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Olga Krupková

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LEPAGEAN 2-FORMS IN HIGHER ORDER HAMILTONIAN MECHANICS II. INVERSE PROBLEM*

OLGA KRUPKOVÁ

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Abstract. The theory of Lepagean 2-forms is used to formulate and investigate the inverse problem in higher order Hamiltonian mechanics. Necessary and sufficient conditions for a system of first order ordinary differential equations to coincide with a system of equations for Hamilton extremals related with a system of (higher order) variational equations are found. The geometrical interpretation of the equations for Hamilton extremals in terms of a distribution is proposed and the relation of the Hamiltonian inverse problem to the problem of "variational integrating factors" is studied.

Key-words. Hamilton equations, regularity, inverse problem, variationality conditions, Hamiltonian vector field, variational integrating factors.

MS Classification. 58 F 05, 70 H 05

The present paper is a continuation of paper [16]. We go on in the notations, numbering of sections, theorems, examples, and references.

8. INVERSE PROBLEM FOR THE HAMILTON EQUATIONS

In Secs. 4 and 6 the equations for Hamilton extremals associated to variational equations were introduced. In this section we discuss the inverse problem related to such equations. The formulation of the problem is analogous to that in [21] where Hamilton equations of a lagrangian were considered; the solution is based on the theory of Lepagean 2-forms.

Theorem 9. *Let $\eta \in \Omega^{1,1}_{j^{s-1}Y}(j^1(j^{s-1}Y))$ be a form. The following conditions are equivalent.*

(1) *η is a Hamilton form associated with a locally variational form $E \in \Omega^{1,1}_Y(j^sY))$*

(2) *There exists a 2-contact form $G \in \Omega^2(j^1(j^{s-1}Y))$ and a Lepagean form $\alpha \in \Omega^2(j^{s-1}Y)$ such that $\eta + G = (\pi_{s-1})^*_{1,0}\alpha$.*

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* The results of Sec. 8 were presented at the Conference on Differential Equations and Their Applications EQUADIFF 6, Brno, August 1985.

(3) *There exists a Lepeagean form* $\alpha \in \Omega^2(j^{s-1}Y)$ *such that* $\bar{h}(A\alpha)$, *where* A *is defined by* (2.8), *is a* (*local*) *lagrangian for* η.

Proof. (2) is obtained by putting $i_{j^1\xi}G = \bar{p}(i_\xi \alpha_E)$ for every π_{s-1}-vertical vector field ξ on $j^{s-1}Y$, where α_E is the Lepagean equivalent of E.

(3) is obvious since $\eta + G$ is Lepagean and the Tonti lagrangian $\varkappa$ of η is of the form $\varkappa = A\eta = \bar{h}(A\alpha)$.

Finally suppose η be locally variational with a (local) lagrangian $\bar{h}(A\alpha)$ and denote by E the 1-contact part of α. Evidently $\bar{h}(A\alpha) = \bar{\lambda}$ where $\lambda = AE$ is the Tonti lagrangian of E. Hence using $d\Theta_{\bar{\lambda}} = (\pi_{s-1})^*_{1,0}\, d\Theta_\lambda$ and $d\Theta_\lambda = \alpha$ we obtain $i_{j^1\xi}\eta = \bar{h}(i_\xi \alpha)$ for each π_{s-1}-vertical vector field ξ on $j^{s-1}Y$, which proves (1).

Let (V, ψ), $\psi = (t, q^\sigma)$, $1 \leqq \sigma \leqq m$ be a fiber chart on Y, and let

$$\eta = \sum_{i=0}^{s-1} H^i_\sigma \bar{\omega}^\sigma_i \wedge dt, \tag{8.1}$$

where H^i_σ, $0 \leqq i \leqq s-1$, $1 \leqq \sigma \leqq m$ are functions on $(V_{s-1})_1$, be the chart expression of a form $\eta \in \Omega^{1,1}_{j^{s-1}Y}(j^1(j^{s-1}Y))$.

Corollary. *The form* (8.1) *is a Hamilton form associated with a locally variational form* $E \in \Omega^{1,1}_Y(j^sY)$ *iff the functions* H^i_σ *satisfy the following relations for* $1 \leqq \sigma, \nu \leqq m$:

$$\frac{\partial H^i_\sigma}{\partial q^\nu_{k,1}} + \frac{\partial H^k_\nu}{\partial q^\sigma_{i,1}} = 0, \tag{8.2}$$

$$\frac{\partial H^i_\sigma}{\partial q^\nu_k} - \frac{\partial H^k_\nu}{\partial q^\sigma_i} + \frac{d}{dt}\frac{\partial H^k_\nu}{\partial q^\sigma_{i,1}} = 0, \qquad 0 \leqq i, k \leqq s-1,$$

$$H^i_\sigma - \sum_{k=0}^{s-1} \frac{\partial H^i_\sigma}{\partial q^\nu_{k,1}}(q^\nu_{k,1} - q^\nu_{k+1}) = 0, \qquad 1 \leqq i \leqq s-1. \tag{8.3}$$

Moreover, E is regular iff η *is regular, i.e. iff* $\det(\partial H_\sigma/\partial q^\nu_{s-1,1}) \neq 0$.

Proof. Let η be given by (8.1). The relations (8.2) ensure that η is locally variational. Hence there exists the Lepagean equivalent α_η of η. It holds $d(\pi^*_{s,s-1}\alpha) = 0$ where α is the $(\pi_{s-1})_{1,0}$-projection of α_η, and (8.3) implies $\pi^*_{s,s-1}\alpha = E + F$ where

$$E = \left[H_\sigma + \sum_{k=0}^{s-1} \frac{\partial H_\sigma}{\partial q^\nu_{k,1}}(q^\nu_{k,1} - q^\nu_{k+1})\right]\omega^\sigma \wedge dt, \tag{8.4}$$

and F is 2-contact. This means that α is Lepagean, i.e. η is a Hamilton form associated with (8.4).

The converse assertion follows from Theorem 3 and the definition of the Hamilton form (relations (4.4)–(4.5) and (3.4)).

The regularity of E is equivalent with the regularity of η since by (5.3) and (8.4)

$$(8.5)\qquad \det\left(\frac{\partial H^i_\sigma}{\partial q^\nu_{k,1}}\right) = \det(F^{ik}_{\sigma\nu}) \neq 0 \Leftrightarrow \det\left(\frac{\partial E_\sigma}{\partial q^\nu_s}\right) = \det\left(\frac{\partial H_\sigma}{\partial q^\nu_{s-1,1}}\right) \neq 0$$

where $F^{ik}_{\sigma\nu}$ are defined by (3.20).

Remark. It is easy to show that (8.2) and (8.3) imply $\partial H^i_\sigma/\partial q^\nu_{k,1} = 0$ for $s \leqq i+k \leqq 2s-2$.

A Hamilton form associated with a locally variational form E being given, there arises a question under what conditions E is (globally) variational (the global inverse problem). The following theorem shows that this problem has a trivial solution.

Theorem 10. *Let $\eta \in \Omega^{1,1}_{j^{s-1}Y}(j^1(j^{s-1}Y))$ be the Hamilton form associated with a locally variational form $E \in \Omega^{1,1}_Y(j^sY)$. η is variational iff E is variational.*

Proof. Suppose η be variational. Then, according to a theorem by Anderson and Duchamp [1], there exists a (global) lagrangian $\varkappa$ for η defined on $j^1(j^{s-1}Y)$. Since η is a Hamilton form of E there exists a covering of $j^1(j^{s-1}Y)$ by open sets W and an extended lagrangian $\bar{\lambda}' = \bar{h}(\Theta_{\lambda'})$ on each W such that (locally) $E = E_{\lambda'}$ and $\varkappa|_W = \bar{\lambda}' + \bar{h}(df)$ for a function f defined on $(\pi_{s-1})_{1,0}W$. This implies that the lagrangian $\varkappa$ is an extended lagrangian and $\varkappa = \bar{h}(\Theta_\lambda)$ where λ is a (global) lagrangian of order s for E.

The converse is obvious.

