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CANONICAL FORMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

Dedicated to Otakar Borůvka

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Abstract. A solution of two classical Halphen's problems of equivalence and classification of OLDE is given. Transformation theory of the n-th order OLDE is constructed on algebraic base using the method of factorization of differential operators. Invariants of OLDE are obtained as consistent conditions of overdetermined system of nonlinear algebraic differential equations. The differential Euclidean algorithm and differential resultant are introduced and used. Representations for iterative equations are given by means of factorization of self-adjoints OLDE. The one-to-one correspondence of the canonical Halphen and Forsyth forms is found. There is pointed a connection between problems of equivalence and classification of OLDE and those of integrating linear and associated nonlinear equations.

Key words. Ordinary linear differential equations, transformation, factorization, invariant, equivalence, classification, canonical form.

MS Classification. 34 C 20, 34 A 05.

INTRODUCTION

Acad. O. Borůvka has drawn attention of modern mathematicians to a classical problem of Kummer (1834) of reducing second order ordinary linear differential equations (OLDE) to a given form [9, 10]. A natural generalization of the Kummer problem is two problems of Halphen (1884) of local equivalence and classification of the n-th order OLDE [16]. Putting the problems is associated with investigations of Laguerre [20] and Brioshi [12] on invariants of OLDE. Halphen studied only the cases $n = 3$ [16] and $n = 4$ [17] himself. Special role was assigned in his works to equations which are locally equivalent to the simplest differential equation $z^{(n)}(t) = 0$. Such equations are also called iterative (Hustý [18]) and are self-adjoint and reducible (Berkovich [2]). Forsyth (1894) [13] constructed a canonical form different from the Halphen's ones. The classical stage in development of the transformation theory of OLDE has been reflected in Wilczynski's book [29].

Defects of the stage are the following. A relationship between Halphen's and Forsyth's forms was not considered. There was much of artificial in techniques for finding invariants; the studies had local character. It was not completely taken into account importance of semi-invariants with respect to dependent and independent variables; they were studied in parallel and regardless of developing the invariant theory. Notice that semi-invariants with respect to dependent variable transformation was constructed by Bohl [8] and with respect to independent one – by Peyovitch [25].

During the recent decades O. Borůvka has caused and deeply developed the global transformation theory of the second order OLDE. He applied algebraic and, particularly, group-theoretic approach himself [11]. Geometric and algebraic methods were used in the global transformation theory of the n-th order OLDE by Neuman [22, 23]. Hustý, employing iterative equations, has obtained constructive results on classification and invariants of OLDE. Canonical forms were used by Hustý, Greguš [15], Šeda [26] and others to study oscillatory properties of OLDE solutions.

For transformations of higher order OLDE see (Berkovich [1], Šeda [27], Suchomel [28]).

1. THE KUMMER PROBLEM

1.1. Statement of the problem. The equations

$$y'' + 2a_1(x)\,y' + a_2(x)\,y = 0, \tag{1.1}$$

$$\ddot{z} + 2b_1(t)\,\dot{z} + b_2(t)\,z = 0, \tag{1.2}$$

where $a_1 \in C^1(i)$, $a_2 \in C(i)$ and $b_1 \in C^1(j)$, $b_2 \in C(j)$ are real-valued functions of x and t respectively, i and j are open (finite or not) intervals. It is to find the set of all transformations $T = (f(t), \varphi(t))$, where

$$(T) \qquad \begin{array}{lllll} f : j \to R, & f \in C^2(j), & f(t) \neq 0, & t \in j, & \\ \varphi : j \to R, & \varphi(j) = i, & \varphi \in C^2(j), & \mathrm{d}\varphi/\mathrm{d}t \neq 0, & \forall\, t \in j, \end{array}$$

so that solutions $y(x)$ and $z(t)$ of (1.1), (1.2) are related by the ratio

$$z(t) = f(t)\,y(\varphi(t)). \tag{1.3}$$

Equations (1.1) and (1.2) are globally transformed into each other by the transformation T if (1.3) holds on the whole intervals i and j. Otherwise, (1.1) and (1.2) are transformed into each other locally.

For the purpose of the work a local transformability is sufficient. And instead of (T) we shall consider the inverse to that $X = (v(x), t(x))$, where

$$(X)\qquad \begin{array}{c} v: i \to R, \quad v \in C^2(i), \quad v(x) \neq 0, \quad x \in i, \\ t: i \to R, \quad t(i) = j, \quad t \in C^3(i), \quad u(x) = \mathrm{d}t/\mathrm{d}x \neq 0, \quad x \in i \end{array}$$

and $y(x) = v(x)\, z(\int u(x)\, \mathrm{d}x)$ is satisfied. The transformation X corresponds to the variable change

$$y = v(x)\, z, \qquad \mathrm{d}t = u(x)\, \mathrm{d}x; \tag{1.4}$$

we shall call (X) (as well as (T)) Kummer–Liouville (KL) transformation and the functions $v(x)$, $t(x)$ and $u(x)$—*the multiplier, the transformer* (*parametrization*) and *kernel* of the transformation (X) respectively. The global and local transformabilities have an equivalence relation. In addition, (1.1) is locally equivalent to any given equation (1.2), i.e., an oscillatory equation can be transformed, e.g., into a non-oscillatory one, and inversely. For local transformability the coefficients of (1.2) are permitted to be complex-valued functions.

