Anton Dekrét

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ON CONNECTIONS ON THE SECOND ITERATED TANGENT BUNDLE

ANTON DEKRÉT

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Abstract. A sector connection Γ on $T_2M = TTM$ is introduced as a double linear section from T_2M into JT_2M. It is shown that Γ can be stated both by the groupoid of the invertible 2-quasi-jets on M and by a linear section from T^*M into the space of all 3-sector forms on M. The class of the sector connections having geodesics on M and some relations between Γ and the first natural prolongations of linear connections on M are described.

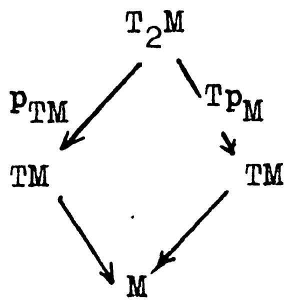
Key words. Quasi-jet, sector connection, sector form, geodesic, natural first order prolongation of linear connections.

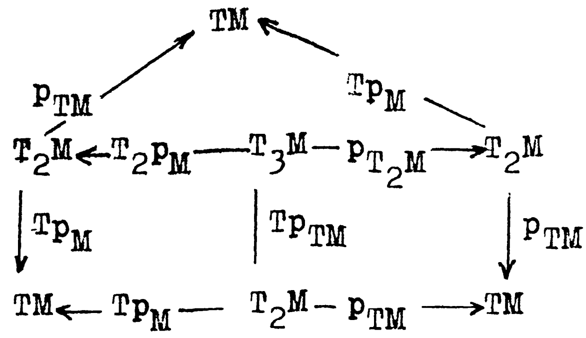
MS Classification. 53 C 05, 58 A 20

In the paper [1] by means of the canonical structure properties of the iterated tangent bundle $T_rM := T \dots TM$ the concept and basic properties of a quasi-jet of order r have been introduced, for $r = 2$ see also [6]. Quasi-jets of order two provide a useful tool for studying connections on T_2M which are closely connected with the structure of T_2M. In the first part of the present paper we recall some basic properties of quasi-jets of order two and three and introduce a Q-connection on T_2M induced by a connection on the groupoid of all invertible quasi-jets of order two on M. Further we define 2-sector connection on T_2M and find the one-two-one correspondence between the set of all Q-connections and the set of all 2-sector connections. Then some geometrical objects connected with a 2-sector connection are modeled, as for example the torsion and the curvature form. The relations between sector 3-forms on T_3M and 2-sector connections are stated. In the last part we deal with geodesics of an 2-sector connection. All presented results are discussed from the point of view of natural first order prolongations of a linear connection on TM.

1. Let (π) be the short denotation of a fibre bundle $\pi : Y \to M$ and let $p_M: TM \to M$ or $Tf: TM \to TN$ denote the tangent bundle of a manifold M or the tangent mapping of a differentiable map $f: M \to N$ respectively. There exist two or three canonical vector bundle structures (p_{TM}), (Tp_M) or (p_{T_2M}), (Tp_{TM}), (T_2p_M) on T_2M or on T_3M

respectively such that the diagrams





are commutative.

We will use the following induced charts. Let (x^i) be a chart on M. Let $X \in T_x M$, $X = j^1_x((t) \to x^i(t)) = \left(x^i_0 = x^i(0), x^i_1 = \dfrac{\mathrm{d}x^i(0)}{\mathrm{d}t}\right.$. Then (x^i_0, x^i_1) is the induced chart on TM. Let $Y = j^1_0((t) \to (x^i_{00}(t), x^i_1(t))) = \left(x^i_{00} = x^i_0(0), x^i_{10} = x^i_1(0), x^i_{01} = \dfrac{\mathrm{d}x^i_0(0)}{\mathrm{d}t},\right.$ $\left. x^i_{11} = \dfrac{\mathrm{d}x^i_1(0)}{\mathrm{d}t}\right) \in T(TM$. It gives the induced chart $(x^i_{00}, x^i_{10}, x^i_{01}, x^i_{11})$ on $T_2 M$. Iterating this construction we get the induced chart on $T_r M$. The geometrical sence of 0- and 1-subscripts is clear.

Let us recall that a map $\varphi\colon (T_2 M_x \to (T_2 N)_y$ or $\varphi\colon (T_3 M)_x \to (T_3 N)_y$ is a quasi-jet of order two or three if it is a vector bundle morphism (shortly v.b.m.) both from (p_{TM}) into (p_{TN}) and from (Tp_M) into (Tp_N) or from $(p_{T_2 M})$ into $(p_{T_2 N})$, from (Tp_{TM}) into (Tp_{TN}) and from $(T_2 p_M)$ into $(T_2 p_N)$ respectively. Let $QJ^2(M, N)$ or $QJ^3(M, N)$ be the manifold of all quasi-jets of order two or three from M into N. Then there exist fibre bundle projections $\varkappa_i\colon QJ^2(M, N) \to QJ^1(M, N)$, $i = 1, 2$, or $\varkappa_k\colon QJ^3(M, N) \to QJ^2(M, N)$, $k = 1, 2, 3$, where $\varkappa_1 \varphi$, $\varkappa_2 \varphi$ or $\varkappa_1 \varphi$, $\varkappa_2 \varphi$, $\varkappa_3 \varphi$ are the base maps of $\varphi\colon (Tp_M) \to (Tp_N)$, $(p_{TM}) \to (p_{TN})$ or $\varphi\colon (T_2 p_M) \to (T_2 p_N)$, $(Tp_{TM}) \to (Tp_{TN})$, $(p_{T_2 M}) \to (p_{T_2 N})$ respectively.

Let $q\colon E \to M$ be a vector bundle and VE_0 be the set of all vertical vectors on E at the points of the zero-section $O\colon M \to E$, $Tq(VE_0) = 0 \subset TM$, $p_E(VE_0) = 0 \subset E$. Denote by V_0 the injection $E \to TE$ determined by $V_0(a) = j^1_0(ta)$, $t \in R$. It is clear that $V_0(E) = VE_0$. In the case of the vector bundles on iterated tangent bundles we add some subscripts to the notations V_0. The injection $V_{01}\colon TM \to T_2 M$, induced by $p_M\colon TM \to M$, determines the fibre projection $\varkappa^1_1\colon QJ^2(M, N) \to J(M, N)$, $\varkappa^1_1 \varphi = (V_{01})^{-1} \cdot \varphi \cdot V_{01}$. Quite analogously the injections $V^1_{02}, V^2_{02}\colon T_2 M \to T_3 M$ induced by the vector bundle structures (Tp_M), (p_{TM}) and the injection TV_{01} give the projections $\varkappa^2_2, \varkappa^1_2, \varkappa^1_1\colon QJ^3(M, N) \to QJ^2(M, N)$. By [1], the set $J^r(M, N)$ of all non-holonomic jets from M into N is a submanifold of $QJ^r(M, N)$. As a special case of Propositions 3 and 4 in [1] we introduce

Lemma 1. *A quasi-jet $A \in QJ^2(M, N)$ is a non-holonomic or semi-holonomic jet iff $\varkappa_1^1 A = \varkappa_2 A$ or $\varkappa_1^1 A = \varkappa_2 A = \varkappa_1 A$ respectively. A quasi-jet $A \in QJ^3(M, N)$ is a non-holonomic or semi-holonomic if $\varkappa_2^1 A = \varkappa_2^2 A = \varkappa_3 A$, $\varkappa_1^1 A = \varkappa_2 A$ or $\varkappa_2^1 A = \varkappa_2^2 A = \varkappa_1^1 A = \varkappa_1 A = \varkappa_2 A = \varkappa_3 A$ respectively.*

In the induced chart on $T_2 M$ the canonical involution i_2 on $T_2 M$, see [3], has the following coordinate form: $i_2(x^i, x^i_{10}, x^i_{01}, x^i_{11} = (x^i, x^i_{01}, x^i_{10}, x^i_{11})$. In the case of $T_3 M$, two involutions both i_3 induced by the structure $T_2(TM)$ and Ti_2 generate the group $I_3 = [Ti_2, i_3]$ of diffeomorphisms on $T_3 M$. In general there is a group I_r of diffeomorphisms on $T_r M$ which is issomorphic with the group of all permutations of the set $\{1, \ldots, r\}$. Propositions 5 and 6 of [1] give

Lemma 2. *If A is a semi-holonomic 2-jet or 3-jet then A is holonomic if $i_2 \,.\, A \,.\, i_2 = A$ or $g^{-1} \,.\, A \,.\, g = A$ for every $g \in I_3$ respectively.*

Let $A \in QJ^r_x(M, N)_y$, $B \in QJ^r_y(N, Z)_z$. Then $B \,.\, A \in QJ^r_x(M, Z)_z$ will denote the composition of quasi-jets A and B. A quasi-jet $A \in QJ^r_x(M, N)_y$ is said to be invertible if there exists $B \in QJ^r_y(N, M)_x$ such that $B \,.\, A = Id_{(T_r M)_x}$. By the standard procedure it can be shown that $QL^2_m := \operatorname{Inv} QJ^2_0(R^m, R^m)_0$ or $QH^r M := \operatorname{Inv} QJ^r_0(R^m M)$, $m = \dim M$, or $Q\pi^2 M := \operatorname{Inv} QJ^r(M, M)$ is a Lie group or a principal bundle with the structure group QL^r_m or a Lie grupoid of operators on $T_r M$ which is a fibre bundle associated with $QH^r M$.

