ldots, b_k, b, (w), b_1$, where $L(w)$ is adjacent to b in G_i and the vertices $F(w)$ and b have in G_1 the distance 2 (a consequence of Theorem 3). Hence G_1 is long with respect to b and because $G_2, \ldots, G_{k-1}$ are short with respect to b, G is long with respect to b.

Corollary 2. *Let G be a cactus with at least three vertices and let $B = b_1, \ldots, b_k, b$ be an arbitrary cycle such that $V(B) \cap V(G) = \{b\}$. Then G is unusable with respect to b if and only if either G or $G \cup B$ includes a 3-C-diad as its BL-subgraph or at least two b-fragments of G include a C-diad with a root b as its BL-subgraph.*

Proof. Follows immediately from Corollary 1.

Theorem 5. *Let G be a cactus and let $B = b_1, \ldots, b_k$ be a block in G such that all vertices of its are the cut vertices. For each $i \in \{1, \ldots, k\}$, let the cacti G_i^* and G_i be defined in the following way: G_i^* is the union of all B, b_i-fragments of G and $G_i = $*

$$= G_i^* \cup B \cup \bigcup_{j=1}^{k} \bigcup_{A \in BL^G(B, b_j)} A.$$

Then G includes a 3-C-diad as its BL-subgraph if and only if there is an index $t \in \{1, \ldots, k\}$ such that G_t includes a 3-C-diad as its BL BL-subgraph.

Proof. Suppose G includes a 3-C-diad as its BL-subgraph and suppose there is an index $t \in \{1, \ldots, k\}$ such that either $G_t^* \cup B$ includes a 3-C-diad as its BL-subgraph or at least two B, b_t-fragments of G include a C-diad with a root b as its

BL-subgraph. As $B \cup \bigcup_{\substack{j=1 \\ j \neq t}}^{k} \bigcup_{A \in BL^G(B,b_j)} A$ includes a prime C-diad with a root b_t as its BL-subgraph, G_t must include a 3-C-diad as its BL-subgraph. If there is no such index, then, according to Corollary 1, for each $i \in \{1, \ldots, k\}$ there is in $(G_i^*)^2 - b_i$ a Hamiltonian path p_i such that $F(p_i)$ is adjacent to b_i in G_i^* and the vertices $L(p_i)$ and b_i have the distance at most 2. Then $b_k, (p_1), b_1, \ldots, (p_{k-1}), b_{k-1}, (p_k), b_k$ is a Hamiltonian circuit in G^2. This is not possible, hence an index t exists.

The converse implication is obvious.

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S. Říha
Helfertova 16
613 00 Brno
Czechoslovakia