1.2. Solving the problem. An effective solving of the local Kummer problem has been given in (Berkovich [4, 5]) on the base of factorization of (1.1) and (1.2):

$$Ly = [D - v'/v - u'/u - r_2(t)\, u]\,[D - v'/v - r_1(t)\, u]\, y = 0, \qquad D = \mathrm{d}/\mathrm{d}x,$$
$$Mz = [D_t - r_2(t)]\,[D_t - r_1(t)]\, z = 0, \qquad D_t = \mathrm{d}/\mathrm{d}t,$$

where r_1 and r_2 satisfy the Riccati equations

$$\dot{r}_1 + r_1^2 + 2b_1 r_1 + b_2 = 0, \qquad \dot{r}_2 - r_2^2 - 2b_1 r_2 + 2\dot{b}_1 - b_2 = 0.$$

Theorem 1.1. *The set of all the transformations* (X), *giving solution of the local Kummer problem, is described by formulae of the form*

$$v(x) = |\, t'\,|^{-1/2} \exp(-\textstyle\int a_1\, \mathrm{d}x + \int b_1\, \mathrm{d}t),$$

where $t = t(x)$ *is the general solution of so called Kummer–Schwartz third order equation* (KS-3)

$$\{t, x\} + B_2(t)\, t'^2 = A_2(x), \tag{1.5}$$

$\{t, x\} = \frac{1}{2} t'''/t' - \frac{3}{4}(t''/t')^2$ *is the Schwartz derivative*, $A_2(x) = a_2 - a_1^2 - a_1'$, $B_2(t) = b_2 - b_1^2 - \dot{b}_1$.

The general solution of (1.5) *can be expressed in implicit form* $W_0(t) = (C_1 + C_2 w_0(x))/(C_3 + C_4 w_0(x))$, $C_1C_4 - C_2C_3 \neq 0$, *where* $w_0(t) = \tau$ *is a solution of the equation* $\{\tau, t\} = B_2(t)$, *and* $w_0(x)$ *is a solution of* $\{\xi, x\} = A_2(x)$, *i.e. in the form*

$$\int \exp(-2\textstyle\int b_1 \mathrm{d}t)\, z_1^{-2}(t)\, \mathrm{d}t\, \dfrac{C_1 + C_2 \int \exp(-2\int a_1 \mathrm{d}x)\, y_1^{-2} \mathrm{d}x}{C_3 + C_4 \int \exp(-2\int a_1 \mathrm{d}x)\, y_1^{-2}\, \mathrm{d}x},$$

here $y_1(x)$ *and* $z_1(t)$ *are some particular solutions of* (1.1) *and* (1.2) *respectively.*

2. TWO PROBLEMS OF HALPHEN

Thus, as one sees, e.g., from § 1, under sufficiently general assumptions every second order OLDE can be reduced to a given form by the KL transformation.

However, for equations of order $n \geqq 3$

$$y^{(n)} + \sum_{k=1}^{n} \binom{n}{k} a_k y^{(n-k)} = 0, \qquad a_k \in C^{n-k}(i) \tag{2.1}$$

the result is not valid.

In the following, instead of (2.1), we consider *semi-canonical form*

$$y^{(n)} + \sum_{k=2}^{n} \binom{n}{k} A_k(x)\, y^{(n-k)} = 0, \qquad A_k \in C^{n-k}(i) \tag{2.2}$$

to which it is easy to come substituting $y = \exp(-\int a_1\, dx)\, z$ and then replace z by y.

Together with (2.2) we consider the equations

$$z^{(n)}(t) + \sum_{k=2}^{n} \binom{n}{k} B_k(t)\, z^{(n-k)}(t) = 0, \qquad B_k \in C^{n-k}(j), \tag{2.3}$$

j is an open interval of t axis.

On the set of equations (2.2) let us determine an equivalence relation with the transformation group G

$$G : (v(x), \textstyle\int u(x)\, dx), \quad v(x) \in C^n(i), \quad u(x) \in C^n(i), \quad u \neq 0, \quad v \neq 0, \quad \forall\, x \in i. \tag{2.4}$$

We need subgroups of G as well:

$$G_1: \ (v(x), \text{id.}); \qquad G_2: \ (\text{id.}, \textstyle\int u(x)\, dx).$$

We call the equations (2.2) and (2.3) equivalent if a transformation $g \in G$ exists such that (2.2) $\xrightarrow{g}$ (2.3).

We call a mapping of coefficients of (2.2), constant on the equivalence classes of OLDE by (2.4), an invariant of (2.2).

More concretely, such a rational differential function $I(A, A', \ldots)$, where $A = (0, A_2, \ldots)$, that

$$I(A, A', \ldots) = \lambda(u)\, I(B, \dot{B}, \ldots) \qquad (\text{with } t = \textstyle\int u(x)\, dx)$$

is called an invariant of the equation (2.2) in respect of G.

If $\lambda(u) = 1$ then I is an *absolute invariant*, and if $\lambda(u) \neq$ const then I is a *relative* one.

Similarly, notions of absolute and relative invariants are introduced for subgroups G_1 and G_2. For instance, the coefficients A_k are absolute invariants of (2.1) regarding the subgroup G_1, i.e. $A_k(a, a', \ldots) = B_k(b, \dot{b}, \ldots)$.