Theorem 9 and its Corollary give necessary and sufficient conditions for a system of first order O.D.E. to be identical with a system of equations for Hamilton extremals. However they do not answer the question whether the given system of equations for Hamilton extremals is given in the canonical form, i.e. whether the given coordinates are the Legendre coordinates of the associated locally variational form E. We shall deal with this question now.

In what follows $\pi : Y \to I$ is a fiber manifold such that $I \subset R$ is an open interval and $Y = I \times B$ where $B \subset R^m$ is an open ball with center at the origin. Fiber coordinates on Y (resp. the associated coordinates on $j^{s-1}Y$) will be denoted by (t, q^σ), $1 \leqq \sigma \leqq m$ (resp. (t, q^σ_i), $1 \leqq \sigma \leqq m$, $0 \leqq i \leqq s-1$). Recall that c denotes the integer $(s/2) - 1 < c \leqq s/2$.

Lemma 3. *Let p^k_ν, $1 \leqq \nu \leqq m$, $0 \leqq k \leqq s-c-1$ be functions defined on $j^{s-1}Y$, let $E \in \Omega^{1,1}_Y(j^sY)$ be a variational form. The following conditions are equivalent:*

(1) *(t, q^σ_i, p^k_ν), $1 \leqq \sigma$, $\nu \leqq m$, $0 \leqq i \leqq c-1$, $0 \leqq k \leqq s-c-1$ are Legendre coordinates of E.*

(2) *It holds* $\det(\partial p^c_\sigma/\partial q^\nu_c) \neq 0$ *if $s = 2c+1$, E is regular and there exists a function H on $j^{s-1}Y$ such that the Lepagean equivalent α_E of E is of the form*

$$(8.6)\qquad \alpha_E = -dH \wedge dt + \sum_{k=0}^{s-c-1} dp^k_\nu \wedge dq^\nu_k.$$

Proof. The assertion follows from Theorems 6 and 5.

Lemma 4. *Let $E \in \Omega_Y^{1,1}(j^sY)$ where $s \geqq 2$ be a regular variational form, let $(T, Q^\sigma, \ldots, Q^\sigma_{c-1}, P_\sigma^{s-c-1}, \ldots, P_\sigma)$, $1 \leqq \sigma \leqq m$ be Legendre coordinates on $j^{s-1}Y$ associated with E. Then the mapping $j^{s-1}Y \in (T, Q^\sigma, \ldots, Q^\sigma_{c-1}, P_\sigma^{s-c-1}, \ldots, P_\sigma) \to \to (t, q^\sigma, \ldots, q^\sigma_{s-1}) \in j^{s-1}Y$ defined by*

$$(8.7) \qquad t = T, \quad q^\sigma = Q^\sigma, \quad q_i^\sigma = \frac{\partial H}{\partial P_\sigma^{i-1}}, \quad 1 \leqq i \leqq c,$$

$$q^\sigma_{c+k} = \frac{d^k}{dt^k}\frac{\partial H}{\partial P_\sigma^{c-1}}, \quad 1 \leqq k \leqq s-c-1, \quad 1 \leqq \sigma \leqq m,$$

where H is the Hamilton function of E, is the inverse to the Legendre transformation.

Proof. It is sufficient to show that (8.7) is a coordinate transformation on $j^{s-1}Y$ and that $\Lambda \,.\, \Sigma = 1$ where Λ (resp. Σ) is the Jacobi matrix of the Legendre transformation (resp. of (8.7)).

Let $s = 2c + 1$. Lemma 1 and 2 and Theorem 8 ensure the existence of a lagrangian $\lambda^0_{\min} = L^0_{\min}\, dt \in \Omega^1_X(j^{c+1}Y)$ such that $\det(\partial^2 L^0_{\min}/\partial q^\sigma_{c+1}\,\partial q^\nu_c) \neq 0$ at each point of $j^{c+1}Y$; then the momenta and the Hamilton function of E are of the form $P^k_\nu = (f^0_{\min})^{k+1}_\nu$ with $(f^0_{\min})^i_\nu$ defined by (2.2), and

$$(8.8) \qquad H = -L^0_{\min} + \sum_{i=0}^{s-2} P^i_\sigma Q^\sigma_{i+1} + P^{c-1}_\sigma q^\sigma_c + P^c_\sigma q^\sigma_{c+1},$$

respectively, where $L^0_{\min}$, q^σ_c and q^σ_{c+1} are considered as functions of the Legendre coordinates. Computing dq^σ_c/dt we obtain that the matrix $(\partial q^\sigma_c/\partial P^c_\nu)$ is regular and inverse to $(\partial P^c_\varrho/\partial q^\lambda_c)$, and that

$$(8.9) \qquad \frac{\partial q^\sigma_c}{\partial P^k_\nu} = 0, \quad 0 \leqq k \leqq c-1$$

holds. Using (8.9) we arrive at

$$(8.10) \qquad \frac{\partial H}{\partial Q^\nu_k} = -\frac{dP^k_\nu}{dt} - \frac{dP^c_\sigma}{dt}\frac{\partial q^\sigma_c}{\partial Q^\nu_k}, \quad 1 \leqq k \leqq c-1, \quad \frac{\partial H}{\partial P^k_\nu} = Q^\nu_{k+1},$$
$$0 \leqq k \leqq c-2,$$

$$\frac{\partial H}{\partial P^{c-1}_\nu} = q^\nu_c, \quad \frac{\partial H}{\partial P^c_\nu} = q^\nu_{c+1} - \frac{dP^c_\sigma}{dt}\frac{\partial q^\sigma_c}{\partial P^c_\nu}, \quad 1 \leqq \nu \leqq m.$$

The matrix Σ is of the form

$$(8.11) \qquad \Sigma = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & J \end{pmatrix}, \quad J = \begin{pmatrix} \left(\dfrac{\partial q^\sigma_c}{\partial P^c_\varrho}\right) & \cdots & \left(\dfrac{\partial q^\sigma_c}{\partial P_\varrho}\right) \\ \vdots & & \\ \left(\dfrac{\partial q^\sigma_{2c}}{\partial P^c_\varrho}\right) & \cdots & \left(\dfrac{\partial q^\sigma_{2c}}{\partial P_\varrho}\right) \end{pmatrix}.$$

Let us introduce the following matrices: $C = (C_{\sigma\nu})$, $N = (N_{\sigma\nu})$, $B = (B_{\sigma\nu})$, where

$$(8.12)\ C_{\sigma\nu} = \frac{\partial P^c_\sigma}{\partial q^\nu_c} = \frac{\partial^2 L^0_{\min}}{\partial q^\sigma_{c+1}\,\partial q^\nu_c} \qquad N_{\sigma\nu} = \frac{\partial P^c_\nu}{\partial q^\sigma_{c]}} - \frac{\partial P^c_\sigma}{\partial q^\nu_c}, \qquad B_{\sigma\nu} = \frac{\partial E_\sigma}{\partial q^\nu_{2c+1}}\,.$$

Obviously, $\det C \neq 0$, $N_{\sigma\nu} = (-1)^c B_{\sigma\nu}$ and $\det N \neq 0$ (c.f. (6.12), (5.3), (5.1)). Computing the elements of J we obtain that J is a triangle matrix with 0 over the diagonal and with the matrices $C^{-1} = (\partial^2 H/\partial P^c_\sigma\,\partial P^{c-1}_\nu)$, $N^{-1}, \ldots, (-1)^{c-1}N^{-1}$ on the diagonal. Since Λ is the matrix (see (6.12), (5.3))

$$(8.13)\qquad \Lambda = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & K \end{pmatrix}, \qquad K = \begin{pmatrix} C & & & \\ & N & & 0 \\ & & \ddots & \\ & & & (-1)^{c-1}N \end{pmatrix},$$

it holds $\Sigma \,.\, \Lambda = 1$.

For $s = 2c$ the proof is similar; the diagonal of J is of the form $(-1)^c B^{-1}, \ldots,$ $(-1)^{2c-1}B^{-1}$ where $(-1)^c B^{-1}_{\sigma\nu} = \partial^2 H/\partial P^{c-1}_\sigma\,\partial P^{c-1}_\nu$, $(-1)^c B_{\sigma\nu} = (-1)^c\,\partial E_\sigma/\partial q^\nu_{2c} =$ $= \partial^2 L_{\min}/\partial q^\sigma_c\,\partial q^\nu_c$.