Now, by Ehresmann's approach to connections we introduce a special connection on $T_2 M$. Let a: $QJ^r(M, N) \to M$ or b: $QJ^r(M, N) \to N$) be the source or target projection. Let U be a neighbourhood of x, $x \in M$. Denote $Q_x \pi^2 M = \{A \in Q\pi^2 M,$ $a(A) = x\}$. Let $\gamma : U \to Q_x \pi^2 M$ be a cross-section of (b) such that $\gamma(x) = Id_{(T_2 M)x}$. Then the jet $j^1_x \gamma$ is called an element of connection on $Q\pi^2 M$ at x. Let $C_x Q\pi^2 M$ be the set of all elements of connections at x and $CQ\pi^2 M$ be the space of all elements of connections on $Q\pi^2 M$. Then a connection on $Q\pi^2 M$ is a global cross-section Γ: $M \to CQ\pi^2 M$ of the fibre bundle a: $CQ\pi^2 M \to M$. Every connection Γ on $Q\pi^2 M$ induces the connection $\Gamma_{T_2 M}$: $T_2 M \to JT_2 M$, $\Gamma_{T_2 M}(u) = j^1_x(z \mapsto \gamma(z)(u)$, where $\Gamma(x) = j^1_x \gamma)$ and JT_2 is the first-jet prolongation of $T_2 M \to M$.

Definition 1. *A connection λ: $T_2 M \to JT_2 M$ on $T_2 M$ is called a Q-connection if there exists a connection Γ on $Q\pi^2 M$ such that $\lambda = \Gamma_{T_2 M}$.*

In the induced chart $(x^i, x^i_{10}, x^i_{01}, x^i_{11})$ on $T_2 M$ the equations of $A \in Q\pi^2 M$ have the following coordinate form

(1) $$y^i_{10} = c^i_{10j} x^j_{10}, \qquad y^i_{01} = c^i_{01j} x^j_{01}, \qquad y^i_{11} = c^i_{jk} x^j_{10} x^k_{01} + c^i_{11j} x^j_{11}\,.$$

It induces a chart $(x^i, c^i_{10j}, c^i_{01j}, c^i_{11j}, c^i_{jk}, y^i$ on $Q\pi^2 M$.

Let $\Gamma\colon M \to CQ\pi^2M$ be a connection on $Q\pi^2M$. In the induced chart let $\Gamma(x) = = j^1_x((z) \mapsto (x^i, c^i_{10j}(z), c^i_{01j}(z), c^i_{11j}(z), c^i_{jk}(z), z^i)$, where $c^i_{\varepsilon j}(x) = \delta^i_j$, $\varepsilon = 10, 01, 11$. Then

$$\Gamma_{T_2M}(x^i, x^i_{10}, x^i_{01}, x^i_{11}) =$$
$$= j^1_x\ ((z^i) \mapsto (z^i, c^i_{10j}(z)\, x^j_{10}, c^i_{01j}(z)\, x^i_{01}, c^i_{jk}(z)\, x^j_{10} x^k_{01} + c^i_{11j}(z)\, x^j_{11}) =$$
$$= (x^i_{10}, x^i_{01}, x^i_{11}, {}^{10}\Gamma^i_{jk}(x)\, x^j_{10}, {}^{01}\Gamma^i_{jk}(x)\, x^j_{01}, \Gamma^i_{jku}(x)\, x^j_{10} x^k_{01} + {}^{11}\Gamma^i_{ju} x^u_{11}),$$

i.e.

(2)
$$x^i_{\varepsilon u} = {}^{\varepsilon}\Gamma^i_{ju} x^j_{\varepsilon}, \qquad \varepsilon = 10{,}01,$$
$$x^i_{11u} = \Gamma^i_{jku} x^j_{10} x^k_{01} + {}^{11}\Gamma^i_{ju} x^j_{11}.$$

Let $\tilde{\pi}^2M$ or $\bar{\pi}^2M$ or π^2M be the Lie groupoid of all invertible non-holonomic or semi-holonomic or holonomic 2-jets from M into M. Let us recall that $\tilde{\pi}^2M, \bar{\pi}^2M, \pi^2M$ are submanifolds of $Q\pi^2M$. A Q-connection λ on T_2M is called non-holonomic or semi-holonomic or holonomic if its determining connection Γ is a connection on $\tilde{\pi}^2M$ or $\bar{\pi}^2M$ or π^2M. Lemmas 1 and 2 imply

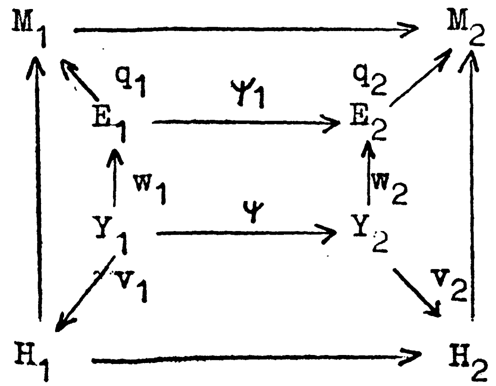
Proposition 1. *Let λ be a Q-connection determined by a connection Γ on $Q\pi^2M$. Then λ is non-holonomic or semi-holonomic or holonomic iff $\varkappa_1\Gamma = \varkappa^1_1\Gamma$ or $\varkappa_1\Gamma = = \varkappa_2\Gamma = \varkappa^1_1\Gamma$ or $\varkappa_1\Gamma = \varkappa_2\Gamma = \varkappa^1_1\Gamma$ and $i_2\Gamma i_2 = \Gamma$ respectively, where $i_2\Gamma i_2(x) = = j^1_x(i_2\gamma(z)\, i_2)$, $\varkappa_i\Gamma(x) = j^1_x\varkappa_i\gamma$, $\Gamma(x) = j^1_x\gamma$.*

In general, a connection on T_2M is a cross-section $\lambda : T_2M \to JT_2M$. We will construct a special connection on T_2M from this point of view. At first we recall some properties of vector bundles. The following one is well known.

Lemma 3. *Let $q_1\colon E \to M$, $q_2\colon Y \to E$ be vector bundles. Let JY be the first-jet prolongation of the fibre bundle $q_1 . q_2\colon Y \to M$. Then $Jq_2\colon JY \to JE$, $Jq_2(h) = = Jq_2(j^1_x f) = j^1_x(q_2 f)$ is a vector bundle.*

Since $p_{TM}, Tp_M\colon T_2M \to TM$ and $p_M\colon TM \to M$ are vector bundles then by Lemma 3 $Jp_{TM}, JTp_M\colon JT_2\mathrm{M} \to J T\mathrm{M}$ are vector bundles, too.

Lemma 4. *Let the diagram*



where (q_i), (v_i), $i = 1, 2$, are vector bundles, ψ is a v.b.m. from (v_1) into (v_2) and w_i is a v.b.m. from Y_i onto E_i, be commutative. Then ψ_1 is a v.b.m. from (q_1) into (q_2).

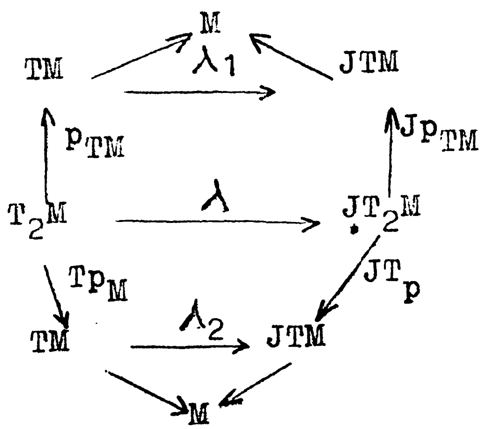
Proof. Let $u_1, u_2 \in (E_1)_x$. Then there exist $\bar{u}_1, \bar{u}_2 \in (v_1)$ such that $w_1(\bar{u}_i) = u_i$, $i = 1, 2$. Then $\psi_1(t_1u_1 + t_2u_2) = w_2 \,.\, \psi(t_1\bar{u}_1 + t_2\bar{u}_2) = t_1w_2 \,.\, \psi(\bar{u}_1) + t_2w_2 \,.\, \psi(\bar{u}_2) = = t_1\psi_1(u_1) + t_2\psi_1(u_2)$.