The equations of order $n = 2$ have no invariants but only *semiinvariants*. The equations of order $n = 3$ have only one invariant (relative).

For equations of order $n \geqq 4$, moreover, it is to introduce notions of *pseudo-invariants* and *conditional invariants*.

We call such a rational differential function $J(A, A', \ldots)$ that

$$J(A, A', \ldots) = \lambda(u, u')\, I_0(A, A', \ldots) + \mu(u)\, J(B, \dot{B}, \ldots)$$

a *pseudoinvariant* of equation (2.2).

A limitation of $J(A, A', \ldots)$, fulfilled for $I_0(A, A', \ldots) = 0$, i.e.

$$I_1(A, A', \ldots) = J(A, A', \ldots)\,|_{I_0=0}, \qquad I_1(A, A', \ldots) = \mu(u)\, I_1(B, \dot{B}, \ldots)$$

is called a conditional invariant of (2.2).

There are two problems associated with Halphen's name.

Problem 1. To find the necessary and sufficient conditions of equivalence of equations (2.2) and (2.3).

Problem 2. To give a classification of the equations of the form (2.2).

3. FACTORIZATION AND EQUIVALENCE CRITERION

We use the method of factorization of differential operators to find conditions of equivalence of equations (2.2) and (2.3) under the KL transformation [1]. One distinguishes two basic forms of factorization: complex-valued and real-valued.

Proposition 3.1. (Mammana [21]). *Let the* OLDE

$$L = D^n + \sum_{k=2}^{n} \binom{n}{k} A_k D^{n-k}, \qquad a_k \in C^{n-k}(i) \tag{3.1}$$

be given corresponding to (2.2). *It is always possible and moreover by means of infinite number of ways, to present* (3.1) *as a factorization with first order operators*

$$L = \prod_{k=n}^{1} (D - \alpha_k) = (D - \alpha_n) \ldots (D - \alpha_2)(D - \alpha_1), \tag{3.2}$$

where $\alpha_k(x)$ are, perhaps complex-valued, functions of x.

Proposition 3.2. (Mammana [21]). *The necessary and sufficient condition for operator* (3.1) *is to be decomposable into a product of real first order factors, is that every integral of equation* (2.1) *vanishes in the interval i not more that $n - 1$ times.*

Similarly to (3.1), the operator M corresponding to (2.2) permits the factorization

$$M = \prod_{k=n}^{1} (D_t - \beta_k) = (D_t - \beta_n) \ldots (D_t - \beta_2)(D_t - \beta_1),$$

where $\beta_k(t)$ are complex-or real-valued functions, depending on the form of the factorization.

The following statement presents a criterion of equivalence of (2.1) and (2.2).

Theorem 3.1 [6]. *For equivalence of* (2.1) *and* (2.2) *it is necessary and sufficient* *that a factorization*

$$L = \prod_{k=n}^{1} [D - v'/v - (k-1)u'/u - \beta_k(t(x))u]$$

is fulfilled.

Proposition 3.3 (A differential analogue of Viete's formulas).
There are the following relations between the "roots" α_k *of factorization* (3.2) *and the coefficients* A_k:

(3.3) $$0 = -\Sigma\alpha_k, \qquad k = \overline{1, n},$$

$$\binom{n}{2}A_2 = \sum_{i\neq j}^{n} \alpha_i(x)\alpha_j(x) - \sum_{k=1}^{n-1}(n-k)\alpha_k'$$

(other relations are more cumbersome).

Note that (3.3) coincides with the corresponding relation for algebraic polynomials. For all $\alpha_k = \text{const}$ Viete's differential formulae coincide with the algebraic ones.

Proposition 3.4. *The multiplier* $v(x)$ *and the kernel* $u(x)$ *of the transformation* (1.4) *are coupled with equation*

$$v'/v + (n-1)/2u'/u = 0$$

and finite relations as well:

(3.4) $$v(x) = |u(x)|^{-(n-1)/2}, \qquad u(x) = v^{-2/(n-1)}.$$

Proposition 3.5. *In order to reduce* (2.2) *to* (2.3) *by means of transformation of type* (1.4), *it is necessary and sufficient for* (1.4) *to have the form*

(3.5) $$(u^{-(n-1)/2}, \qquad \int u(x)\,dx),$$

where $t(x) = \int u(x)\,dx$ *satisfies the* KS-3

$$\{t, x\} + \frac{3}{n+1}B_2(t)\,t'^2 = \frac{3}{n+1}A_2(x).$$

4. ASSOCIATED NONLINEAR EQUATIONS AND EQUIVALENCE CONDITIONS

Applying transformation (3.5) we obtain the transformed form of equation (2.2):