Let $(T, Q^\sigma_i, P^k_\sigma)$, $1 \leqq \sigma \leqq m$, $0 \leqq i \leqq c-1$, $0 \leqq k \leqq s-c-1$ be fiber coordinates on $j^{s-1}Y$, and denote by $(T, Q^\sigma_i, P^k_\sigma, Q^\sigma_{i,1}, P^k_{\sigma,1})$ the associated coordinates on $j^1(j^{s-1}Y)$. A form $\eta \in \Omega^{1,1}_{j^{s-1}Y}(j^1(j^{s-1}Y))$ has the chart expression

$$(8.14)\qquad \eta = \sum_{i=0}^{c-1} K^i_\sigma\,\mathrm{d}Q^\sigma_i \wedge \mathrm{d}T + \sum_{i=0}^{s-c-1} \bar{K}^\sigma_i\,\mathrm{d}P^i_\sigma \wedge \mathrm{d}T$$

in this chart, where K^i_σ, $\bar{K}^\sigma_i$ are functions defined on $j^1(j^{s-1}Y)$. Lemma 3 and (4.2) obviously imply that a necessary condition for (8.14) to be the canonical form of a regular Hamilton form is

$$(8.15)\qquad K^i_\sigma = F^i_\sigma - P^i_{\sigma,1} - \sum_{k=c}^{s-c-1} R^{i\nu}_{\sigma k} P^k_{\nu,1}, \qquad \bar{K}^\sigma_i = G^\sigma_i + Q^\sigma_{i,1},$$
$$\bar{K}^\sigma_c = G^\sigma_c + \sum_{k=0}^{c-1} \bar{R}^{\sigma k}_{c\nu} Q^\nu_{k,1} + M^{\sigma\nu}P^c_{\nu,1}, \qquad 0 \leqq i \leqq c-1,\ 1 \leqq \sigma \leqq m,$$

where F^i_σ, $R^{i\nu}_{\sigma c}$, $\bar{R}^{\sigma i}_{c\nu}$, $M^{\sigma\nu}$, G^σ_k, $0 \leqq i \leqq c-1$, $0 \leqq k \leqq c$, $1 \leqq \sigma, \nu \leqq m$ are functions on $j^{s-1}Y$.

Theorem 11. (1) *Let $s = 2c$, $s \geqq 2$. Let $(T, Q^\sigma_i, P^i_\sigma)$ $0 \leqq i \leqq c-1$, $1 \leqq \sigma \leqq m$ be fiber coordinates on $j^{2c-1}Y$, let*

$$(8.16)\qquad \eta = \sum_{i=0}^{c-1} [(F^i_\sigma - P^i_{\sigma,1})\,\mathrm{d}Q^\sigma_i + (G^\sigma_i + Q^\sigma_{i,1})\,\mathrm{d}P^i_\sigma] \wedge \mathrm{d}T$$

be the chart expression of $\eta \in \Omega^{1,1}_{j^{2c-1}Y}(j^1(j^{2c-1}Y))$. η is a Hamilton form associated *with a regular variational form E and* $(T, Q_i^\sigma, P_\sigma^i)$ *are Legendre coordinates of E iff the following conditions are satisfied:*

(a) η *is variational*

(b) *the functions* G_{c-1}^σ *do not depend on* P_ν^k, $0 \leqq k \leqq c-2$

(c) *the mapping* $j^{2c-1}Y \ni (T, Q^\sigma, \ldots, Q_{c-1}^\sigma, P_\sigma^{c-1}, \ldots, P_\sigma) \to (t, q^\sigma, \ldots, q_{2c-1}^\sigma) \in j^{2c-1}Y$ *defined by*

$$t = T, \quad q_i^\sigma = Q_i^\sigma, \quad 0 \leqq i \leqq c-1, \quad q_c^\sigma = -G_{c-1}^\sigma, \tag{8.17}$$
$$q_{c+k+1}^\sigma = \frac{\partial q_{c+k}^\sigma}{\partial T} - \sum_{l=0}^{c-1} \frac{\partial q_{c+k}^\sigma}{\partial Q_l^\varrho} G_l^\varrho + \sum_{l=c-1-k}^{c-1} \frac{\partial q_{c+k}^\sigma}{\partial P_\varrho^l} F_\varrho^l,$$
$$0 \leqq k \leqq c-2, \; 1 \leqq \sigma \leqq m$$

is a coordinate transformation on $j^{2c-1}Y$

(d) *in the coordinates* (t, q_i^σ) *the relations*

$$F_\sigma^i = \frac{\mathrm{d}P_\sigma^i}{\mathrm{d}t}, \quad 1 \leqq i \leqq c-1, \qquad G_k^\sigma = -q_{k+1}^\sigma, \quad 0 \leqq k \leqq c-2 \tag{8.18}$$

hold.

(2) *Let* $s = 2c+1$, $s \geqq 2$. *Let* $(T, Q_i^\sigma, P_\sigma^k)$, $0 \leqq i \leqq c-1$, $0 \leqq k \leqq c$, $1 \leqq \sigma \leqq m$ *be fiber coordinates on* $j^{2c}Y$, *let*

$$\eta = \{\sum_{i=0}^{c-1} [(F_\sigma^i - P_{\sigma,1}^i - R_{\sigma c}^{i\nu} P_{\nu,1}^c)\,\mathrm{d}Q_i^\sigma + (G_i^\sigma + Q_{i,1}^\sigma)\,\mathrm{d}P_\sigma^i] + \tag{8.19}$$
$$+ (G_c^\sigma + \sum_{k=0}^{c-1} R_{c\nu}^{\sigma k} Q_{k,1}^\nu + M^{\sigma\nu} P_{\nu,1}^c)\,\mathrm{d}P_\sigma^c\} \wedge \mathrm{d}T,$$

be the chart expression of $\eta \in \Omega_{j^{2c}Y}^{1,1}(j^1(j^{2c}Y))$. η *is a Hamilton form associated with a regular variational form E and* $(T, Q_i^\sigma, P_\sigma^k)$ *are Legendre coordinates of E iff the following conditions are satisfied:*

(a) η *is variational*

(b) *the functions* G_{c-1}^σ *do not depend on* P_ν^k, $0 \leqq k \leqq c-1$

(c) *it holds*

$$R_{\nu c}^{k\sigma} = R_{c\nu}^{\sigma k} = -\frac{\partial G_{c-1}^\sigma}{\partial Q_k^\nu}, \qquad 0 \leqq k \leqq c-1, \tag{8.20}$$
$$M^{\sigma\nu} = -\frac{\partial G_{c-1}^\sigma}{\partial P_\nu^c} + \frac{\partial G_{c-1}^\nu}{\partial P_\sigma^c}, \qquad \det(M^{\sigma\nu}) \neq 0$$

(d) *the mapping* $j^{2c}Y \ni (T, Q^\sigma, \ldots, Q_{c-1}^\sigma, P_\sigma^c, \ldots, P_\sigma) \to (t, q^\sigma, \ldots, q_{2c}^\sigma) \in j^{2c}Y$ *defined by*

$$t = T, \quad q_i^\sigma = Q_i^\sigma, \quad 0 \leqq i \leqq c-1, \quad q_c^\sigma = -G_{c-1}^\sigma, \quad q_{c+k+1}^\sigma = \tag{8.21}$$
$$= \frac{\partial q_{c+k}^\sigma}{\partial T} - \sum_{l=0}^{c-1} \frac{\partial q_{c+k}^\sigma}{\partial Q_l^\varrho} G_l^\varrho + \sum_{l=c-k}^{c-1} \frac{\partial q_{c+k}^\sigma}{\partial P_\varrho^l} (F_\varrho^l + R_{c\varrho}^{\nu l} T_\nu) - \frac{\partial q_{c+k}^\sigma}{\partial P_\nu^c} T_\nu,$$

$$0 \leqq k \leqq c-1, \quad 1 \leqq \sigma \leqq m,$$

where

(8.22)
$$T_\nu = M_{\nu\lambda}(G_c^\lambda + \sum_{i=0}^{c-1} R_{c\varkappa}^{\lambda i} q_{i+1}^\varkappa)$$

and $(M_{\nu\lambda}) = M^{-1}$, *is a coordinate transformation on* $j^{2c}Y$

(e) *in the coordinates* (t, q_i^σ) *the relations*

(8.23)
$$G_k^\sigma + q_{k+1}^\sigma = 0, \quad 0 \leqq k \leqq c-2, \qquad G_c^\sigma + \sum_{i=0}^{c-1} R_{c\nu}^{\sigma i} q_{i+1}^\nu +$$
$$+ M^{\sigma\nu} \frac{\mathrm{d}P_\nu^c}{\mathrm{d}t} = 0, \qquad F_\sigma^i - \frac{\mathrm{d}P_\sigma^i}{\mathrm{d}t} - R_{c\sigma}^{\nu i} \frac{\mathrm{d}P_\nu^c}{\mathrm{d}t} = 0, \quad 1 \leqq i \leqq c-1$$

hold.