Definition 2. *A connection λ: $T_2M \to JT_2M$ is called a sector connection if λ is a v.b.m. both from (p_{TM}) into (Jp_{TM}) and from (Tp_M) into (JTp_M).*

Let λ be a sector connection on T_2M. Denote by $\lambda_1 = \pi_1\lambda$ or $\lambda_2 = \pi_2\lambda$ the underlying map of λ from (p_{TM}) into (Jp_{TM}) or from (Tp_M) into (JTp_M).

Proposition 2. *If λ is a sector connection on T_2M then λ_1 and λ_2 are linear connections on TM.*

Proof. It is clear that diagram



is commutative. Recall that p_{TM} or Tp_M is a v.b.m. from (Tp_M) onto (p_M) or from (p_{TM}) onto (p_M) respectively. Obviously Jp_{TM} and JTp_M are vector bundle morphisms. Then by Lem ma 4 λ_1 and λ_2 are linear.

In the ind uced charts $(x^i, x^i_{10}, x^i_{01}, x^i_{11})$ on T_2M and $(x^i, x^i_{10}, x^i_{01}, x^i_{11}, x^i_{10j}, x^i_{01j}, x^i_{11j})$ on JT_2M we obtain the following coordinate equations of a sector connection λ:

(3) $$x^i_{10u} = {}^1F^i_{ju}(x)x^j_{10},\; x^i_{01u} = {}^2F^j_{ju}(x)x^j_{01},\; x^i_{11u} = F^i_{jku}(x)x^j_{10}x^k_{01} + {}^3F^i_{ju}(x)x^j_{11}.$$

The quadruple $({}^1F^i_{ju}, {}^2F^i_{ju}, {}^3F^i_{ju}, F^i_{jku})$ is called the Christoffel's functions of λ.

Let $JV_0 : JTM \to JT_2M$, $JV_0(x^i, x^i_1, x^i_{1j}) = (x^i, 0, 0, x^i_1, 0, 0, x^i_{1j})$, be the first-jet prolongation of the canonical injection $V_0 : TM \to VTM \subset T_2M$. Hence JV_0 is a v.b.m. from JTM into (Jp_{TM}) as well as from JTM into (JTp_M). Let $(JV_0)^{-1}$ be the inverse map to $JV_0 : JTM \to J(VTM_0)$.

Lemma 5. *Let λ be a sector connection on T_2M. Then $\varkappa_1^1\lambda \equiv \lambda_3 := (JV_0)^{-1} . \lambda . V_0 : TM \to J^1TM$ is a linear connection on TM.*

Proof follows from the coordinate equations $x_{1j}^i = {}^3F_{uj}^i x_1^u$ of λ_3. Hence every vector connection λ determines three linear connections $\lambda_1, \lambda_2, \lambda_3$ on TM the Christoffel's functions of which are ${}^1F_{jk}^i, {}^2F_{jk}^i, {}^3F_{jk}^i$.

Comparing (2) with (3) we get

Proposition 3. *There exists the* (1,1)*-correspondence between the set of all sector connections and the set of all Q-connections on T_2M.*

We say that a sector connection λ is non-holonomic or semi-holonomic or holonomic if the corresponding Q-connection is non-holonomic or semi-holonomic or holonomic respectively. Then, in the non-holonomic and semi-holonomic cases, Proposition 1 can be reformulated in the following way.

Proposition 4. *A sector connection λ on T_2M is non-holonomic or semi-holonomic if the linear connections on TM determined by λ satisfy the conditions $\lambda_1 = \lambda_3$ or $\lambda_1 = \lambda_2 = \lambda_3$.*

The canonical involution $i_2 : T_2M \to T_2M$ induces the involution $Ji_2 : JT_2M \to JT_2M$. If λ is a sector connection then the map $Ji_2 . \lambda . i_2 : T_2M \to JT_2M$ is the sector connection on T_2M determined by the equations:

$$x_{10u}^i = {}^1F_{ju}^i x_{10}^j, \qquad x_{01u}^i = {}^2F_{ju}^i x_{01}^j,$$
$$x_{11u}^i = F_{jkju}^i(x)\, x_{10}^j x_{01}^k + {}^3F_{ju}^i x_{11}^j,$$

Then the assertion of Proposition 1 on holonomic Q-connections can be rephrased in the following way:

Proposition 5. *A sector connection λ on T_2M is holonomic if is semi-holonomic and $Ji_2 . \lambda . i_2 = \lambda$.*

This result coincides with [8].

A sector connection λ is called projectable or 1-symmetric if $\lambda_1 = \lambda_2$ or if it is semi-holonomic and its underlying connection λ_1 is without torsion.

Now we will construct some vector fields of a sector connection λ on T_2M. Before we recall that every connection $\gamma : Y \to JY$ on a fibre bundle $\pi : Y \to M$ can be interpreted as a map (γ-lift) $H\gamma : Yx_MTM \to TY$ such that $T\pi . H\gamma(X) = X$ and $H\gamma(y) : \{y\}xT_xM \to T_yY$ is linear. Hence γ determines the decomposition $TY = VY \oplus H\gamma$, where $VY \to Y$ is the vector bundle of all vertical vectors on (π) and $H\gamma \to Y$ is the vector bundle of all γ-horizontal vectors, i.e. of all images under the γ-lift $H\gamma$. For $X \in TY$ we have $X = v_\gamma(X) + H_\gamma(X)$, where $v_\gamma(X)$ or $H_\gamma(X)$ denotes the vertical or horizontal part of X.

Let λ be a sector connection on T_2M. Let $u \in T_2M$. Set $S_1^\lambda(u) := H\lambda(u)\,[p_{TM}(u)]$, $S_2^\lambda(u) := H\lambda(u)\,[Tp_M(u)]$. Obviously $u \mapsto S_1^\lambda(u)$, $u \mapsto S_2^\lambda(u)$ are vector fields on T_2M In local charts it holds

$$S_1^\lambda(x^i, x_{10}^i, x_{01}^i, x_{11}^i) = x_{10}^i\,\partial/\partial x^i + {}^1F_{jk}^i x_{10}^j x_{10}^k\,\partial/\partial x_{10}^i +$$
$$+ {}^2F_{jk}^i x_{01}^j x_{10}^k\,\partial/\partial x_{01}^i + [F_{jku}^i x_{10}^j x_{01}^k x_{10}^u + {}^3F_{jk}^i x_{11}^j x_{10}^k]\,\partial/\partial x_{11}^i,$$
$$S_2^\lambda(x^i, x_{10}^i, x_{01}^i, x_{11}^i) = x_{01}^i\,\partial/\partial x^i + {}^1F_{jk}^i x_{10}^j x_{01}^k\,\partial/\partial x_{10}^i +$$
$$+ {}^2F_{jk}^i x_{01}^j x_{01}^k\,\partial/\partial x_{01}^i + (F_{jku}^i x_{10}^j x_{01}^k x_{01}^u + {}^3F_{jk}^i x_{11}^j x_{01}^k\,\partial/\partial x_{11}^i.$$

We see that S_1 coincides with S_2 on the submanifold of all velocities of order two, $x_{10}^i = x_{01}^i$.