(4.1) $$z^{(n)}(t) + \binom{n}{2}\left(A_2u^{-2} - \frac{n+1}{6}u''u^{-3} + \frac{n+1}{4}u'^2u^{-4}\right)z^{(n-2)}(t) + \binom{n}{3}\times$$
$$\times\left[A_3u^{-3} - 3A_2u'u^{-4} - \frac{n+1}{4}u'''u^{-4} + \frac{3(n+1)}{2}u'u''u^{-5} - \frac{3}{2}(n+1)u'^3u^{-6}\right]\times$$
$$\times z^{(n-3)}(t) + \binom{n}{4}\left[A_4u^{-4} - 6A_3u'u^{-5} + \frac{3(n+11)}{2}A_2u'^2u^{-6} - \right.$$

$$- (n+5) A_2 u'' u^{-5} + \frac{3(n+1)(n+59)}{16} u'^4 u^{-8} - \frac{(n+1)(n+59)}{4} u'^2 u'' u^{-7} +$$

$$+ \frac{(n+1)(n+23)}{12} u''^2 u^{-6} + 3(n+1) u''' u' u^{-6} - \frac{3(n+1)}{10} u^{\mathrm{IV}} u^{-5} \Big] z^{(n-4)}(t) + \ldots$$

$$\ldots + u^{-(n+1)/2} \left[(u^{-(n-1)/2})^{(n)} + \sum_{k=2}^{n-1} \binom{n}{k} A_k (u^{-(n-1)/2})^{(n-k)} \right] z = 0.$$

Note that by the connection (3.4) equation (4.1) can have differential expression of v as its coefficients.

In virtue of (4.1) reduction of (2.2) to (2.3) leads to associated nonlinear equations.

Lemma 4.1. *For equivalence of* (2.2) *to* (2.3) *it is necessary and sufficient that the following overdetermined system of nonlinear equations in* $t(x)$;

$$\{t, x\} + \frac{3}{n+1} B_2 t'^2 = \frac{3}{n+1} A_2, \tag{4.2'}$$

$$t^{\mathrm{IV}}/t' - 6t''t'''/t'^2 + 6(t''/t')^3 + \frac{12}{n+1} A_2 t''/t' + \frac{4}{n+1} B_3 t'^3 = \frac{4}{n+1} A_3, \tag{4.2''}$$

$$t^{v}/t' - 10t^{\mathrm{IV}}t''/t'^2 - \frac{5(n+23)}{18}(t''/t')^2 + \frac{5(n+59)}{6} t''^2 t'''/t'^3 - \tag{4.2'''}$$

$$- \frac{5(n+59)}{8}(t''/t')^4 - \frac{5(n+11)}{n+1} A_2 (t''/t')^2 + A_2 \frac{10(n+5)}{3(n+1)} t'''/t' +$$

$$+ \frac{20}{n+1} A_3 t''/t' + \frac{10}{3(n+1)} B_4 (t')^4 = \frac{10}{3(n+1)} A_4,$$

$$\left[(t')^{-\frac{n-1}{2}}\right]^{(n)} + \sum_{k=2}^{n} \binom{n}{k} A_k \left[(t')^{-\frac{n-1}{2}}\right]^{(n-k)} - B_n (t')^{-\frac{n-3}{2}} = 0 \tag{4.2^{n-1}}$$

is consistent.

Lemma 4.2. *For equivalence of* (2.2) *and* (2.3) *it is necessary and sufficient that the following overdetermined system of nonlinear equations in* $v(x)$

$$v'' - \frac{n-2}{n-1} v'^2/v + 3\frac{n-1}{n+1} A_2 v - \frac{3(n-1)}{n+1} B_2 v^{\frac{n-5}{n-1}} = 0, \tag{4.3'}$$

$$v''' - \frac{3(n-3)}{n-1} v'v''/v + \frac{2(n-2)(n-3)}{(n-1)^2} v'^3/v^2 + \frac{12}{n+1} A_2 v' + \frac{2(n-1)}{n+1} A_3 v -$$

$$- \frac{2(n-1)}{n+1} B_3 v^{\frac{n-7}{n-1}} = 0, \tag{4.3''}$$

$$v^{\mathrm{IV}} - 4\frac{n-4}{n-1} v'v'''/v - \frac{22(n-4)}{9(n-1)} v''^2/v + \frac{2}{9}\frac{(49n-125)(n-4)}{(n-1)^2} v'^2 v''/v^2 - \tag{4.3'''}$$

$$- \frac{(49n-125)(n-2)(n-4)}{9(n-1)^3} v'^4/v^3 - \frac{10(n-4)(n+7)}{3(n-1)(n+1)} A_2 v'^2/v +$$

$$+\frac{10}{3}\frac{n+5}{n+1}A_2v''+\frac{20}{n+1}A_3v'+\frac{5(n-1)}{3(n+1)}B_4v^{\frac{n-2}{n-1}}=0,$$

$$v^{(n)}+\sum_{k=2}^{n}\binom{n}{k}A_kv-B_nv^{\frac{n-3}{n-1}}=0 \tag{4.3^{n-1}}$$

is consistent.

Equation (4.3^{n-1}), generalizing the Ermakov equation, was studied in Berkovich [3].

