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Archivum Mathematicum, Vol. 23 (1987), No. 2, 95--107

Persistent URL: `http://dml.cz/dmlcz/107285`

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 23, No. 2 (1987), 95–108

GENERALIZED STURM—LIOUVILLE EQUATIONS

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(Received November 22, 1985)

Abstract. The Sturm–Liouville equation is treated within the frame of generalized ordinary differential equations. This concept allows discontinuities in the solutions as well as in their derivatives. An integral identity is derived for obtaining comparison results of Sturm type for generalized Sturm–Liouville systems.

Key words. Generalized linear differential equations, Sturm–Liouville system, strong impulses, zeros of solutions, comparison theorem for generalized Sturm–Liouville systems.

MS Classification. 34 A 30, 34 C 10.

INTRODUCTION

Recently there have been some efforts at generalizing the concept of the classical Sturm–Liouville equation

(1) $$-(m(t)\,v')' + p(t)\,v = 0$$

with a positive function m in such a way that the requirement of continuity of the derivative v' of a solution or even of continuity of the solution itself have been withdrawn. For results in this direction see e.g. [1], [2], [4], [5], [6].

For example K. Kreith in [4] studies the system

(2) $$v' = \left[\frac{1}{m(t)} + \sum_{i=1}^{n} Q_i\delta(t - t_i)\right] z,$$

$$z' = [p(t) + \sum_{i=1}^{n} R_i\delta(t - t_i)]\, v,$$

$t \in (a, b)$, $a < t_1 < \ldots < t_n < b$ where the terms $R_i\delta(t - t_i)$ are called "weak" linear impulses leading to discontinuities in z (i.e. in the derivative) and the terms $Q_i\delta(t - t_i)$ are called "strong" linear impulses which produce discontinuities in the solution v. Of course, a precise meaning of the concept of the solution of such a system has to be given. K. Kreith in [4] solved this problem via the generalized Prüfer transformation and the corresponding transformed equations.

Our aim is to present generalizations in this direction for the original equation (1) using the theory of generalized linear differential equations which has been developed in [7].

Let us write (1) in the usual form of a system

$$v' = \frac{1}{m(t)} z,$$

$$z' = p(t)\, v$$

or in the integral form

$$v(t) = v(c) + \int_c^t \frac{1}{m(\tau)} z(\tau)\,\mathrm{d}\tau. \tag{3}$$

$$z(t) = z(c) + \int_c^t p(\tau)\, v(\tau)\,\mathrm{d}\tau, \qquad c, t \in (a, b).$$

The coefficients m and p are assumed to be elements of $L^1_{loc}(a, b)$ where $(a, b) \subset R$ is an interval and the function m is positive. Denoting

$$R(t) = \int_c^t \frac{1}{m(\tau)}\,\mathrm{d}\tau, \qquad P(t) = \int_c^t p(\tau)\,\mathrm{d}\tau, \qquad t \in (a, b)$$

we can rewrite the system (3) to the Stieltjes form

$$v(t) = v(c) + \int_c^t z(\tau)\,\mathrm{d}R(\tau), \tag{4}$$

$$z(t) = z(c) + \int_c^t v(\tau)\,\mathrm{d}P(\tau), \qquad c, t \in (a, b).$$

The system (4) is equivalent to the original equation (1) in the following sense: if $v : [\alpha, \beta] \to R$ is a solution of (1) on an interval $[\alpha, \beta] \geqq (a, b)$ and if we take $z(t) = v'(t)\, m(t)$ $\tau \in [\alpha, \beta]$ ($m(t)\, v'(t)$ is absolutely continuous on $[\alpha, \beta]$) then the couple (v, z) satisfies (4) for every $c, t \in [\alpha, \beta]$, and vice versa, if the couple (v, z) satisfies (4) for every $c, t \in [\alpha, \beta]$ then v and z are absolutely continuous on $[\alpha, \beta]$ and v satisfies (1) almost everywhere on $[\alpha, \beta]$.

It should be mentioned that in our case the functions R and P belong to the class $\mathrm{AC}_{loc}(a, b)$ and R is strictly increasing.

Now we generalize the system (4) by imposing on the "coefficients" R and P the following assumptions:

$$R, P \in \mathrm{BV}_{loc}(a, b) \tag{5}$$

$$R \text{ is strictly increasing.} \tag{6}$$

Let us denote $A(t) = \begin{pmatrix} 0, & R(t) \\ P(t), & 0 \end{pmatrix}$; $A(t)$ is a 2×2-matrix, $\operatorname{var}_\alpha^\beta A < \infty$ for $[\alpha, \beta] \subset (a, b)$ and following the notation used in [7] the system (4) can be written simply in the vector form

$$\mathrm{d}x = \mathrm{d}[A]\, x, \tag{7}$$

where $x = (v, z)^T \in R^2$.