Proof. We shall sketch the proof for $s = 2c + 1$; the case $s = 2c$ is dealt with in a similar way.

Suppose (a)–(e). Variationality of η ensures the existence of the Lepagean equivalent α_η of η. It holds

(8.24)
$$\alpha_\eta = \eta + \sum_{i=0}^{c-1} (\pi_\sigma^i \wedge \omega_i^\sigma + R_{c\nu}^{\sigma i} \pi_\sigma^c \wedge \omega_i^\nu) + \frac{1}{2} M^{\sigma\nu} \pi_\sigma^c \wedge \pi_\nu^c$$

where $\omega_i^\sigma = \mathrm{d}Q_i^\sigma - Q_{i,1}^\sigma \,\mathrm{d}T$, $\pi_\sigma^k = \mathrm{d}P_\sigma^k - P_{\sigma,1}^k \,\mathrm{d}T$, $0 \leqq i \leqq c-1$, $0 \leqq k \leqq c$, $1 \leqq \sigma \leqq m$. Denote by α the $(\pi_{2c})_{1,0}$-projection of α_η. We shall transform α to the coordinates (t, q_i^σ). Using (e) we obtain $\alpha = E + F$ where

(8.25)
$$E = \left(F_\sigma - \frac{\mathrm{d}P_\sigma}{\mathrm{d}t} - R_{c\sigma}^{\nu 0} \frac{\mathrm{d}P_\nu^c}{\mathrm{d}t} \right) \omega^\sigma \wedge \mathrm{d}t$$

and F is 2-contact. Since $\mathrm{d}\alpha = 0$, α is the Lepagean equivalent of E. According to Theorem 9 η is the Hamilton form of (8.25). We shall show that the assumptions of Lemma 3 are satisfied. From (b) and (a) we obtain $\partial q_{c+k}^\sigma / \partial P_\nu = 0$, $0 \leqq k \leqq c-1$, which easily leads to the condition $\det(\partial P_\sigma / \partial q_{2c}^\nu) \neq 0$. Since the second condition of (8.23) implies $\partial P_\sigma^c / \partial q_{2c}^\nu = 0$, we arrive at

(8.26)
$$\det\left(\frac{\partial E_\sigma}{\partial q_{2c+1}^\nu}\right) = -\det\left(\frac{\partial P_\sigma}{\partial q_{2c}^\nu}\right) \neq 0,$$

i.e. E is regular. Regularity of (d) implies that $\det(\partial G_{c-1}^\sigma / \partial P_\nu^c) \neq 0$ holds. Finally, variationality of η and (c) ensure the existence of a function H on $j^{2c}Y$ such that for $0 \leqq i \leqq c-1$, $1 \leqq \sigma \leqq m$

(8.27)
$$F_\sigma^i = -\frac{\partial H}{\partial Q_i^\sigma}, \qquad G_i^\sigma = -\frac{\partial H}{\partial P_\sigma^i}, \qquad G_c^\sigma + \frac{\partial G_{c-1}^\sigma}{\partial T} = -\frac{\partial H}{\partial P_\sigma^c};$$

H is obtained by putting

$$(8.28)\qquad H = -\sum_{i=0}^{c-1} Q_i^\sigma \int_0^1 F_\sigma^i(T, uQ_k^\nu, uP_\nu^k)\,du - \sum_{i=0}^{c-1} P_\sigma^i \int_0^1 G_i^\sigma(T, uQ_k^\nu, uP_\nu^k)\,du - $$
$$- P_\sigma^c \int_0^1 \left(G_c^\sigma + \frac{\partial G_{c-1}^\sigma}{\partial T}\right)(T, uQ_k^\nu, uP_\nu^k)\,du.$$

Thence (using (c)) we obtain

$$(8.29)\qquad \alpha = -dH \wedge dT + \sum_{i=0}^{c-1} dP_\sigma^i \wedge dQ_i^\sigma + dP_\sigma^c \wedge dq_c^\sigma$$

which proves that $(T, Q_i^\sigma, P_\sigma^k)$ are Legendre coordinates of E.

Conversely, suppose (8.19) be the Hamilton form of a regular variational form E, where $(T, Q_i^\sigma, P_\sigma^k)$ are Legendre coordinates of E. Considering the inverse of the Legendre transformation (8.7) we obtain using (8.9) and (8.10) that it has the properties (b) and (d), and that (e) and (c) hold. The matrix $M = (M^{\sigma\nu})$ is regular since $M = CNC^T$ where C, N are defined by (8.12) and $C_{\sigma\nu}^T = C_{\nu\sigma}$. Variationality of η is ensured by Theorem 3.

This completes the proof.

Remarks. (1) The ADK-conditions (3.2) for (8.16) have the following form: for $0 \leqq i, k \leqq c-1$, $1 \leqq \sigma, \nu \leqq m$

$$(8.30)\qquad \frac{\partial F_\sigma^i}{\partial Q_k^\nu} - \frac{\partial F_\nu^k}{\partial Q_i^\sigma} = 0, \qquad \frac{\partial F_\sigma^i}{\partial P_\nu^k} - \frac{\partial G_k^\nu}{\partial Q_i^\sigma} = 0, \qquad \frac{\partial G_i^\sigma}{\partial P_\nu^k} - \frac{\partial G_k^\nu}{\partial P_\sigma^i} = 0.$$

Notice that the conditions (8.30) (meaning that the equations $dP_\sigma^i/dT = F_\sigma^i$, $dQ_i^\sigma/dT = -G_i^\sigma$, $0 \leqq i \leqq c-1$, $1 \leqq \sigma \leqq m$ are variational as first order equations) ensure the existence of a function H on $j^{2c-1}Y$ such that $F_i^\sigma = -\partial H/\partial Q_i^\sigma$, $G_i^\sigma = -\partial H/\partial P_\sigma^i$; it is sufficient to put

$$(8.31)\qquad H = -\sum_{i=0}^{c-1} Q_i^\sigma \int_0^1 F_\sigma^i(T, uQ_k^\nu, uP_\nu^k)\,du - \sum_{i=0}^{c-1} P_\sigma^i \int_0^1 G_i^\sigma(T, uQ_k^\nu, uP_\nu^k)\,du.$$

However (8.30) do not ensure that these equations are Hamilton canonical equations (c.f. [22], [18]).