Theorem 4.1. (2.2) *is equivalent to* (2.3) (*the systems* (4.2) *and* (4.3) *are compatible*), *if and only if* $n-2$ *relations between invariants*

$$I_0(A)=u^3I_0(B), \tag{4.4}$$

$$J_{n,1}(A)=6\frac{u'}{u}I_0(A)+u^4J_{n,1}(B),$$

$$J_{n,2}(A)=-30\left(\frac{u'}{u}\right)^2I_0(A)+10\frac{u'}{u}J_{n,1}(A)+u^5J_{n,2}(B),$$

. . . .

$$J_{n,n-3}(A)=\alpha_0\left(\frac{u'}{u}\right)^{n-3}I_0(A)+\sum_{k=1}^{n-4}\alpha_k\left(\frac{u'}{u}\right)^{n-3-k}J_{n,k}(A)+$$

$+u^nJ_{n,n-3}(B)$, *depending on* $n-3$ *parameters* $\alpha_0, \alpha_1, \dots, \alpha_{n-4}$,

where

$$I_0(A)=A_3-\frac{3}{2}A_2', \tag{4.5}$$

$$J_{n,1}(A)=A_4-2A_3'+\frac{6}{5}A_2''-\frac{5(5n+7)}{3(n+1)}A_2^2,$$

$$J_{n,2}(A)=A_5-\frac{5}{2}A_4'+\frac{15}{7}A_3''-\frac{5}{7}A_2'''-\frac{10(7n+13)}{7(n+1)}A_2I_0(A)$$

are fulfilled.

If $I_0(A)=0$ then systems (4.2) and (4.3) can be shortened since in the case equations (4.2″) and (4.3″) are consequences of (4.2′) and (4.3′) respectively and, hence, they can be omitted.

If $I_0(A)=0$ then the pseudoinvariant $J_{n,1}(A)$ becomes the conditional invariant $I_{n,1}(A)=J_{n,1}|_{I_0=0}$.

If $I_0(A)=I_{n,1}(A)=0$ then equations (4.2‴) and (4.3‴) can be omitted from systems (4.2) and (4.3) as well. If $I_0(A)=I_{n,1}(A)=0$ then pseudoinvariant $J_{n,2}(A)$ becomes the conditional invariant $J_{n,2}|_{I_0=I_{n,1}=0}=I_{n,2}$, and we have

$$I_{n,1}(A)=A_4-\frac{9}{5}A_2''-\frac{3}{5}\frac{5n+7}{n+1}A_2^2,$$

$$I_{n,2}(A)=A_5-\frac{5}{2}A_4'+\frac{5}{2}A_2'''.$$

5. REDUCED AND CANONICAL FORMS OF OLDE

In this section Halphen's (H) and Forsyth's (F) canonical forms are constructed. They belong to the reduced form (R) which occurred before. Schematically it can be represented as follows

$$(R)-\left|\begin{array}{l}\to (H)\\ \quad\leftrightarrow\\ \to (F)\end{array}\right.$$

5.1. The reduced form. The transformed form (4.1) can be presented as follows:

$$\text{(R)} \qquad z^{(n)}(t) + \sum_{k=2}^{n} \binom{n}{k} r_k z^{(n-k)}(t) = 0,$$

where

$$r_2 = r = A_2 u^{-2} - \frac{n+1}{6} u'' u^{-3} + \frac{n+1}{4} u'^2 u^{-4},$$

$$r_3 = \frac{3}{2}\dot{r} + I_0(A) u^{-3},$$

$$r_4 = \frac{9}{5}\ddot{r} + \frac{3(5n+7)}{5(n+1)} r^2 - 6u^{-5} u' I_0(A) + I_{n,1}(A) u^{-4}.$$

Theorem 5.1. (*classificational*). *The set of equations* (2.2) *can be divided into* $n-1$ *classes according to the table* 1.

Table 1

Class	Invariants	Transformation $(u^{(n-1)/2}, \int u\, dx)$	Halphen's canonical forms
Y_0	$I_0 \neq 0$	$u_0 = \sqrt[3]{I_0}$	H_0 is the principal one, it depends on $n-2$ parameters
Y_k, $k = \overline{1, n-3}$	$I_0 = J_{n,1} = \dots = J_{n,k-1} = 0$, $I_{n,k} = J_{n,k} \neq 0$	$u_k = \sqrt[k+3]{I_{n,k}}$	H_k is a degenerate one, it depends on $n-k-2$ parameters
Y_{n-2}	$I_0 = J_{n,1} = \dots = J_{n,n-3} = 0$	$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = \frac{3}{n+1} A_2$	H_{n-2} is the simplest degenerate one: $z^{(n)}(t) = 0$

The coefficients of the canonical forms are absolute invariants (Halphen's).

5.2. Halphen's canonical forms. For those we have

$$(\mathrm{H}_0)\qquad z^{(n)}(t)+\sum_{k=2}^{n}\binom{n}{k}h_{k0}z^{(n-k)}(t)=0,$$

where

$$h_{20}=r\,|_{u=u_0}=h_0,\qquad h_{30}=\frac{3}{2}\dot{h}_0+I,\qquad h_{40}=\frac{9}{5}\ddot{h}_0+\frac{3(5n+7)}{5(n+1)}h_0^2-$$
$$-6u_0'u_0^{-2}+I_{n,1}u_0^{-4},\ldots$$

$$(\mathrm{H}_1)\qquad z^{(n)}(t)+\sum_{k=2}^{n}\binom{n}{k}h_{k1}z^{(n-k)}(t)=0,$$

where

$$h_{21}=r\,|_{u=u_1}=h_1,\qquad h_{31}=\frac{3}{2}\dot{h}_1,\qquad h_{41}=\frac{9}{5}\ddot{h}_1+\frac{3(5n+7)}{5(n+1)}h_2^2+I_{n,1}u_1^{-4},\ldots$$

$$(\mathrm{H}_2)\qquad z^{(n)}(t)+\sum_{k=2}^{n}\binom{n}{k}h_{k2}z^{(n-k)}(t)=0,$$

where

$$h_{22}=r\,|_{u=u_2}=h_2,\qquad h_{32}=\frac{3}{2}\dot{h}_2,\qquad h_{42}=\frac{9}{5}\ddot{h}_2+\frac{3(5n+7)}{5(n+1)}h_2^2+1,\ldots$$