A function $x : [\alpha, \beta] \to R^2$ is a solution of (7) on $[\alpha, \beta]$ if for every $c, t \in [\alpha, \beta]$ the equality

$$x(t) = x(c) + \int_c^t \mathrm{d}[A(\tau)]\, x(\tau)$$

holds.

Let us shortly mention some fundamental results concerning the equation (7); these results can be found in [7].

(i) *Every solution* $x : [\alpha, \beta] \to R^2$ *of* (7) *is a function of bounded variation on* $[\alpha, \beta]$.

(ii) *The initial value problem*

$$\mathrm{d}x = \mathrm{d}[A]\, x, \qquad x(c) = x_0 \in R^2, \qquad c \in [\alpha, \beta] \subset (a, b) \tag{8}$$

has a unique solution on $[\alpha, \beta]$ *for every* $x_0 \in R^2$, $c \in [\alpha, \beta]$ *if and only if the matrices* $I - \Delta^- A(t)$, $I + \Delta^+ A(t)$ *are regular for every* $t \in (\alpha, \beta]$, $t \in [\alpha, \beta)$, *respectively*. (The notation $\Delta^- A(t) = A(t) - A(t-) = A(t) - \lim_{\tau \to t-} A(\tau)$, $\Delta^+ A(t) = A(t+) - A(t) = \lim_{\tau \to t+} A(\tau) - A(t)$ is used here. I is the 2×2 identity matrix.)

(iii) *If* $x : [\alpha, \beta] \to R^2$ *is a solution of* (7) *then all the onesided limits* $x(\alpha +)$, $x(t +)$, $x(\beta -)$, $x(t -)$, $t \in (\alpha, \beta)$ *exist and*

$$x(t+) = [I + \Delta^+ A(t)]\, x(t), \qquad t \in [\alpha, \beta],$$
$$x(t-) = [I - \Delta^- A(t)]\, x(t), \qquad t \in [\alpha, \beta].$$

For technical reasons let us now state an additional condition on (5) and (6). We will assume in the sequel that

$$R(t-) = R(t), \qquad P(t+) = P(t), \qquad t \in (a, b). \tag{9}$$

Using (9) we have

$$I - \Delta^- A(t) = \begin{pmatrix} 1, & -\Delta^- R(t) \\ -\Delta^- P(t), & 1 \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ -\Delta^- P(t), & 1 \end{pmatrix}$$

and

$$I + \Delta^+ A(t) = \begin{pmatrix} 1, & \Delta^+ R(t) \\ \Delta^+ P(t), & 1 \end{pmatrix} = \begin{pmatrix} 1, & \Delta^+ R(t) \\ 0, & 1 \end{pmatrix}.$$

Hence both the matrices $I - \Delta^- A(t)$, $I + \Delta^+ A(t)$ are regular for every $t \in (a, b)$ and by (ii), on every $[\alpha, \beta] \subset (a, b)$ there exists a uniquely determined solution x of the initial value problem (8) for every $x_0 \in R^2$, $c \in [\alpha, \beta]$.

From now on, instead of (7) we will use the more specific notation

$$\begin{aligned} \mathrm{d}v &= z\, \mathrm{d}R, \\ \mathrm{d}z &= v\, \mathrm{d}P. \end{aligned} \tag{10}$$

Let us recall that the couple (v, z) of functions v, z: $[\alpha, \beta] \to R$ is a solution of (10) on $[\alpha, \beta] \subset (a, b)$ if for every $c, t \in [\alpha, \beta]$ the equalities

$$v(t) = v(c) + \int_c^t z(\tau)\,\mathrm{d}R(\tau),$$

$$z(t) = z(c) + \int_c^t v(\tau)\,\mathrm{d}P(\tau)$$

are satisfied.

Moreover, if $x = (v, z)^{\mathrm{T}}$ is a solution of (7) on $[\alpha, \beta]$ then by (iii) we get for the components v, z

$$\begin{aligned} v(t+) &= v(t) + \Delta^+ R(t)\, z(t), \\ z(t+) &= z(t), \qquad\qquad t \in [\alpha, \beta) \end{aligned}$$

and

$$\begin{aligned} v(t-) &= v(t), \\ z(t-) &= z(t) - \Delta^- P(t)\, v(t), \qquad t \in (\alpha, \beta]. \end{aligned}$$

Hence every solution $(v, z)^{\mathrm{T}}$ of (7) exhibits discontinuities at the points of discontinuity of the functions R and P where the component v is continuous from the left and $\Delta^+ v(t) = v(t+) - v(t) = \Delta^+ R(t)\, z(t)$, the component z is continuous from the right and $\Delta^- z(t) = z(t) - z(t-) = \Delta^- P(t)\, v(t)$.