Variationality conditions for (8.19) (where $M^{\sigma\nu}$ and $R_{\nu c}^{k\sigma}$ are defined by (8.20)) are of the form (8.30) and

$$(8.32)\qquad \frac{\partial G_i^\sigma}{\partial P_\nu^c} - \frac{\partial G_c^\nu}{\partial P_\sigma^i} = 0, \qquad \frac{\partial F_\sigma^i}{\partial P_\nu^c} - \frac{\partial G_c^\nu}{\partial Q_i^\sigma} - \frac{\partial^2 G_{c-1}^\nu}{\partial T\,\partial Q_i^\sigma} = 0, \qquad 0 \leqq i \leqq c-1,$$
$$\frac{\partial G_c^\sigma}{\partial P_\nu^c} - \frac{\partial G_c^\nu}{\partial P_\sigma^c} + \frac{\partial^2 G_{c-1}^\sigma}{\partial T\,\partial P_\nu^c} - \frac{\partial^2 G_{c-1}^\nu}{\partial T\,\partial P_\sigma^c} = 0, \qquad 1 \leqq \sigma, \nu \leqq m.$$

(2) The transformation (8.17) (resp. (8.21)) can be defined equivalently by $t = T$, $q_i^\sigma = Q_i^\sigma$, $0 \leqq i \leqq c-1$, $q_c^\sigma = -G_{c-1}^\sigma$, $q_{c+k}^\sigma = -d^k G_{c-1}^\sigma/dt^k$, $1 \leqq k \leqq s-c-1$,

$1 \leqq \sigma \leqq m$. The Jacobi matrix of (8.17) (resp. (8.21)) is a triangle matrix with zeros over its diagonal and with the matrices $(\partial q_c^\sigma / \partial P_\nu^{c-1})$, $-(\partial q_c^\sigma / \partial P_\nu^{c-1})$, ..., $(-1)^{c-1}$ $(\partial q_c^\sigma / \partial P_\nu^{c-1})$ (resp. $(\partial q_c^\sigma / \partial P_\nu^c)$, N, ..., $(-1)^{c-1}N$, where

$$N^{\sigma\varrho} = \frac{\partial G_{c-1}^\sigma}{\partial P_\varkappa^c} M_{\varkappa\lambda} \frac{\partial G_{c-1}^\lambda}{\partial P_\varrho^c}, \tag{8.33}$$

$N = (N^{\sigma\varrho})$) on the diagonal. Thence the regularity condition of (8.17) (resp. (8.21)) is of the form $\det(\partial G_{c-1}^\sigma / \partial P_\nu^{c-1}) \neq 0$ (resp. $\det(\partial G_{c-1}^\sigma / \partial P_\nu^c) \neq 0$ and $\det M \neq 0$).

(3) For $s = 2$ we can formulate the assertions (a)–(d) of Theorem 11 in the following form: η is variational and $\det(\partial G^\sigma / \partial P_\nu) \neq 0$.

(4) If $s = 1$ we obtain the following trivial assertion: Let (T, P_σ), $1 \leqq \sigma \leqq m$ be fiber coordinates on Y, let $\eta = (G^\sigma + M^{\sigma\nu} P_{\nu,1})\, dP_\sigma \wedge dT$ be the chart expression of $\eta \in \Omega_Y^{1,1}(j^1 Y)$. η is the Hamilton form associated with a regular variational form E iff $M^{\sigma\nu} = -M^{\nu\sigma}$, $\partial G^\sigma / \partial P_\nu - \partial G^\nu / \partial P_\sigma + \partial M^{\nu\sigma} / \partial T = 0$, $\partial M^{\sigma\nu} / \partial P_\varrho + \partial M^{\varrho\sigma} / \partial P_\nu + \partial M^{\nu\varrho} / \partial P_\sigma = 0$, and $\det(M^{\sigma\nu}) \neq 0$. (Clearly $E = \eta$ and the inverse to the Legendre transformation is any coordinate transformation $(T, P_\sigma) \to (t, q^\sigma)$ such that $\det(\partial q^\sigma / \partial P_\nu - \partial q^\nu / \partial P_\sigma) \neq 0$ and $M^{\sigma\nu} = \partial q^\sigma / \partial P_\nu - \partial q^\nu / \partial P_\sigma$).

Example 7. Let $Z \subset R^6$, $I \subset R$ be open sets, consider a fiber manifold $\pi : Z \times I \to I$, denote by $(t, z_1, \ldots, z_6)$ fiber coordinates on Z. Consider a system of equations

$$\begin{gathered} \dot z_1 = 0, \quad \dot z_2 = 0, \quad \dot z_3 - z_6 = 0, \quad \dot z_4 + z_5 = 0, \\ -2\dot z_5 - z_1 + z_6 = 0, \quad 2\dot z_6 + z_2 + z_5 = 0 \end{gathered} \tag{8.25}$$

for sections $\delta : I \to Z$. We shall show that there exist integers m, s, and a regular variational form E defined on $j^s Y$ where $\dim Y = m + 1$ such that (8.25) are Hamilton equations of E and $(t, z_1, \ldots, z_6)$ are Legendre coordinates associated with E. Obviously it is sufficient to apply Theorem 11 to the case $Z \times I = j^{s-1} Y$, $\dim Y = m + 1$, where $s = 1$, $m = 6$, resp. $s = 2$, $m = 3$, resp. $s = 3$, $m = 2$, resp. $s = 6$, $m = 1$. A (unique) solution is obtained for $s = 3$, $m = 2$, and $z_1 = P_x^0$, $z_2 = P_y^0$, $z_3 = Q_x$, $z_4 = Q_y$, $z_5 = P_x^1$, $z_6 = P_y^1$ (i.e. $M_{xy} = -M_{yx} = 2$, $M_{xx} = M_{yy} = 0$); it is of the form $E = E_x\, dx \wedge dt + E_y\, dy \wedge dt$, where $E_x = -\ddot x - 2\dddot y$, $E_y = -\ddot y + 2\dddot x$ in a fiber chart (t, x, y) on Y. The inverse to the Legendre transformation is defined by $x = Q_x$, $y = Q_y$, $\dot x = -G_x^0 = P_y^1$, $\dot y = -G_y^0 = -P_x^1$, $\ddot x = F_y^1 = -(1/2)(P_y^0 + P_x^1)$, $\ddot y = -F_x^1 = -(1/2)(P_y^1 - P_x^0)$.

9. THE EULER – LAGRANGE DISTRIBUTION

In [10] the Euler–Lagrange distribution of a lagrangian was introduced. In this section, using analogous constructions, we shall associate a distribution (= differential system) to the equations for Hamilton extremals. Clearly in the case of regular problems this distribution will provide a geometrical description of extremals of a given locally variational form. In the second part of this section we associate a distribution to an arbitrary system of regular higher order equations linear in the highest derivatives and we show that variationality of this distribution means the existence of the "variational integrating factors" for the system of equations.

Let $\pi : Y \to X$ be a fiber manifold, $\dim X = 1$. Let $E \in \Omega^{1,1}_Y(j^s Y)$ be a locally variational form (not $\pi_{s,k}$-projectable for any $k < s$), $\alpha_E \in \Omega^2(j^{s-1}Y)$ its Lepagean equivalent, denote by $E = E_\sigma \omega^\sigma \wedge dt$ the chart expression of E in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y. Let ξ be a π_{s-1}-vertical vector field on $j^{s-1}Y$, $\xi = \xi^\nu_k \partial/\partial q^\nu_k$ (summation over $0 \leqq k \leqq s-1$) its chart expression in the chart (V, ψ). Using (3.21) we obtain

$$i_\xi \alpha_E = \sum_{i=0}^{s-1} \xi^\sigma_i \eta^i_\sigma \tag{9.1}$$

where

$$\pi^*_{s,s-1}\eta_\sigma = E_\sigma\, dt + \sum_{k=0}^{s-1} 2F^{0k}_{\sigma\nu}\omega^\nu_k, \qquad \eta^i_\sigma = \sum_{k=0}^{s-1-i} 2F^{ik}_{\sigma\nu}\omega^\nu_k, \tag{9.2}$$

$$1 \leqq i \leqq s-1, \quad 1 \leqq \sigma \leqq m.$$

It is easily seen that the forms η^i_σ, $0 \leqq i \leqq s-1$, $1 \leqq \sigma \leqq m$ on V_{s-1} define a distribution on $Tj^{s-1}Y$, which in general has not a constant dimension. This distribution will be denoted by Δ_E and called the *Euler-Lagrange distribution* associated with the locally variational form E.