Let λ_s be a linear connection on TM determined by λ. Let $S_s : b \mapsto H\lambda_s(b)\,(b) = x_1^i\,\partial/\partial x^i + {}^sF_{jk}^i x_1^j x_1^k\,\partial/\partial x_1^i$ be the spray of λ_s. Being a natural first order prolongation functor, T determines the vector field TS_s on T_2M. In the induced coordinates,

$$TS_s = x_{10}^i\,\partial/\partial x^i + {}^sF_{jk}^i x_{10}^j x_{10}^k\,\partial/\partial x_{10}^i + x_{11}^i\,\partial/\partial x_{01}^i +$$
$$+ ({}^sF_{jk,u}^i x_{10}^j x_{10}^k x_{01}^u + {}^sF_{jk}^i x_{11}^j x_{10}^k + {}^sF_{jk}^i x_{10}^j x_{11}^k)\,\partial/\partial x_{11}^i,$$

where we use $F_{jk,u}^i := \partial F_{jk}^i/\partial x^u$. Let $X \in T_xM$. There exists a unique vector $\bar{X}$ of the spray of λ_2 such that $p_{TM}(\bar{X}) = X$. As $(p_{T_2M}, Tp_{TM}, T_2p_M) : T_3M \to B_3M \subset x_{TM}^3 T_2M$ is an affine bundle associated with TM, see [9], therefore

(4) $$\beta_\lambda(X) := [S_1^\lambda(\bar{X}) - TS_1(\bar{X})] = (F_{jku}^i + {}^3F_{tk}^i\,{}^2F_{ju}^t - {}^1F_{jk,u}^i - {}^1F_{jt}^i\,{}^2F_{ku}^t)\,x_1^j x_1^k x_1^u\,\partial/\partial x^i, \qquad X = (x^i, x_1^i),$$

is a tangent vector in T_xM. A geometrical relation of β to λ will be given later.

Further it will be useful to find some sector connections which are connected with three linear connections $\lambda_1, \lambda_2, \lambda_3$ determined by a sector connection λ in a natural way. Recall the well known "pull-back" construction of connections. Let $\pi_i : Y_i \to X_i$, $i = 1, 2$, be two fibre bundles. Let $(\Phi, \varphi) : Y_1 \to Y_2$ be a fibre morphism such that $\Phi|_{(Y_1)_x}$ is a diffeomorphism for every $x \in X_1$. Let $\Gamma : Y_2 \to JY_2$ be a connection on Y_2. Let $\Phi^*\Gamma$ denote a connection on Y_1 such that $\Phi^*\Gamma(h = j_x^1(z \to \Phi^{-1}\psi\varphi(z))$, where $\Gamma(\Phi(h)) = j_{\varphi(x)}^1\psi$, $h \in (Y_1)_x$. Let λ_s, $s = 1, 2, 3, 4$, be linear connections on TM. The projection $v_L : VTM = TMx_MTM \to TM$ on the second summand is a v.b.m. over $p_M : TM \to M$ such that $v_L^*\lambda_3$ is a connection on $p_{TM} : VTM \to TM$. Let $H\lambda_4 \to TM$ be a vector bundle of all λ_4-horizontal vectors on TM. Then $(Tp_M\lambda_4) := Tp_M|_{H\lambda_4} : H\lambda_4 \to TM$ is a v.b.m. over p_M such that the connection $(Tp_M\lambda_4)\,{}^*\lambda_2$ on $H\lambda_4 \to TM$ can be constructed. In local charts let

$$\lambda_s : (x^i, x_1^i) \mapsto j_x^1((z^i) \mapsto (z^i, {}^sh_j^i(z)\,x_1^j)) = (x^i, x_1^i, x_{1j}^i = {}^sF_{jk}^i x_1^j),$$
$${}^sh_j^i(x) = \delta_j^i.$$

As

$$v_L(x^i, x_{10}^i, 0, x_{11}^i) = (x^i, x_1^i = x_{11}^i),$$

then

$$v_L^*\lambda_3(x^i, x_{10}^i, 0, x_{11}^i) = j_{(x^i, x_1^i)}^1[(z^i, z_1^i) \mapsto (z^i, z_{10}^i = z_1^i, 0, z_{11}^i = {}^3h_j^i(z)\,x_{11}^j)].$$

Since

$$(Tp_M\lambda_4)(u = (x^i, x_{10}^i, x_{01}^i, x_{11}^i = {}^4F_{jk}^i x_{10}^j x_{01}^k)) = (x^i, x_1^i = x_{01}^i),$$

then

$$(Tp_M\lambda_4)^*\,\lambda_2(u) = j^1_{(x^i,x^i_{10})}(z^i,\,z^i_1,\,{}^2h^i_j(z)\,x^j_{01},\,{}^4F^i_{jk}(z)\,z^j_{10}\,{}^2h^k_t x^t_{01}).$$

Composing these connections with λ_1 we get

$$v^*_L\lambda_3\lambda_1(x^i,\,x^i_{10},\,0,\,x^i_{11}) = j^1_x((z)\mapsto(z^i,\,z^i_{10} = {}^1h^i_j(z)\,x^j_{10},\,0,\,z^i_{11} = {}^3h^i_j(z)\,x^j_{11})),$$

$$(Tp_m\lambda_4)^*\,\lambda_2(u)\,\lambda_1 = j^1_x\;((z^i)\mapsto(z^i,\,{}^1h^i_j(z)\,x^j_{10},\,{}^2h^i_j x^j_{01},\,{}^4F^i_{jk}(z)\,{}^1h^j_w(z)\,x^w_{10}\,{}^2h^k_t(z)\,x^t_{01}).$$

Let $v^*_L\lambda_3\lambda_1 \oplus_{\lambda_1} (Tp_M\lambda_4)^*\lambda_2\lambda_1$ denote a connection on $T_2M \to M$ determined by

$$\begin{aligned}&(x^i,\,x^i_{10},\,x^i_{01},\,x^i_{11}) =\\ &= [(x^i,\,x^i_{10},\,0,\,x^i_{11} - {}^4F^i_{jk}x^j_{10}x^k_{01}) +_{p_{TM}} (x^i,\,x^i_{10},\,x^i_{01},\,{}^4F^i_{jk}x^j_{10}x^k_{01})]\to\\ &\to j^1_x((z)\mapsto(z^i,\,{}^1h^i_j(z)\,x^j_{10},\,{}^2h^i_j(z)\,x^j_{01},\,{}^3h^i_j(z)\,(x^j_{11} - {}^4F^j_{tk}(x)\,x^t_{10}x^k_{01}) +\\ &+ {}^4F^i_{jk}(z)\,{}^1h^j_w(z)\,x^w_{10}\,{}^2h^k_t(z)\,x^t_{01}),\end{aligned}$$

i.e. by the following equations

$$\text{(5)}\qquad \begin{aligned}&x^i_{10u} = {}^1F^i_{ju}x^j_{10},\; x^i_{01u} = {}^2F^i_{ju}x^j_{01}\\ &x^i_{11u} = ({}^4F^i_{jk,u} + {}^4F^i_{tk}\,{}^1F^t_{ju} + {}^4F^i_{jt}\,{}^2F^t_{ku} - {}^3F^i_{tu}\,{}^4F^t_{jk})\,x^j_{10}x^k_{01} + {}^3F^i_{ju}x^j_{11}.\end{aligned}$$

Comparing (5) with (3) we get.

Lemma 6. *The connection $v^*_L\lambda_3\lambda_1 \oplus_{\lambda_1} (Tp_M\lambda_4)^*\lambda_2\lambda_1$ is a sector connection on T_2M which is non-holonomic or semiholonomic iff $\lambda_1 = \lambda_3$ or $\lambda_1 = \lambda_2 = \lambda_3$ respectively.*

We can reformulate the Janyška result [4] in the following way. He has stated an 8-parameter family J of the sector semi-holonomic connections which are the natural first order prolongations of a linear connection γ on TM projectable over γ. Let us recall the Christoffel's functions of two connections of this family established earlier by Kolář [5] and by Oproiu [7]:

$$\text{(6)}\qquad \begin{aligned}&F^i_{jku} = F^i_{jk,u} + F^i_{tk}F^t_{ju} + F^i_{jt}F^t_{ku} - F^i_{tu}F^t_{jk}, \qquad \text{Kolář}\\ &F^i_{jku} = F^i_{ju,k} + F^i_{jt}F^t_{ku}, \qquad \text{Oproiu}\end{aligned}$$

where F^i_{jk} are the Christoffel's functions of γ. Now (5) and (6) immediately give in the case $\lambda_1 = \lambda_2 = \lambda_3 = \gamma$ the following assertion.

Proposition 6. *The connection $v^*_L\gamma\,.\,\gamma \oplus_\gamma (Tp_M\gamma)^*\gamma\,.\,\gamma$ is the Kolář's prolongation of γ.*

Let λ be a sector connection on T_2M and $\lambda_1, \lambda_2, \lambda_3$ be the linear connections on TM determined by λ. Then the sector connection $\lambda^*_s = v^*_L\lambda_3\lambda_1 \oplus_{\lambda_1} (Tp_M\lambda_s)^*\lambda_2\lambda_1$, $s = 1, 2, 3$, is called s-conjugated to λ.