Taking this construction into consideration we can have a short look back at the situation described by K. Kreith in [4]: the equations in question have the form (2). For a given $d \in (a, b)$ let us define $H_d^-(t) = 0$ for $t \leqq d$, $H_d^-(t) = 1$ for $t > d$ and $H_d^+(t) = 0$ for $t < d$, $H_d^+(t) = 1$ for $t \geqq d$.

For $t \in (a, b)$ denote

$$R(t) = \int_c^t \frac{1}{m(\tau)}\,\mathrm{d}\tau + \sum_{i=1}^n Q_i H_{t_i}^-(t),$$

$$P(t) = \int_c^t p(\tau)\,\mathrm{d}\tau + \sum_{i=1}^n R_i H_{t_i}^+(t),$$

where $c \in (a, b)$ is fixed.

The assumptions (5) and (9) are obviously satisfied in this situation and if $m \in L_{loc}^1(a, b)$ is positive and $Q_i \geqq 0$ then also (6) holds. Let us mention that the requirement $Q_i \geqq 0$, $i = 1, \dots, n$ is also stated in [4]. The above mentioned properties of a solution of (7) imply that in the case of this choice of the "coefficients" R and P, the solution exhibits discontinuities and (10) describes the case of both "weak" and "strong" linear impulses for the generalized Sturm-Liouville equation. Finally, let us mention that in the frame of the theory of generalized linear differential equations (see [7]) used here the order of jumps for the two components is in a sense pre-determined. This depends in our case on the assumption (9), which causes the "first" jump (i.e. a discontinuity from the left) for the z-component and the "second" jump (discontinuity from the right) for the v-component of a solution. If we assumed $R(t+) = R(t)$, $P(t-) = P(t)$ for $t \in (a, b)$ instead of (9) then this order would be inverse. If it is not assumed that the coefficients R and P have onesided discontinuities from different sides for R and P, the theory also works but technical difficulties can occur for algebraic reasons given by the requirement of the regularity of the matrices $I + \Delta^+ A(t)$, $I - \Delta^- A(t)$ (see (ii)).

SOME PROPERTIES OF SOLUTIONS OF THE SYSTEM (10)

Lemma 1. *Assume that* (v, z) *is a nontrivial solution of the system* (10) *on an interval* $[\alpha, b]$ *such that* $v(\alpha) = 0$, $v(t) > 0$ *for* $t \in (\alpha, \beta)$ *where* $\beta \in (\alpha, b)$ *and* $v(\beta +) \leqq 0$. *Then* $z(\alpha) > 0$ *and* $z(\beta) < 0$.

Proof. 1. Clearly $z(\alpha) \neq 0$. Otherwise the solution (v, z) would be trivial. Assume that $z(\alpha) < 0$. Then also $z(\alpha +) = z(\alpha) < 0$ and there exists a $\delta > 0$ such that $z(\tau) < 0$ for $\tau \in [\alpha, \alpha + \delta]$ and $v(t) = v(\alpha) + \int_\alpha^t z(\tau)\,\mathrm{d}R(\tau) = \int_\alpha^t z(\tau)\,\mathrm{d}R(\tau) < 0$ for $t \in [\alpha, \alpha + \delta]$ since R is strictly increasing. This contradicts the assumption $v(t) > 0$ for $t \in (\alpha, \beta)$ and consequently $z(\alpha) > 0$.

2. Since $v(t) > 0$ for $t \in (\alpha, \beta)$ and v is continuous from the left we have $v(\beta) = v(\beta -) \geqq 0$.

a) Assume that $z(\beta) = 0$. Then $v(\beta +) = v(\beta) + \Delta^+ R(\beta)\, z(\beta) = v(\beta) \leqq 0$ by he assumption and $v(\beta) = 0$. Hence the solution (v, z) is trivial, a contradiction.

b) Assume that $z(\beta) > 0$. We have

$$v(\beta +) = v(\beta) + \Delta^+ R(\beta)\, z(\beta).$$

If $\Delta^+ R(\beta) > 0$ then this relation yields the contradictory inequality $v(\beta +) > 0$. If $\Delta^+ R(\beta) = 0$ then by the same relation we obtain $v(\beta +) = v(\beta) = 0$ and $z(\beta -) = z(\beta) - \Delta^- P(\beta)\, v(\beta) = z(\beta) > 0$. Hence z is continuous from the left at the point β and consequently there exists $\delta > 0$ such that $z(\tau) > 0$ for $\tau \in [\beta - \delta, \mathrm{q}]$. By definition of a solution of (10) we have in this case

$$v(t) = v(\beta) + \int_\beta^t z(\tau)\,\mathrm{d}R(\tau) = -\int_t^\beta z(\tau)\,\mathrm{d}R(\tau) < 0$$

for $t \in [\beta - \delta, \beta]$ since R is strictly increasing. This again contradicts the assumption $v(t) > 0$ for $t \in (\alpha, \beta)$.