Theorem 12. *Let $U \subset j^{s-1}Y$ be an open set.* $\dim \Delta_E = 1$ *on U iff E is regular at each point $j^{s-1}_x\gamma \in U$.*

Proof. Let $\dim \Delta_E = 1$ on U. Then η^i_σ, $0 \leqq i \leqq s-1$, $1 \leqq \sigma \leqq m$ are linearly independent at each point of U, i.e. the rank of the $ms \times m(s+1)$ – matrix A whose columns are coefficients at dt, $dq^1, \ldots, dq^m, \ldots, dq^m_{s-1}$ in (9.2) is maximal. Since A is equivalent with

$$\begin{pmatrix} F^{00}_{\sigma\nu} & \cdots & F^{0,s-1}_{\sigma\nu} & E_\sigma \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ F^{s-1,0}_{\sigma\nu} & & 0 & \end{pmatrix} \tag{9.3}$$

where σ (resp. ν) labels rows (resp. columns) and (5.3) holds, we arrive at $\det(F^{0,s-1}_{\sigma\nu}) \neq 0$, i.e. E is regular.

The converse is obvious.

Remarks. (1) Let $\Delta_E \subset Tj^{s-1}Y$ be an Euler–Lagrange distribution of dimension 1, $E = E_\sigma\omega^\sigma \wedge dt$ the chart expression of E in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y. Then locally

(9.4) $$\Delta_E = \text{span}\,\{\omega_i^\sigma, 0 \leqq i \leqq s-2, A_\sigma\,\mathrm{d}t + B_{\sigma\nu}\,\mathrm{d}q^\nu_{s-1}, 1 \leqq \sigma \leqq m\} = = \text{span}\left\{\frac{\partial}{\partial t} + \sum_{i=0}^{s-2} q^\sigma_{i+1}\frac{\partial}{\partial q^\sigma_i} - B^{\sigma\varrho}A_\varrho\frac{\partial}{\partial q^\sigma_{s-1}}\right\}.$$

where $A_\sigma, B_{\sigma\nu}$ are defined by (3.23). If $(\bar{V}_{s-1}, \bar{\varphi}_{s-1})$, $\bar{V}_{s-1} \subset j^{s-1}Y$, $\varphi_{s-1} = (T, Q_i^\sigma, P_\nu^k)$, $0 \leqq i \leqq c-1$, $s-c-1 \geqq k \geqq 0$ are Legendre coordinates of E one obtains

(9.5) $$\Delta_E = \text{span}\left\{-\frac{\partial H}{\partial Q_k^\nu}\mathrm{d}T - \mathrm{d}P_\nu^k - \frac{\partial q_c^\sigma}{\partial Q_k^\nu}\mathrm{d}P_\sigma^c, -\frac{\partial H}{\partial P_\nu^k}\mathrm{d}T + \mathrm{d}Q_k^\nu,\right.$$

$$\left. 0 \leqq k \leqq c-1, -\frac{\partial H}{\partial P_\nu^i}\mathrm{d}T + \mathrm{d}q_i^\nu - \frac{\partial q_c^\sigma}{\partial P_\nu^i}\mathrm{d}P_\sigma^c, c \leqq i \leqq s-c-1, 1 \leqq \nu \leqq m\right\} =$$

$$= \text{span}\left\{\frac{\partial}{\partial t} + \sum_{k=0}^{c-1}\frac{\partial H}{\partial P_\nu^k}\frac{\partial}{\partial Q_k^\nu} - \sum_{k=0}^{c-1}\left(\frac{\partial H}{\partial Q_k^\nu} + \frac{\partial q_c^\sigma}{\partial Q_k^\nu}\frac{\mathrm{d}P_\sigma^c}{\mathrm{d}t}\right)\frac{\partial}{\partial P_\nu^k} + \right.$$
$$\left. + \sum_{k=c}^{s-c-1}\frac{\mathrm{d}P_\nu^k}{\mathrm{d}t}\frac{\partial}{\partial P_\nu^k}\right\}$$

where q_c^σ and $\mathrm{d}P_\sigma^c/\mathrm{d}t$ are defined by (8.7) and (8.10). Each (local) vector field generating Δ_E is called *Hamiltonian vector field* and denoted by ζ. Notice that (a) $i_\zeta\alpha_E = 0$, (b) there exists a global generator ζ of Δ_E iff there exists a nowhere zero function f on $j^{s-1}Y$ such that in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y

(9.6) $$\zeta = f(t)\left(\frac{\partial}{\partial t} + \sum_{i=0}^{s-2} q^\sigma_{i+1}\frac{\partial}{\partial q^\sigma_i} - B^{\sigma\varrho}A_\varrho\frac{\partial}{\partial q^\sigma_{s-1}}\right);$$

evidently, the integral curves of ζ generally do not coincide with Hamilton extremals of E.

(2) If $\Delta_E \subset Tj^{s-1}Y$ is the Euler–Lagrange distribution of a regular locally variational form E then Δ_E is integrable, and the collection of all maximal connected integral manifolds of Δ_E is a 1-dimensional regular foliation of $j^{s-1}Y$. Let Z be an integral manifold of Δ_E. Then there does not exist any point of $j^{s-1}Y$ such that Z would be tangent to a fiber $\pi_{s-1}^{-1}(x)$, $x \in X$, at this point. (If $z \in \pi_{s-1}^{-1}(x)$ would be such a point it should hold $\Delta_E(z) = T_z\pi_{s-1}^{-1}(x)$, i.e. a vector $\xi(z) \in \Delta_E(z)$ would be vertical which contradicts (9.4)).

Lemma 5. *Let $\delta : I \to j^{s-1}Y$ be a section defined on an open set $I \subset X$. δ is a Hamilton extremal of a locally variational form $E \in \Omega_Y^{1,1}(j^sY)$ iff δ is an integral mapping of Δ_E.*

Proof. One has to show that $\delta^*\eta_\sigma^i = 0$, $0 \leqq i \leqq s-1$, $1 \leqq \sigma \leqq m$ iff $\delta^*i_\xi\alpha_E = 0$ for every π_{s-1}-vertical vector field ξ on $j^{s-1}Y$, where η_σ^i are generators of Δ_E defined by (9.2). This is easily done with help of (9.1) and (4.13).

Thence the image of each Hamilton extremal δ of E is an integral manifold of Δ_E.

We shall show that the Euler–Lagrange distribution of a regular form E has no other integral manifolds.

Theorem 13. *Let $\Delta_E \subset Tj^{s-1}Y$ be the Euler–Lagrange distribution of a regular locally variational form $E \in \Omega_Y^{1,1}(j^sY)$, let $Z \subset j^{s-1}Y$ be a maximal connected integral manifold of Δ_E. Then to each point $z \in Z$ there exists a neighbourhood $W \subset Z$, $W \ni z$ and a section $\delta : I \to j^{s-1}Y$ defined on an open set $I \subset X$ such that δ is a regular Hamilton extremal of E and $\delta(I) = W$.*

Proof. Let $Z \subset j^{s-1}Y$ be a maximal connected integral manifold of the Euler–Lagrange distribution Δ_E associated with a regulăr locally variational form E, let $z \in Z$ be a point. Denote by $f : U \to Z$ where $U \subset X$ is an open set an integral mapping of Δ_E such that $z \in f(U)$. By the Inverse Mapping Theorem and Remark (2) there exist a neighbourhood W of z, $W \subset Z$ and a mapping $f^{-1} : W \to U_0$ where $U_0 \subset U$ such that the set $I = \pi_{s-1}(W)$ is open in X and $f|_{U_0} \circ f^{-1} = \mathrm{id}_W$. Define a mapping $g : I \to U_0$ by $g(x) = f^{-1}(\pi_{s-1}^{-1}(x) \cap W)$ and put $\delta = f|_{U_0} \circ g$. Obviously $\delta : I \to j^{s-1}Y$ satisfies $(\pi_{s-1} \circ \delta)(x) = x$, i.e. δ is a section of π_{s-1}, and $\delta(I) = W$. Since $W \subset Z$ is an integral manifold of Δ_E Lemma 5 implies that δ is a Hamilton extremal of E.