Theorem 5.2. *Equations* (2.2) *and* (2.3) *belong to the same class* (*not being equivalent*) *if and only if*

$$(\mathrm{Y}_0)\qquad I_0(A)=u^3I_0(B),\qquad I_0\neq 0,$$

$$(\mathrm{Y}_1)\qquad I_{n,1}(A)=u^4I_{n,1}(B),\qquad I_0=0,\qquad I_{n,1}\neq 0,$$

$$(\mathrm{Y}_2)\qquad I_{n,2}(A)=u^5I_{n,2}(B),\qquad I_0=I_{n,1}=0,\qquad I_{n,2}\neq 0,$$

$$(\mathrm{Y}_{n-3})\qquad I_{n,n-3}(A)=u^nI_{n,n-3}(B),\qquad I_0=I_{n,1}=\ldots=I_{n,n-4}=0,\qquad I_{n,n-3}\neq 0,$$

$$(\mathrm{Y}_{n-2})\qquad I_0=I_{n,1}=\ldots=I_{n,n-3}=0.$$

In case of belonging to classes Y_{n-3} and Y_{n-2} equations (2.2) and (2.3) are equivalent.

5.3. Forsyth's canonical forms.

Theorem 5.3. (*classificational*). *The set of equations* (2.2) *can be divided into* $n-1$ *classes according to the table* 2:

Table 2

Class	Invariants	Transformation $(u^{-(n-1)/2}, \int u\,dx)$	Forsyth's canonical forms
Y_0	$I_0 \neq 0$	$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = \frac{3}{n+1}A_2$	F_0 is the principal one, it depends on $n-2$ parameters
Y_k $k = \overline{1, n-3}$	$I_0 = J_{n,1} = \dots =$ $= J_{n,k-1} = 0$ $I_{n,k} = J_{n,k} \neq 0$		F_k is a degenerate one, it depends on $n-k-2$ parameters
Y_{n-2}	$I_0 = J_{n,1} = \dots =$ $= J_{n,n-3} = 0$		F_{n-2} is the simplest degenerate one: $F_{n-2} = H_{n-2}$.

Here we have:

$$(\mathrm{F}_0) \qquad z^{(n)}(t) + \sum_{k=3}^{n} \binom{n}{k} f_{k0} z^{(n-k)}(t) = 0,$$

where

$$f_{30} = I_0(A)\,u^{-3}, \qquad f_{40} = -6u^{-5}u' I_0(A) + I_{4,1}u^{-4}, \dots,$$

$$(\mathrm{F}_1) \qquad z^{(n)}(t) + \sum_{k=4}^{n} \binom{n}{k} f_{k1} z^{(n-k)}(t) = 0,$$

where $f_{41} = I_{4,1}u^{-4}, \dots$.

Thus, the one–to–one correspondence is established between the main and degenerate forms of Halphen and Forsyth.

6. ITERATIVE EQUATIONS

6.1. Iterative (formally antiself–adjoint and reducible) operator of the odd order $2n+1$ can be presented as the factorization

$$L = \prod_{k=1}^{n}\left(D + \frac{n+1-k}{n}\alpha\right) D \prod_{k=n}^{n}\left(D - \frac{n+1-k}{n}\alpha\right).$$

Theorem 6.1. *The operator L can be expressed in the form of $(2n+1)$–multiple iteration of the first order operator:*

$$\exp\left(\frac{2n+1}{n}\int \alpha\,dx\right)L = \left[\exp\left(\frac{1}{n}\int \alpha\,dx\right)(D-\alpha)\right]^{2n+1},$$

where α satisfies the Riccati equation

$$\alpha' + \frac{1}{2n}\alpha^2 + \frac{3n}{n+1}A_2 = 0,$$

moreover, the corresponding second order equation

$$y'' + \frac{3}{2n+2}A_2 y = 0$$

and the equation Ly = 0 are reduced to the simplest ones $\dddot{z} = 0$ *and* $z^{(2n+1)}(t) = 0$ *by means of the transformations*

$$\left(\exp\left(\frac{1}{2n}\int\alpha\,dx\right),\qquad \int\exp\left(-\frac{1}{n}\int\alpha\,dx\right)dx\right),$$

$$\left(\exp\left(\int\alpha\,dx\right),\qquad \int\exp\left(-\frac{1}{n}\int\alpha\,dx\right)dx\right),$$

respectively.