Let $X_1, X_2, X_3 \in T_xM$. There exists $u \in (T_3M)_x$ such that $p_{TM}\,.\,p_{T_2M}(u) = X_1$, $Tp_M\,.\,p_{T_2M}(u) = X_2$, $Tp_M\,.\,Tp_{TM}(u) = X_3$. Since $H_\lambda(u)$, $H_{\lambda^*_s}(u)$ lie in the same fibre of the affine bundle $(p_{T_2M}, Tp_{TM}, TTp_M) : T_3M \to B_3M$, see [9], we set

$$\nabla^\lambda_s(X_1, X_2, X_3) := H_\lambda(u) - H_{\lambda^*_s}(u) \in TM.$$

In induced charts using (3) and (5) we get

(7) $$\nabla^\lambda_s(X_1, X_2, X_3) = (F^i_{jku} + {}^3F^i_{tu}{}^sF^t_{jk} - {}^sF^i_{jk,u} - {}^sF^i_{tk}{}^1F^i_{ju} - {}^sF^i_{jt}{}^2F^t_{ku})\, x^j_1 x^k_2 x^u_3 .$$

This means that ∇^λ_s is a cross-section $M \to TM \oplus (\oplus^3 T^*M)$. It is called the λ_s-difference of λ.

From (4) and (7) we obtain

Lemma 7. *If λ is projectable, i.e. $\lambda_1 = \lambda_2$, then $\beta_\lambda(X) = \nabla^\lambda_{\lambda_1}(X, X, X)$.*

Clearly $\nabla^{\lambda^*}_s = 0$. Therefore if λ is the Kolář's prolongation of a linear connection γ on TM then $\nabla^\lambda = 0$ and conversally if $\lambda \in J$ and $\nabla^\lambda = 0$ then λ is the Kolár's counection. In the case of the Oproiu's prolongation we have.

Lemma 8. *If λ is the Oproiu's prolongation of a linear connection γ then $\nabla^\lambda = -\Phi_\gamma$, where Φ_γ is the curvature form of γ.*

Proof. Using (7) we get

$$\nabla^\lambda = (F^i_{ju,k} - F^i_{jk,u} + F^i_{tu}F^t_{jk} - F^i_{tk}F^t_{ju})\, x^j_1 x^k_2 x^u_3 .$$

2. On a torsion form of a sector connection λ. Let λ be a sector connection and let $\lambda_1, \lambda_2, \lambda_3$ be linear connections on TM determined by λ. Let

$$A = a^i(x^j, x^j_1)\, \partial/\partial x^i + {}^1F^i_{jk} x^j_1 a^k \partial/\partial x^i_1 ,$$
$$B = b^i(x^j, x^j_1)\, \partial/\partial x^i + {}^1F^i_{jk} b^k x^j_1\, \partial/\partial x^l_1$$

be λ_1-horizontal vector fields on TM. Let $TA : T(TM) \to T(T_2M)$ be the tangent map of $A : TM \to T(TM)$. Then the λ-vertical part $\nabla_B A := v_\lambda TA(B)$ of $TA(B)$ determines a vector field on TM which will be called the absolute derivative of A with respect to B according to λ. Using the Lie bracket $[A, B]$ we get

(8) $$\nabla_A B - \nabla_B A - [A, B] = {}^2F^i_{jk}(a^j b^k - b^j a^k)\, \partial/\partial x^i + \\ + (F^i_{jks} + {}^3F^i_{us}{}^1F^u_{jk})\, x^j_1 (a^k b^s - b^k a^s)\, \partial/\partial x^i_1$$

Let $a, b, c \in T_xM$. Let $\bar{A}, \bar{B}$ be vector fields on M such that $\bar{A}(x) = a$, $\bar{B}(x) = b$. Let A or B be the λ_1-lift of $\bar{A}$ or $\bar{B}$ respectively. Put

$$\tau^\lambda(a, b, c) = (\nabla_A B - \nabla_B A - [A, B])_{(c)} \in T_cTM \oplus \wedge^2 T^*_xM.$$

It is obvious by (8) that we have a cross-section $\tau^\lambda : M \to T(TM) \oplus (T^*M \wedge^2 T^*M)$ which will be called the torsion form of λ.

Lemma 9. *If the underlying connection λ_2 of a sector connection λ is without torsion then τ^λ is a cross-section $M \to TM \oplus T^*M \wedge^2 T^*M$.*

Proof follows from (8).

Let $a, b, c \in T_xM$. Set $\tau^\lambda_s(a, b, c) = v_{\lambda_s}\tau^\lambda(a, b, c)$. By (8) we have

(9) $$\tau^\lambda_s(a, b, c) = (F^i_{jku} + {}^3F^i_{tu}{}^1F^t_{jk} - {}^sF^i_{jt}{}^2F^t_{ku})\, c^j_1 (a^k b^u - b^k a^u)\, \partial/\partial x^i .$$

Hence τ_s^λ is a cross-section $M \to TM \oplus T^*M \wedge^2 T^*M$ which is called the torsion of λ with respect to λ_s.

Proposition 7. *Let λ be a sector connection on T_2M. Then the torsion of the sector connection λ_1^*, 1-conjugated to λ, coincides with the curvature form of λ_1, i.e. $\tau_1^{\lambda_1^*} = \Phi_{\lambda_1}$.*

Proof follows from (5) and (9).

Corollary. *If λ is the Kolar's prolongation of a linear connection γ on TM then $\tau_1^\lambda = \Phi_\gamma$.*

Remark 1. If λ is 1-symmetric then the antisymmetrisation $A\tau_1^\lambda$ of τ_1^λ has the following coordinate form

$$A\tau_1^\lambda = 2F^i_{jku}\,\mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^u \oplus \partial/\partial x^i.$$

On a curvature form of λ. In general, the curvature form of a connection Γ on a fibre bundle $\pi : Y \to M$ is the section $\Phi_\Gamma : Y \to VY \oplus \wedge^2 T^*M$, where $\Phi_\Gamma(y, a, b) = v_\Gamma([H\Gamma A, H\Gamma B]_y)$, A, B are vector fields on M such that $A(\pi y) = a, B(\pi(y)) = b$.

Let λ be a sector connection on T_2M the underlying connections λ_1 and λ_2 o which are intergrable. Let $a, b, c, d \in T_xM$. There exists λ_s-horizontal vector $h \in T(TM)$, $s = 1, 2, 3$, such that $p_{TM}(h) = c$, $Tp_M(h) = d$. Set $\varphi_s(c, d, a, b) = \Phi_\lambda(h, a, b)$. In the induced charts we get

$$\varphi_s(c, d, a, b) =$$
$$= (F^i_{tkw}\,{}^1F^t_{uj} + F^i_{utw}\,{}^2F^t_{kj} + {}^3F^i_{tw}F^t_{ukj} + {}^3F^i_{tw}\,{}^3F^t_{vj}\,{}^sF^v_{uk} + F^i_{tw,j}\,{}^sF^t_{uk} + F^i_{ukw,j})\times$$
$$\times c^u d^k (a^w b^j - a^j b^w \quad \partial/\partial x^i.$$

This yields

Lemma 10.. *φ_s is a cross-section $M \to TM \oplus (\oplus^2 T^*M)\wedge^2 T^*M$. Quite analogously, putting*

$$\psi_{sq}(c, d, a, b) = \Phi_\lambda(h, a, b) - \Phi_{\lambda_q^*}(h, a, b)$$

we get a cross-section $\psi_{sq} : M \to TM \oplus (\oplus^2 T^*\mathrm{M}) \wedge^2 T^*M$ in the case of any sector connection λ on T_2M.