Corollary. (a) *Let $E \in \Omega_Y^{1,1}(j^sY)$ be a regular locally variational form. Then to each point $y \in Y$ there exists an extremal $\gamma : I \to Y$ of E defined on an open set $I \subset X$ such that $\gamma(x) = y$ for some $x \in I$.*

(b) *Let $\delta_1 : I_1 \to j^{s-1}Y$, $\delta_2 : I_2 \to j^{s-1}Y$ where $I_1 \cap I_2 \neq \emptyset$ be two Hamilton extremals of a regular locally variational form E, suppose $\delta_1(I_1) \cap \delta_2(I_2) = \emptyset$. Denote by $\gamma_1 : I_1 \to Y$, $\gamma_2 : I_2 \to Y$ the corresponding extremals (i.e. $\gamma_i = \pi_{s-1,0} \circ \delta_i$, $i = 1, 2$). Then either $\gamma_1(I_1) \cap \gamma_2(I_2) = \emptyset$ or $\gamma_1(x_0) = \gamma_2(x_0)$ at a point $x_0 \in I_1 \cap I_2$ and there exists a neighbourhood I of x_0 such that $\gamma_1(x) \neq \gamma_2(x)$ for each $x \in I$, $x \neq x_0$.*

Proof. (a) follows directly from Theorem 13, Remark (2) and the definition of a regular Hamilton extremal.

(b) Let $U \subset I_1 \cap I_2$ be a set such that $\gamma_1(x) = \gamma_2(x)$ for each $x \in U$. Then for each open subset $U_0 \subset U$, $U_0 \neq \emptyset$ the section $\gamma : U_0 \to Y$ defined by $\gamma = \gamma_1|_{U_0} = \gamma_2|_{U_0}$ satisfies $\delta_1(U_0) = j^{s-1}\gamma(U_0) = \delta_2(U_0)$ which contradicts our assumption. Hence $U_0 = \emptyset$ which means that either $U = \emptyset$ or U is a union of one-point sets, i.e. for each $x_0 \in U$ there exists an open set $I \subset X$, $I \ni x_0$ such that $\gamma_1(x) \neq \gamma_2(x)$ for all $x \in I$, $x \neq x_0$.

Let $E \in \Omega_Y^{1,1}(j^sY)$ be a form such that in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y $E = E_\sigma \omega^\sigma \wedge dt$,

$$E_\sigma = A_\sigma + B_{\sigma\nu} q_s^\nu \tag{9.7}$$

where the functions A_σ, $B_{\sigma\nu}$, $1 \leqq \sigma, \nu \leqq m$ do not depend on q^ϱ_s. Obviously this remains true for each fiber chart (V, ψ) on Y. Denote by $\Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ the module of such forms; it holds $\Omega^{1,1\,\mathrm{lin}}_Y(j^sY) \subset \Omega^{1,1}_Y(j^sY)$. We say that a form $E \in \Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ is *regular* if

$$\det\left(\frac{\partial E_\sigma}{\partial q^\nu_s}\right) = \det(B_{\sigma\nu}) \neq 0. \tag{9.8}$$

Evidently, if $E \in \Omega^{1,1}_Y(j^sY)$ is locally variational then $E \in \Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ and the concepts of regularity (5.1) and (9.8) coincide.

Let $w : j^{s-1}Y \to j^sY$ be a (global) section. There arises a distribution Δ_w such that in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y

$$\Delta_w = \mathrm{span}\,\{\omega^\sigma_i, 0 \leqq i \leqq s-1, -w^\sigma\,\mathrm{d}t + \mathrm{d}q^\sigma_{s-1}, 1 \leqq \sigma \leqq m\} \tag{9.9}$$

where $w^\sigma = q^\sigma_s \circ w$. Δ_w is 1-dimensional thus integrable, and to every integral section $\delta \in \Gamma(\pi_{s-1})$ of Δ_w there exists a section $\gamma \in \Gamma(\pi)$ such that $\delta = j^{s-1}\gamma$ and it holds $w \circ \delta = j^s\gamma$.

Let $E \in \Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ be a regular form, $w : j^{s-1}Y \to j^sY$ a section, Δ_w its distribution. We say that E and Δ_w are *related* if

$$w^*E = 0. \tag{9.10}$$

Locally $w^\sigma = -B^{\sigma\nu}A_\nu$, or equivalently, $E_\sigma = B_{\sigma\nu}(q^\nu_s - w^\nu)$. It is easily seen that a section $\gamma \in \Gamma(\pi)$ is a solution of the equation $E \circ j^s\gamma = 0$ iff $j^{s-1}\gamma$ is an integral section of the related to E distribution Δ_w. Two regular forms E, $\bar{E} \in \Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ are called *equivalent* if their related distributions Δ_w, $\Delta_{\bar{w}}$ coincide. Obviously, E and $\bar{E}$ are equivalent iff for each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y where $E_\sigma = A_\sigma + B_{\sigma\nu}q^\nu_s$, $\bar{E}_\sigma = \bar{A}_\sigma + \bar{B}_{\sigma\nu}q^\nu_s$ there exist functions G^ν_σ, $1 \leqq \sigma, \nu \leqq m$ defined on V_{s-1} such that $\det(G^\nu_\sigma) \neq 0$ at each point of V_{s-1} and it holds $\bar{E}_\sigma = G^\nu_\sigma E_\nu$. Obviously $G^\nu_\sigma = \bar{B}_{\sigma\varrho}B^{\varrho\nu}$ where $(B^{\varrho\nu})$ denotes the inverse matrix to $(B_{\varrho\nu})$. In this way we obtain a splitting of the submodule of all regular forms of $\Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ into equivalence classes of forms (such that for equivalent forms the sets of solutions of the corresponding equations coincide). A section $w : j^{s-1}Y \to j^sY$ (resp. its distribution $\Delta_w \subset Tj^{s-1}Y$) is called *variational* if there exists a regular locally variational form $E \in \Omega^{1,1\,\mathrm{lin}}_Y(j^sY)$ related with Δ_w; in this case the regular $m \times m$ – matrix $(B_{\sigma\nu})$, $B_{\sigma\nu} = \partial E_\sigma/\partial q^\nu_s$, is called *variational integrating factor* for Δ_w. Notice that a variational distribution related with a regular locally variational form E is precisely the (regular) Euler–Lagrange distribution Δ_E of E.

Remark. As a simple consequence of the definitions and Theorem 9 we obtain the following characterization of variational distributions $\Delta_w \subset Tj^{s-1}Y$:

(1) For each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y there exist 1-forms η^i_σ, $1 \leqq \sigma \leqq m$, $0 \leqq i \leqq s-1$ defined on V_{s-1} such that $\Delta_w = \mathrm{span}\,\{\eta^i_\sigma\}$, and the form α defined

by $i_\xi \alpha = \xi^\sigma_i \eta^i_\sigma$ (summation over $0 \leqq i \leqq s-1$) for each π_{s-1}-vertical vector field ξ on $j^{s-1}Y$ is Lepagean.

(2) For each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y there exist 1-forms η^i_σ, $1 \leqq \sigma \leqq m$, $0 \leqq i \leqq s-1$ defined on V_{s-1} such that $\Delta_w = \operatorname{span}\{\eta^i_\sigma\}$, and the form η defined by the relation $\eta = \omega^\sigma_i \wedge \bar{h}(\eta^i_\sigma)$ is a Hamilton form.

(3) Each (local) vector field ζ generating Δ_w is Hamiltonian, i.e. there exists a Lepagean form $\alpha \in \Omega^2(j^{s-1}Y)$ such that $i_\zeta \alpha = 0$.

(4) For each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y the system of equations $(q^\nu_s - w^\nu) \circ \circ j^s\gamma = 0$ for sections $\gamma \in \Gamma(\pi)$ has a variational integrating factor.

The existence of a variational integrating factor for Δ_w is equivalent with the existence of a regular matrix $(B_{\sigma\nu})$ which is a solution of a system of partial differential equations which arise from the ADK-conditions (3.2). The set of all equivalent locally variational forms related with Δ_w is then obtained by finding all the solutions. As concerns the problem of finding certain „variationality conditions" for Δ_w (for $s = 2$) we refer to e.g. [14], [20], [19], and references cited therein.