6.2. Iterative (formally self-adjoint and reducible) operator of the even order $2n$ can be presented as the factorization

$$L = \prod_{k=1}^{n}\left(D + \frac{2n+1-2k}{2n-1}\alpha\right)\prod_{k=n}^{1}\left(D - \frac{2n+1-2k}{2n-1}\alpha\right).$$

Theorem 6.2. *The operator L can be expressed in the form of 2n-multiple iteration of the first order operator*

$$\exp\left(\frac{4n}{2n-1}\int\alpha\,dx\right)L = \left[\exp\left(\frac{2}{2n-1}\int\alpha\,dx\right)(D-\alpha)\right]^{2n}.$$

where α satisfies the Riccati equation

$$\alpha' + \frac{1}{2n-1}\alpha^2 + \frac{3(2n-1)}{2n+1}A_2 = 0$$

moreover, the corresponding second order equation

$$y'' + \frac{3}{2n+1}A_2 y = 0$$

and the equation Ly = 0 are reduced to the simplest ones $\dddot{z} = 0$ *and* $z^{(2n)}(t) = 0$ *by means of the transformations*

$$\left(\exp\left(\frac{1}{2n-1}\int\alpha\,dx\right),\qquad \int\exp\left(-\frac{2}{2n-1}\int\alpha\,dx\right)dx\right),$$

$$\left(\exp\left(\int\alpha\,dx\right),\qquad \int\exp\left(-\frac{2}{2n-1}\int\alpha\,dx\right)dx\right)$$

respectively.

7. THE EUCLIDEAN DIFFERENTIAL ALGORITHM AND ITS APPLICATION TO FINDING INVARIANT I_0

The relative invariant I_0 can be also obtained on the basis of using and developing the known analogy between algebraic polynomials and OLDO (see Berkovich [4]).

7.1. Differential operator ring. Let us consider the set $K[D]$ of operators of the form $L = \sum a_k D^{n-k}$ of an arbitrary order n with the coefficients from a differential field K. The addition operation is introduced in $K[D]$ in the natural way, and the multiplication operation is characterized by the following Leibniz formula

$$D^s b = \sum_{k=0}^{s} \binom{s}{k} b^{(s-k)} D^k.$$

It is easy to find out that $K[D]$ is an associative ring but it is not a commutative one. It contains the unity and has no zero divisors.

Propositon 7.1. *The ring $K[D]$ is Euclidean.*

It means existence of the Euclidean algorithm of division with remainder (let us consider the right-definite case) in $K[D]$, i.e., for any two operators L and M, $\operatorname{ord} L \geqq \operatorname{ord} M$, the equality $L = QM + S$, where Q is a right quotient and S is a right remainder, is valid. Then the division with remainder is single-valued.

7.2. Factorization of OLDO in the principal differential field.

Definition 7.1. (Frobenius [14].) We call the equation $Ly = 0$ *undecomposable* in K if it has not a common integral with any other OLDE of less order with coefficients from K.

Otherwise, the equation $Ly = 0$ is called *decomposable* in K.

Proposition 7.2. *The necessary and sufficient condition for decomposability of $Ly = 0$ is the factorization $L = QP$,* ($\operatorname{ord} L = \operatorname{ord} Q + \operatorname{ord} P$).

Proposition 7.3. *The system of two equations*

$$Ly = 0, \qquad My = 0 \tag{7.1}$$

is nontrivially consistent, if and only if such an operator N ($\operatorname{ord} N \geqq 1$) *exists which is the right greatest common divisor* (RGCD) *of the operators L and M,* i.e.,

$$N = \mathrm{RGCD}(L, M), \qquad (L = Q_1 N, M = Q_2 N). \tag{7.2}$$

7.3. The right differential remainder theorem.

Proposition 7.4. *The remainder of division on the right of the n-th order operator L_n on the first order operator D-α has the form* $S = \exp(-\int \alpha\, dx)\, L_n \exp(\int \alpha\, dx)$.

Consequence 7.1. *If the equation $L_n y = 0$ has a solution $y = y(x)$ then the factorization*

$$L_n = L_{n-1}(D - y'/y) \Leftrightarrow L_n \equiv 0(\text{mod}\,(D - y'/y)) \textit{ holds.}$$

7.4. Generalization of the notion of undecomposabilisy of OLDE.

In the next we need the following, practically forgotten, generalization of the notion of undecomposability of OLDE going back to Koenigsberger [19].

Definition 7.2. The equation $Ly = 0$ is called *undecomposable* in K if either

a) it is not decomposable according to the definition 7.1, or

b) it has not a common solution with any nonlinear algebraic differential equation of less order having coefficients from K.

Otherwise, the equation $Ly = 0$ is called *decomposable* in K in the generalized sense.

Remark 7.1. The main reason for the idea of undecomposability in the sense of the definition 7.2 was not applied, is evidently that the theory of OLDO divisibility has not been expanded on nonlinear algebraic differential equations. To make such an expansion possible, it is necessary associate the nonlinear equation

$$\sum_{k=0}^{n} a_k(x, y, y', \ldots, y^{(k)}) = 0$$

with the OLDO

$$L = \sum_{k=0}^{n} a_k(x, y, y', \ldots, y^{(k)})\, D^{n-k}$$

and develop a theory analogous to that for OLDE.

7.5. Finding I_0. Let us find I_0 combining the differential Euclidean algorithm with the differential remainder theorem (we omit the adjective "right" for brevity): we shall find I_0 as a condition of compatibility of the overdetermined system (4.3′) and (4.3″). To simplify calculations, but without less generality, let us consider the system

$$v'' - \frac{n-2}{n-1} v'^2/v + 3\frac{n-1}{n+1} A_2 v - \frac{3(n-1)}{n+1} B_2 u^2 v = 0, \tag{7.2}$$

$$v''' - \frac{3(n-3)}{n-1} v'v''/v + \frac{2(n-2)(n-3)}{(n-1)^2} v'^3 v^{-2} + \frac{12}{n+1} A_2 v' + \frac{2(n-1)}{n+1} A_3 v - \frac{2(n-1)}{n+1} B_3 u^3 v = 0, \tag{7.3}$$

the compatibility condition of which is at the same time the necessary condition for equivalence of the equations (2.2) and (2.3) under the transformations of type (2.4), (3.4).