3. On relations between sector connections and sector forms. Let $\tau^r M \to M$ denote the vector bundle of all sector r-forms on M, see [9]. Let us recall that $f \in \tau_x^r M$ is a function $(T_rM)_x \to R$ linear according to all canonical vector bundle structures on T_rM. As every $g \in I_r$ is a v.b.m. from a v.b. structure on T_rM into another one therefore the group I_r acts on $\tau^r M$ by $gf(X) = f(g(X))$, $X \in T_rM$, $g \in I_r$, $f \in \tau^r M$. We are interested in the cases $r = 2, 3$. The extension of our considerations for an arbitrary integer $r > 0$ is only a technical matter. In the induced charts $(x^i, x^i_{10}, x^i_{01}, x^i_{11})$ on T_2M and $(x^i, x^i_{100}, x^i_{010}, x^i_{110}, x^i_{001}, x^i_{101}, x^i_{011}, x^i_{111})$ on T_3M the following coordinate form of f holds, see [9]:

$$f \in \tau^2 M, \qquad f = a_{ij}x^i_{10}x^j_{01} + a_i x^i_{11}, \qquad f = (x^i, a_i, a_{ij}),$$

$$f \in \tau^3 M, \qquad f = a_{ijk} x^i_{100} x^j_{010} x^k_{001} + a_{ij} x^i_{100} x^j_{011} + b_{ij} x^i_{101} x^j_{010} + \\ + c_{ij} x^i_{110} x^j_{001} + a_i x^i_{111} .$$

The above introduced canonical injection $V_{01} : TM \to T_2 M$, $(x^i, x^i_1) \mapsto (x^i, 0, 0, x^i_1)$, induces the fibre bundle structure $\varkappa_1 : \tau^2 M \to T^* M$, $\varkappa_1 f := f . V_{01} = a_i x^i_1$. Set $(\tau^2 M)_0 = \{f \in \tau^2 M, \varkappa_1 f = 0 \in T^* M\}$. Let $X_1, X_2 \in T_x M$. Then there exists $X \in T_2 M$ such that $p_{TM}(X) = X_1$, $Tp_M(X) = X_2$. Let $f \in (\tau^2 M)_0$. Setting $\bar{f}(X_1, X_2) = f(X) = a_{ij} x^i_1 x^j_2$ we get the identification $(\tau^2 M)_0 = T^* M \oplus T^* M$, $f \mapsto f$.

Lemma 10. *The fibre bundle $\varkappa_1 : \tau^2 M \to T^* M$ is an affine bundle associated with $T^* M \oplus T^* M$.*

Proof follows from the fact that every couple $f_1 = (x^i, a_i, a^1_{ij})$, $f_2 = (x^i, a_i, a^2_{ij})$ such that $\varkappa_1 f_1 = \varkappa_1 f_2$ determines the unique element $f_1 - f_2 = (x^i, 0, a^1_{ij} - a^2_{ij}) \in \in (\tau^2 M)_0 \equiv T^* M \oplus T^* M$.

In the case of the canonical involution i_2 on $T_2 M$, $i_2 f := f . i_2$ is a sector 2-form such that $\varkappa_1(i_2 f) = \varkappa_1 f$. Therefore $\Delta f := f - i_2 f = (x^i, 0, a_{ij} - a_{ji}) \in \Lambda^2 T^* M$. It will be called the difference of f. The sector 2-form $i_2 f$ is said to be transposed to f. We say that f is symmetric if $f = i_2 f$, i.e. if Δf vanishes.

Quite analogously, in the case $r = 3$, three injections from $T_2 M$ into $T_3 M$:

$$V^2_{02} : (x^i, x^i_{10}, x^i_{01}, x^i_{11}) \mapsto (x^i, x^i_{10}, 0, 0, 0, 0, x^i_{01}, x^i_{11}),$$
$$V^1_{02} : (x^i, x^i_{10}, x^i_{01}, x^i_{11}) \mapsto (x^i, 0, x^i_{01}, 0, 0, x^i_{10}, 0, x^i_{11}),$$
$$V^1_{01} : (x^i, x^i_{10}, x^i_{01}, x^i_{11}) \mapsto (x^i, 0, 0, x^i_{10}, x^i_{01}, 0, 0, x^i_{11}),$$

determine three submersions $\tau^2 M \to \tau^2 M$:

$$\varkappa^2_2 f : = f . V^2_{02} = a_{ij} x^i_{10} x^j_{01} + a_i x^i_{11},$$
$$\varkappa^1_2 f : = f . V^1_{02} = b_{ij} x^i_{10} x^j_{01} + a_i x^i_{11},$$
$$\varkappa^1_1 f : = f . TV^1_{01} = c_{ij} x^i_{10} x^j_{01} + a_i x^i_{11}.$$

Let $\omega \varepsilon (\tau^3_x M)_0 := \{f \in \tau^3 M, \varkappa^2_2 f = \varkappa^1_2 f = \varkappa^1_1 f = 0\}$. Let $X_1, X_2, X_3 \in T_x M$. Then there exists $X \in (T_3 M)_x$ such that $p_{TM} p_{T_2 M}(X) = X_1$, $Tp_M p_{T_2 M}(X) = X_2$, $Tp_M Tp_{TM}(X) = = X_3$. It is easy to see that the map $\omega \mapsto \bar{\omega}$, $\bar{\omega}(X_1, X_2, X_3) = \omega(X) = a_{ijk} x^i_1 x^j_2 x^k_3$ is a vector bundle isomorphism from $(\tau^3 M)_0$ onto $\oplus^3 T^* M$. Denote by B^3 the image of $\tau^3 M$ under the map $(\varkappa^1_1, \varkappa^1_2, \varkappa^2_2) : \tau^3 M \to \tau^2 M x_{T^*M} \tau^2 M x_{T^*M} \tau^2 M$. Let ω_1, ω_2 be two sector 3-forms such that $(\varkappa^1_1, \varkappa^1_2, \varkappa^2_2)(\omega_1) = (\varkappa^1_1, \varkappa^1_2, \varkappa^2_?)(\omega_2)$. Then $\omega_1 - \omega_2 \in \in (\tau^3 M)_0$ and it holds.

Lemma 11. *The fibre bundle $(\varkappa^1_1, \varkappa^1_2, \varkappa^2_2) : \tau^2 M \to B^3$ is an affine bundle associated with $\oplus^3 T^* M$.*

The group I_3 acts on $\tau^3 M$. For example $Ti_2(f)$ is a sector 3-form and it is easy to see that $\varkappa^2_2 . Ti_2(f)$ is transposed to $\varkappa^1_2 f$, $\varkappa^1_2 . Ti_2(f)$ is transposed to $\varkappa^2_2 f$ and $\varkappa^1_1 . Ti_2(f) = \varkappa^1_1 f$. A sector 3-form f is called sub-symmetric if $\varkappa^2_2 f = \varkappa^1_2 f = \varkappa^1_1 f$ is symmetric. In the case of a sub-symmetric sector 3-form f for every $g \in I_3$ it holds

$\varkappa_2^2 g(f) = \varkappa_2^1 g(f) = \varkappa_1^1 g(f) = \varkappa_2^2 f$. Then $\Delta f := \sum_{g \in I_3} (\mathrm{sgng})\, g(f)$, where sgng is 1 or -1 if the permutation g is even or odd, lies in $(\tau_3 M)_0$. In the induced chart $\Delta f = \sum_{g \in I_3} (\mathrm{sgng})\, a_{i_{g(1)} i_{g(2)} i_{g(3)}}$.

Let $p_L : Rx...xR \to R$ be the projection on the last summand. Let $A \in QJ_x^r(M, R)$, $A : (T_r M)_x \to T_r R = x^{2^r} R$. Then obviously $f_A := p_L \cdot A$ is a sector r-form. For every sector r-form f there exists $A \in QJ^r(M, R)_0$ such that $f_A = f$. We will say that a sector r-form f is non-holonomic, semi-holonomic, holonomic if there exists a non-holonomic, semi-holonomic, holonomic r-jet $A \in QJ^r(M, R)_0$ such that $f_A = f$. It is clear that every sector 2-form is semiholonomic and it is holonomic if $i_2 f = f$. As a consequence of Lemma 1 we get

Lemma 12. *A sector 3-form f is non-holonomic or semi-holonomic if $\varkappa_2^1 f = \varkappa_2^2 f$ or $\varkappa_2^1 f = \varkappa_2^2 f = \varkappa_1^1 f$ respectively.*

Lemma 13. *A semi-holonomic sector 3-form f is holonomic if $Ti_2 f = f = i_3 f$.*

Remark 2. If $f \in \tau^2 M$ is holonomic then it is sub-symmetric and $\Delta f = 0$.

Now we turn to the relations between connections and sector forms. At first we recall that the Libermann's identification $L_1 : JT^*M \to \bar{J}^2(M, R)_0$ induces the identification $L_r : \tilde{J}^r T^*M \to [J^{r-1}(\bar{J}^2(M, R)]_0 \subset \tilde{J}^{r+1}(M, R)_0$ with the property $L_r(\bar{J}^r T^*M = \bar{J}^{r+1}(M, R)_0$ where $\tilde{J}^r$ or $\bar{J}^r$ denotes the functor of the non-holonomic or semi-holonomic r-jet prolongation of fibre bundles. It is well known that a linear connection $\gamma : TM \to JTM$ induces the linear connection $\gamma^* : T^*M \to JT^*M = \bar{J}^2(M, R)_0$. It is clear that $\bar{J}^2(M, R)_0 = \tau^2 M$. Therefore every linear connection γ determines the linear cross-section $\bar{\gamma}^* : T^*M \to \tau^2 M$.