Two following examples provide a classification of variational distributions in certain simple cases.

Example 8. Let $\dim Y = 2$ (i.e. $m = 1$), consider a section $w : j^{2c}Y \to j^{2c+1}Y$. Using the ADK-conditions (3.2) we obtain that no distribution $\Delta_w \subset Tj^{2c}Y$ is variational.

Example 9. Suppose $Y = I \times B$ where $I \subset R$ (resp. $B \subset R^m$) is an open interval (resp. an open ball with the center at the origin), denote by (t, q^σ) (resp. $(t, \dot{q}^\sigma, \ddot{q}^\sigma)$) fiber coordinates on Y (resp. the associated coordinates on j^2Y). Let $B_{\sigma\nu} = B_{\sigma\nu}(t, q^\varrho)$, $1 \leqq \sigma, \nu \leqq m$ be functions on Y such that the matrix $B = (B_{\sigma\nu})$ is symmetric and regular at each point $\gamma(x) \in Y$. Put $g = (g_{ij})$, $0 \leqq i, j \leqq m$, where $g_{00} = 1$, $g_{\sigma\nu} = B_{\sigma\nu}$, $g_{\sigma 0} = g_{0\sigma} = 0$. g is a metric on Y, the Christoffel symbols of g are of the form

$$(9.11) \qquad \Gamma_{\sigma\nu\varrho} = \frac{1}{2}\left(\frac{\partial B_{\sigma\nu}}{\partial q^\varrho} + \frac{\partial B_{\sigma\varrho}}{\partial q^\nu} - \frac{\partial B_{\nu\varrho}}{\partial q^\sigma}\right), \qquad \Gamma_{\sigma\nu 0} = \frac{1}{2}\frac{\partial B_{\sigma\nu}}{\partial t},$$
$$\Gamma_{\sigma 00} = 0, \qquad 1 \leqq \sigma, \nu, \varrho \leqq m.$$

Let $w : j^1Y \to j^2Y$ be a section, put $w^\sigma = \ddot{q}^\sigma \circ w$ and $w_\sigma = g_{\sigma\nu}w^\nu = B_{\sigma\nu}w^\nu$. The following proposition characterizes all variational distributions Δ_w of w which arise from the metric g.

Proposition. *The following two conditions are equivalent:*

(1) *The distribution Δ_w is variational, related with a variational form $E = E_\sigma \, dq^\sigma \wedge \wedge \, dt$, $E_\sigma = B_{\sigma\nu}(\ddot{q}^\nu - w^\nu)$ on j^2Y.*

(2) *It holds*

$$(9.12) \qquad -w_\sigma = \sum_{i,j=0}^{m} \Gamma_{\sigma ij}\dot{q}^i\dot{q}^j + A_{[\sigma\nu]}\dot{q}^\nu + C_\sigma, \qquad 1 \leqq \sigma \leqq m,$$

where $\dot q^0 = 1$, $\Gamma_{\sigma ij}$ *are defined by* (9.11) *and* $A_{[\sigma\nu]}$, C_σ *are functions on* Y *satisfying*

$$A_{[\sigma\nu]} = -A_{[\nu\sigma]}, \qquad \frac{\partial A_{[\sigma\nu]}}{\partial q^\varrho} + \frac{\partial A_{[\varrho\sigma]}}{\partial q^\nu} + \frac{\partial A_{[\nu\varrho]}}{\partial q^\sigma} = 0, \tag{9.13}$$
$$\frac{\partial A_{[\sigma\nu]}}{\partial t} = \frac{\partial C_\sigma}{\partial q^\nu} - \frac{\partial C_\nu}{\partial q^\sigma}.$$

Proof. Suppose (1). Since E is variational it satisfies the ADK-conditions (3.2) which reduce to the conditions

$$\frac{\partial B_{\sigma\nu}}{\partial t} + \frac{\partial B_{\sigma\nu}}{\partial q^\varrho}\dot q^\varrho + \frac{1}{2}\left(B_{\sigma\varrho}\frac{\partial w^\varrho}{\partial \dot q^\nu} + B_{\nu\varrho}\frac{\partial w^\varrho}{\partial \dot q^\sigma}\right) = 0, \tag{9.14}$$

$$\frac{\partial}{\partial q^\nu}(B_{\sigma\varrho}w^\varrho) - \frac{\partial}{\partial q^\sigma}(B_{\nu\varrho}w^\varrho) = \tag{9.15}$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial t} + \dot q^\varkappa\frac{\partial}{\partial q^\varkappa}\right)\left(\frac{\partial}{\partial \dot q^\nu}(B_{\sigma\varrho}w^\varrho) - \frac{\partial}{\partial \dot q^\sigma}(B_{\nu\varrho}w^\varrho)\right).$$

Put

$$\Gamma_{\sigma\nu\varrho} = -\frac{1}{2}B_{\sigma\varkappa}\frac{\partial^2 w^\varkappa}{\partial \dot q^\nu \partial \dot q^\varrho}. \tag{9.16}$$

Differentiating (9.14) with respect to $\dot q^\varrho$ we arrive at $\partial B_{\sigma\nu}/\partial q^\varrho - \Gamma_{\sigma\nu\varrho} - \Gamma_{\nu\sigma\varrho} = 0$, hence

$$\Gamma_{\sigma\nu\varrho} = \frac{1}{2}\left(\frac{\partial B_{\sigma\varrho}}{\partial q^\nu} + \frac{\partial B_{\nu\varrho}}{\partial q^\sigma} - \frac{\partial B_{\sigma\nu}}{\partial q^\varrho}\right). \tag{9.17}$$

Since $\partial\Gamma_{\sigma\nu\varrho}/\partial \dot q^\varkappa = 0$, we obtain $B_{\sigma\varkappa}w^\varkappa = -\Gamma_{\sigma\nu\varrho}\dot q^\nu \dot q^\varrho - A_{\sigma\nu}\dot q^\nu - C_\sigma$ where $A_{\sigma\nu}$, C_σ are functions on Y. Substituting into (9.14) we arrive at

$$B_{\sigma\varkappa}w^\varkappa = -\Gamma_{\sigma\nu\varrho}\dot q^\nu \dot q^\varrho - \frac{\partial B_{\sigma\nu}}{\partial t}\dot q^\nu - A_{[\sigma\nu]}\dot q^\nu - C_\sigma, \tag{9.18}$$

where $[\sigma\nu]$ means antisymmetrization in the indices σ, ν. Finally, (9.15) implies that the relations (9.13) hold. Hence (2) is satisfied.

Conversely, suppose (2). Then $E_\sigma = B_{\sigma\nu}(\ddot q^\nu - w^\nu)$ satisfies (3.2), i.e. Δ_w is variational.

Notice that the relations (9.13) are necessary and sufficient for the existence of functions $\alpha_\sigma(t, q^\nu)$, $\varphi(t, q^\nu)$ on Y such that

$$A_{[\sigma\nu]} = \frac{\partial \alpha_\sigma}{\partial q^\nu} - \frac{\partial \alpha_\nu}{\partial q^\sigma}, \qquad C_\sigma = \frac{\partial \varphi}{\partial q^\sigma} + \frac{\partial \alpha_\sigma}{\partial t}; \tag{9.19}$$

hence if $B_{\sigma\nu} = \delta_{\sigma\nu}$ and $\dim Y - 1 = m = 3$ (9.12) becomes the "Lorentz force" (this result has been obtained in [13] and [17]). Similarly, if $B_{\sigma\nu}$ are functions of q^{ϱ} only and $\alpha_{\sigma} = \varphi = 0$, we obtain the result of [15] (Theorem 1).

For $m = \dim Y - 1 = 1$ (9.12) is of the form

$$(9.20) \qquad -\ddot{q} \circ w = \frac{1}{2} \frac{\partial \ln B}{\partial q} \dot{q}^2 + \frac{\partial \ln B}{\partial t} \dot{q} + C(t, q).$$

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O. Krupková
Department of Mathematics
Faculty of Science, J. E. Purkyně University
Janáčkovo nám. 2a
662 95 Brno
Czechoslovakia