We associate the equations (7.2) and (7.3) with the OLDO

$$L_2 = D^2 - \frac{n-2}{n-1} v'v^{-1}D + 3\frac{n-1}{n+1}A_2 - 3\frac{n-1}{n+1}B_2u^2,$$

$$L_3 = D^3 - \frac{3(n-3)}{n-1}v'v^{-1}D^2 + \frac{2(n-2)(n-3)}{(n-1)^2}v'^2v^{-2} + \frac{12}{n+1}A_2D + \\ + \frac{2(n-1)}{n+1}(A_3 - B_3u^3).$$

Theorem 7.1. (2.2) *and* (2.3) *are equivalent, if* (*and only if for* $n = 3$) *the following equivalent conditions are fulfilled:*

a) $N = \mathrm{RGCD}(L, M) = D - v'/v$;

b) *the right remainder in the Euclidean algorithm applied to L and M vanishes:*

$$S = 0 = A_3 - \frac{3}{2}A_2' - \left(B_3 - \frac{3}{2}\dot{B}_2\right)u^3.$$

Now the formula for I_0 follows from Th. 7.1 as a consequence.

8. DIFFERENTIAL RESULTANT AND ITS APPLICATION TO FINDING I_0

The compatibility condition of the system (7.1) can be obtained using the differential resultant (Ore [24], Berkovich and Tzirulik [7]) as well, which can be given in the form of a determinant, similarly to Sylvester's construction of resultant of two algebraic polynomials.

Let L and M be two OLDO of the orders n and m respectively. We shall "multiply" the operator L on the left by $I, D, D^2, \ldots, D^{m-1}$, and the operator M – by $I, D, D^2, \ldots, D^{n-1}$. Obviously, if the system (7.1) is compatible then the generated over-determined system

$$(8.1)\quad \begin{cases} Ly = 0, & DLy = \sum_{k=0}^{n+1} a_{1,k}D^ky = 0, \ldots, D^{m-1}Ly = \sum_{k=0}^{n+m-1} a_{m-1,k}D^ky = 0, \\ My = 0, & DMy = \sum_{k=0}^{m+1} b_{1,k}D^ky = 0, \ldots, D^{n-1}My = \sum_{k=0}^{m+n-1} b_{n-1,k}D^ky = 0, \end{cases}$$

where

$$(8.2)\qquad a_{r,k} = \sum_{s=\max(0,k-n)}^{\min(r,k)} \binom{r}{s} a_{k-s}^{(r-s)}, \qquad k = \overline{0, n+r}$$

and $b_{s,k}$ is calculated using the similar formula, is compatible as well

$$\left(L = \sum_{k=0}^{n} a_kD^k,\ M = \sum_{k=0}^{m} b_kD^k\right).$$

Proposition 8.1. *The homogeneous system is compatible if and only if the rank of the right resultant matrice* $R(a, b)$ *formed from the coefficients of* (8.1) *is less than its order*

$$\operatorname{rank} R < n + m, \tag{8.3}$$

where

$$\begin{Vmatrix} a_{m-1,n+m-1} & a_{m-1,n+m-2} & \cdots & \cdot & \cdot & \cdot & \cdots & a_{m-1,0} \\ 0 & a_{m-2,n+m-2} & \cdots & \cdot & \cdot & \cdot & \cdots & a_{m-2,0} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_n & a_{n-1} & \cdot & \cdots & a_0 \\ b_{n-1,n+m-1} & b_{n-1,n+m-2} & \cdots & \cdot & \cdot & \cdot & \cdots & b_{n-1,0} \\ 0 & b_{n-1,n+m-2} & \cdots & \cdot & \cdot & \cdot & \cdots & b_{n-2,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdot & b_m & b_{m-1} & \cdots & b_0 \end{Vmatrix} = R. \tag{8.4}$$

Inequality (8.3) is easy to obtain by straight replacing the system (8.1) by the corresponding system of linear algebraic equations in $y_k = y^{(k)}$, $k = 0, 1, \ldots, n + m - 1$. Inequality (8.3) is a consequence of the Kroneker–Kapelli differential theorem as well.

Definition 8.1. *We call* $\det R$, *where* R *is constructed according to* (8.4), (8.2), *the right differential resultant* ($R\,\text{Res}$) *of the operators* L *and* M.

Proposition 8.2. *The system* (7.1) *is compatible if and only if* $R\,\text{Res}\,(L, M) = 0$ $(a_n \neq 0, b_m \neq 0)$.

Theorem 8.1. (2.2) *and* (2.3) *belong to the same class* Y_0 (*the system* (7.2), (7.3) *is consistent*) *if and only if* (4.4), (4.5) *hold.*

CONCLUSIONS

The obtained results display fruitfulness of the developed approach using factorization and transformations of differential equations and structure and properties of the associated ones as well. It is a good basis for general theory of OLDE having constructive character. For instance, it gives unified and regular techniques to solve in a natural way problems of integrability and finding exact solutions of differential equations.

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