Proposition 8. *Let $\zeta : T^*M \to \tau^2 M$ be a linear cross-section. Then there exists the unique linear connection $\gamma : TM \to JTM$ such that for any $u \in TM$ and for every $z \in T^*_{pm(u)}M$ the γ-horizontal space $H\gamma_u$ is the kernel of $\zeta(z)$, i.e. $\zeta(z)\,(H\gamma_u) = 0$.*

Proof. In the induced chart let ζ be given by the equations $\bar{z}_i = z_i$, $z_{ij} = \gamma_{ij}^k(x) z_k$. Then for $X \in T_u TM$ $\zeta(z)\,(X) = (\gamma_{ij}^k x_{10}^i x_{01}^j + x_{11}^k\ z_k$. This means that $\zeta(z)\,(X) = 0$ for every $z \in T^*_{pM(u)}$ iff $x_{11}^k = \to \gamma_{ij}^k x_{10}^i x_{01}^j$, i.e. iff X is a horizontal vector of the linear connection γ the Christoffel's functions of which are $-\gamma_{ij}^k(x)$. Clearly, γ is unique.

Remark on the converse of Proposition 8. If $\gamma : TM \to JTM$ is a linear connection with the Christoffel's functions $\gamma_{ij}^k(x)$ then $-\gamma_{ij}^k(x)$ are the Christoffel's functions of γ^* and the induced cross-section $\bar{\gamma}^* : T^*M \to \tau^2 M$ is given by $\bar{z}_i = z_i$, $z_{ij} = -\gamma_{ij}^k(x) z_k$, i.e. $\bar{\gamma}^*(z)\,(H\gamma_u) = 0$.

Corollary. *There exists the* (1,1)*-correspondence between the set of all linear connections on TM and the set of all linear cross-sections $\zeta : T^*M \to \tau^2 M$.*

Remark 3. If a linear connection γ is determined by a linear section $\zeta : T^*M \to \tau^2 M$ then the transposed connection γ^t is determined by the cross-section ζ^t transposed to ζ. Then $\Delta\zeta := \zeta - \zeta^t : T^*M \to \Lambda^2 T^*M$ is a vector bundle morphism and it coincides with the classical torsion tensor $\tau : M \to TM \oplus \wedge^2 T^*M$ of γ. Then γ is without torsion if ζ is holonomic.

Remark 4. Every connection $\varepsilon : T^*M \to JT^*M$ on T^*M determines the cross-section $\bar{\varepsilon} : T^*M \to \tau^2 M$. If ε is not linear, $\bar{\varepsilon} = \varepsilon_{ij}(x, z)\, x^i_{10} x^j_{01} + z_i x^i_{11}$, then $\bar{\varepsilon}(0) \in \in (\tau^2 M)_0 = T^*M \oplus T^*M$.

By similar considerations we get for $r = 3$:

Proposition 9. *Let $h : T^*M \to \tau^2 M$ be a linear cross-section. Let λ_1, λ_2 be the linear connections on TM determined by the cross-sections $\varkappa^?_2 h$, $\varkappa^1_2 h : T^*M \to \tau^2 M$. Then there exists a unique sector connection λ on $T_2 M$ such that $\pi_1 \lambda = \lambda_1$, $\pi_2 \lambda = \lambda_2$ and for every λ-horizontal vector $X \in (\lambda\gamma)_x$, for every $z \in T^*_x M$ $h(z)(X) = 0$ at any $x \in M$.*

Proof. Let h be given by $\bar{z}_i = z_i$, $a_{ij} = {}^1h^k_i(x)\, z_k$, $b_{ij} = {}^2h^k_{ij}(x)\, z_k$, $c_{ij} = {}^2h^k_{ij}(x)\, z_k$, $a_{iju} = h^k_{iju}(x)\, z_k$. Then $x^i_{1k} = -{}^s h^i_{jk} x^j_1$ are the equations of λ_s, $s = 1, 2$. Because of it any sector connection Γ such that $\pi_1 \Gamma = \lambda_1$, $\pi_2 \Gamma = \lambda_2$ has the following equations:

$$x^i_{10k} = -{}^1h^i_{jk} x^i_{10},\ x^l_{01k} = -{}^2h^i_{jk} x^j_{01},$$
$$x^i_{11k} = F^i_{juk}(x)\, x^j_{10} x^u_{01} + F^i_{jk}(x)\, x^j_{11},$$

i.e. $X \in T_w(T_2 M$ is Γ-horizontal iff

$$x^i_{101} = -{}^1h^i_{jk} x^j_{100} x^k_{001}, \qquad x^i_{011} = -{}^2h^i_{jk} x^j_{010} x^k_{001},$$
$$x^i_{111} = (F^i_{juk} x^j_{100} x^u_{010} + F^i_{jk} x^j_{110})\, x^k_{001}.$$

If $X \in H\Gamma$ then

$$h(z)(X) = [(h^i_{juk} - {}^1h^i_{jt}\, {}^2h^t_{uk} - {}^2h^i_{tu}\, {}^1h^t_{jk} + F^i_{juk})\, x^j_{100} x^u_{010} x^k_{001} + + ({}^3h^i_{jk} + F^i_{jk})\, x^j_{110} x^k_{001}]\, z_i.$$

Therefore $h(z)(X) = 0$ for every $z \in T^*_x M$ and every $X \in (H\Gamma)_x$ iff

$$F^i_{juk} = {}^2h^i_{tu}\, {}^1h^t_{jk} + {}^1h^i_{jt}\, {}^2h^t_{uk} - h^i_{juk}, \qquad F^i_{jk} = -{}^3h^i_{jk}. \tag{10}$$

These equations determine the unique sector connection λ of the desired properties.

Remark 5. Any couple of the linear connections induced by three linear sections $\varkappa^1_1 h$, $\varkappa^1_2 h$, $\varkappa^2_2 h : T^*M \to \tau^2 M$ which are determined by a linear cross-section $h : T^*M \to \tau^3 M$ can be chosen as the underlying connections $\pi_1 \lambda$, $\pi_2 \lambda$ of the sector connection λ constructed by h in the sence of Proposition 9. Hence there exist 3^2 sector connections on $T_2 M$ determined by h. If h is semi-holonomic then λ is unique.

Proposition 10. *Let λ be a sector connection on $T_2 M$. Then there is a unique linear cross-section $h : T^*M \to \tau^2 M$ such that $\varkappa^2_2 h$, $\varkappa^1_2 h$ are determined by the linear connections $\pi_1 \lambda$, $\pi_2 \lambda$ on TM and $h(z)(X) = 0$ for every λ-horizontal vector $X \in (H\lambda)_x$ and for every $z \in T^*_x M$ at any $x \in M$.*

Proof is quite analogous to that of Proposition 9.

Remark 6. It is easy to see that in general a sector connection λ determines 3^2 linear cross-sections $h : T^*M \to \tau^3 M$. Nevertheles there is the (1,1)-correspondence $\psi : \lambda \to h$ such that the connections $\pi_1\lambda$, $\pi_2\lambda$ correspond to the cross-sections $\varkappa_2^2 h$, $\varkappa_2^1 h$. At any case, $\varkappa_1^1 h$ induces the connection λ_3 on TM determined by λ. This means that λ is non-holonomic if $\varkappa_1^1 h = \varkappa_2^2 h$. Hence λ induced by a non-holonomic cross-section h, $\varkappa_2^2 h = \varkappa_2^1 h$, is non-holonomic if h is semi-holonomic.

According to the correspondence ψ we introduce the action of the group I_3 on the set of all sector connections on T_2M by $g(\lambda) = g(\psi\lambda)$. For instance if $({}^1F^i_{jk}, {}^2F^i_{jk}, {}^3F^i_{jk}, F^i_{juk})$ are the Christoffel's functions of λ then

$$({}^1F^i_{jk} = {}^2F^i_{kj}, \qquad {}^2F^i_{jk} = {}^1F^i_{kj}, \qquad {}^3F^i_{jk} = {}^3F^i_{jk},$$
$$F^i_{juk} = F^i_{ujk} - {}^1F^i_{ut}{}^2F^t_{jk} - {}^2F^i_{tj}{}^1F^t_{uk} + {}^2F^i_{tj}{}^1F^t_{ku} + {}^1F^i_{ut}{}^2F^t_{kj})$$

are the Christoffel's functions of $Ti_2\lambda$.

Let $X_i \in T_xM$, $i = 1, 2, 3$. There exists $u \in (T_3M)_x$ such that

$$p_{TM}p_{T_2M}(u = X_1, \qquad Tp_M p_{T_2M}(u) = X_2, \qquad Tp_M Tp_{TM}(u) = x_3.$$

If λ is a 1-symmetric sector connection then it is easy to compute that

$$\Delta\lambda(X_1, X_2, X_3) := \sum_{g\in I_3} \operatorname{sgng} H_{g\lambda}(u) = \sum_{g\in P_3} \operatorname{sgng} F^i_{j_{g(1)}j_{g(2)}j_{g(3)}} x_1^{j_{g(1)}} x_2^{j_{g(2)}} x_3^{j_{g(3)}}.$$

where $H_{g\lambda}$ (u) denotes the $g\lambda$-horizontal part of u as it was introduced above. It means that $\Delta\gamma : M \to TM \otimes \wedge^3 T^*M$. Comparing it with Remark 1 we get

Proposition 11. *If λ is a 1-symmetric sector connection on* T_2M *then* $A\tau_1^\lambda = 1/2\,\Delta\lambda$.

Remark on a construction of the Kolář s prolongation of a linear connection on TM. Let $\gamma : TM \to JTM$, $x^i_{1k} = \Gamma^i_{jk}x^j_1$, be a linear connection. Let $\bar\gamma^* : T^*M \to \tau^2 M$, $\bar z_i = z_i$, $z_{jk} = -\Gamma^i_{jk}z_i$, be the cross-section induced by γ. Denote by $f_\gamma : T^*Mx_MT_2M \to R$ the function defined by $f_\gamma(z, t) = \bar\gamma^*(z)(t) = -\Gamma^i_{jk}(x)z_ix^j_{10}x^k_{01} + z_jx^j_{11}$. Then $f^*_\gamma := p_2 \,.\, Tf_\gamma : T(T^*M)\,x_{TM}T(T_2M) \to R$ is a linear form on $T^*Mx_MT_2M$, where $p_2 : TR = R\times R \to R$ is the projection on the second summand. In coordinates we get

$$f^*_\gamma = -\Gamma^i_{jk,u}z_ix^j_{10}x^k_{01}\,dx^u - \Gamma^i_{jk}x^j_{10}x^k_{01}\,dz_i - \Gamma^i_{jk}z_ix^k_{01}\,dx^j_{10} -$$
$$- \Gamma^i_{jk}z_ix^j_{10}\,dx^k_{01} + x^j_{11}\,dz_j + z_j\,dx^j_{11}.$$

Let $H\gamma^*(z) : T_xM \to T_zT^*M$ be the γ^*-horizontal lift, where $\gamma^* : T^*M \to JT^*M$ is the connection induced by γ. Then

$$\bar f^*_\gamma : T^*Mx_MT_3M \to R, \qquad \bar f^*_\gamma(z, v) := f^*_\gamma(H\gamma^*(z)(Tp_M \,.\, Tp_{TM}(v)), v) =$$
$$= (-\Gamma^i_{jk,u} + \Gamma^i_{tu}\Gamma^t_{jk})\, z_ix^j_{100}x^k_{010}x^u_{001} - \Gamma^i_{jk}z_ix^j_{101}x^k_{010} -$$
$$- \Gamma^i_{jk}z_ix^j_{100}x^k_{011} - \Gamma^i_{jk}z_ix^j_{110}x^k_{001} + z_jx^j_{111}$$

is a linear section $T^*M \to \tau^3 M$ the values of which are semi-holonomic 3-forms. By Proposition 9, $\bar f^*_\gamma$ determines the unique semi-holonomic sector connection λ for which the equations (10) give (6), i.e. λ is the Kolář's prolongation of γ.

4. On geodesics of sector connections. Elements of $T_r M$ or cross-sections $\zeta : T_{r-1}M \to T(T_{r-1}M)$ are called r-vectors on M or r-vector fields respectively. Let λ be a sector connection on $T_2 M$. We will say that a 2-vector field ζ on M over a curve c on TM is λ-parallel if its T-lift is λ-horizontal. Let $x^i = c^i(t)$, $x^i_{10} = c^i_{10}(t)$, $x^i_{01} = c^i_{01}(t)$, $x^i_{11} = c^i_{11}(t)$ be a 2-vector field ζ over a curve c. Then ζ is λ-parallel if

$$\frac{dc^i_{10}}{dt} = {}^1F^i_{jk} c^j_{10} \frac{dc^k}{dt}, \qquad \frac{dc^i_{01}}{dt} = {}^2F^i_{jk} c^j_{01} \frac{dc^k}{dt} \tag{11}$$

$$\frac{dc^i_{11}}{dt} = F^i_{jku} c^j_{10} c^k_{01} \frac{dc^k}{dt} + {}^3F^i_{jk} c^j_{11} \frac{dc^k}{dt}.$$

If $i_2\xi = \xi$, i.e. if $Tp_M\xi = p_{TM}\xi$ then ξ is said to be the second velocity. Since in the induced chart, $c^i_{10} = c^i_{01}$ is the condition for ξ to be the second velocity then instead of the second equation of (11) we can use the equation

$$({}^1F^i_{jk} - {}^2F^i_{jk})\, c^j_{10}\, dc^k/dt = 0.$$

It means that if λ is not projectable then at any 2-vector of the second velocity there exists a unique curve c on TM and a unique 2-vector field of the second velocity which is λ-parallel over c.

Let γ be a linear connection on TM. Then in the case of a 2-vector γ-horizontal vector field the third equation of (11) is of the form

$$\gamma^i_{jk,u} c^j_{10} c^k_{01} \frac{dc^u}{dt} + \gamma^i_{jk} \frac{dc^j_{10}}{dt} c^k_{01} + \gamma^i_{jk} c^j_{10} \frac{dc^k_{01}}{dt} = \tag{12}$$

$$= F^i_{jku} c^j_{10} c^k_{01} \frac{dc^u}{dt} + {}^3F^i_{tk} \gamma^t_{ju} c^j_{10} c^u_{01} \frac{dc^k}{dt},$$

where γ^i_{jk} are the Christoffel's functions of γ.

A curve c on M is called a geodesic of a sector connection λ on T_2M if its $T_3 = TTT$-lift T_3c is λ-horizontal, i.e. if T_2c is λ-parallel over Tc. Hence the equations 11) give for a geodesic $x^i = c^i(t)$ the following relations:

$$\frac{d^2c^i}{dt^2} = {}^1F^i_{jk} \frac{dc^j}{dt} \frac{dc^k}{dt} = {}^2F^i_{jk} \frac{dc^j}{dt} \frac{dc^k}{dt},$$

$$\frac{d^3c^i}{dt^3} = F^i_{jku} \frac{dc^j}{dt} \frac{dc^k}{dt} \frac{dc^u}{dt} + {}^3F^i_{jk} \frac{d^2c^j}{dt^2} \frac{dc^k}{dt}.$$

Therefore if c is a geodesic of λ then is a geodesic of its underlying connections λ_1 and λ_2. Hence the question of geodesics there is only in the case of a projectable sector connection. If a curve c on M is a geodesic of a projectable sector connection λ then Tc is λ_1-horizontal and because of it by (12) we get the conditions for c:

$$\frac{d^2c^i}{dt^2} = F^i_{jk} \frac{dc^j}{dt} \frac{dc^k}{dt},$$

$$(F^i_{jku} + {}^3F^i_{tu}F^t_{jk} - F^i_{jk,u} - F^i_{tk}F^t_{ju} - F^i_{jt}F^t_{ku})\frac{dc^j}{dt}\frac{dc^k}{dt}\frac{dc^u}{dt} = 0.$$

It means that every geodesic of λ_1 is not geodesic of λ. The coordinate form (7) of ∇_1^λ gives

Proposition 11. *Let λ be a projectable sector connection. Then every geodesic of the underlying connection λ_1 is a geodesic of λ iff the symmetrisation of ∇_1^λ vanishes.*

A projectable sector connection λ on T_2M is called geodesic if every geodesic of λ_1 is a geodesic of λ. Hence the above introduced connection λ_1^* is geodesic. Consequently the Kolář's and Oproiu's prolongation of a linear connection γ on TM are geodesic. In general using the formula (6) of [4] one can easily calculate that the symmetrisation of ∇_1^λ for arbitrary natural first order prolongation of γ vanishes. Hence it holds.

Proposition 12. *Every natural first order prolongation of a linear connection γ on TM is geodesic.*

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Anton Dekrét
VŠLD
Marxova 24
960 53 Zvolen
Czechoslovakia