Since by a) and b) the assumption $z(\beta) \geqq 0$ leads to contradictions, we necessarily have $z(\beta) < 0$.

Remark 1. Essentially the same reasoning yields also that if (v, z) is a solution of (10) on $[\alpha, \beta]$ such that $v(\alpha) = 0$, $v(t) < 0$ for $t \in (\alpha, \beta)$, $\beta \in (\alpha, b)$ and $v(\beta +) \geqq 0$ then $z(\alpha) < 0$ and $z(\beta) > 0$.

Let us mention that in the situation described by Lemma 1 the point $\beta \in (\alpha, b)$ plays the role of the "right consecutive zero" to the point α for the first component v of the solution of (10). The point $\beta \in (\alpha, b)$ is the first point to the right of α at which the first component of the solution (v, z) of (10) changes its sign.

In fact, if (v, z) is a solution of (10) such that at some point β in the interval of definition of this solution we have $v(t) > 0$ for $t \in (\beta - \delta, \beta)$ where $\delta > 0$ and $v(\beta +) \leqq 0$ then $v(t) < 0$ for $t \in (\beta, \beta + \Delta)$ for some $\Delta > 0$ if $v(\beta +) < 0$. If $v(\beta +) = 0$ then Lemma 1 yields $z(\beta) < 0$ and since $z(\beta +) = z(\beta)$ there is a $\Delta > 0$ such that $z(t) < 0$ for $t \in [\beta, \beta < \Delta]$. Hence for $t \in [\beta, \beta + \Delta)$ we have

$$v(t) = v(\beta) + \int_\beta^t z(\tau)\,\mathrm{d}R(\tau) = v(\beta) + \int_\beta^{\beta+\delta} z(\tau)\,\mathrm{d}R(\tau) + \\ + \int_{\beta+\delta}^t z(\tau)\,\mathrm{d}R(\tau)$$

for every $0 < \delta < \Delta$ and evidently $\int_{\beta+\delta}^t z(\tau)\,\mathrm{d}R(\tau) < \int_{\beta+\delta_1}^t z(\tau)\,\mathrm{d}R(\tau) < 0$ for every δ s.t. $0 < \delta < \delta_1 < t$ where δ_1 is fixed. Passing to $\delta \to 0_+$ we obtain

$$v(t) = v(\beta) + \lim_{\delta\to 0+} \int_\beta^{\beta+\delta} [z(\tau)\,\mathrm{d}R(\tau) + \int_{\beta+\delta}^t z(\tau)\,\mathrm{d}R(\tau)] < \\ < v(\beta) + z(\beta)\,\Delta^+ R(\beta) + \int_{\beta+\delta_1}^t z(\tau)\,\mathrm{d}R(\tau) < v(\beta +) = 0,$$

i.e. on $(\beta, \beta + \Delta)$ the function $v(t)$ is negative and v indeed changes its sign at the point β.

Lemma 2. *Assume that $R, P, \tilde{R}, \tilde{P} : (a, b) \to R$ satisfy* (5), (6) *and* (9). *For a given interval $[\alpha, \beta] \subset (a, b)$, $\alpha < \beta$ let us assume that $R(t) = \tilde{R}(t)$, $P(t) = \tilde{P}(t)$ for $t \in (\alpha, \beta)$.*

Let the couple (v, z) be a solution of (10) *on $[\alpha, \beta]$. Define $\tilde{v}, \tilde{z} : [\alpha, \beta] \to R$ as follows:*

$$\tilde{v}(t) = v(t) \quad \text{for } t \in (\alpha, \beta], \qquad \tilde{v}(\alpha) = v(\alpha +) - \Delta^+ \tilde{R}(\alpha)\, z(\alpha),$$
$$\tilde{z}(t) = z(t) \quad \text{for } t \in [\alpha, \beta), \qquad \tilde{z}(\beta) = z(\beta -) + \Delta^- \tilde{P}(\beta)\, b(\beta).$$

Then the couple of functions $(\tilde{v}, \tilde{z})$ is a solution of the system

$$\mathrm{d}v = z\,\mathrm{d}\tilde{R},$$
$$\mathrm{d}z = v\,\mathrm{d}\tilde{P}$$

on $[\alpha, \beta]$.

Proof. It is clear that for every $c, t \in (\alpha, \beta)$ we have

$$\tilde{v}(t) = \tilde{v}(c) + \int_c^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau), \qquad \tilde{z}(t) = \tilde{z}(c) + \int_c^t \tilde{v}(\tau)\,\mathrm{d}\tilde{P}(\tau)$$

because

$$\int_c^t z(\tau)\,\mathrm{d}R(\tau) = \int_c^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau), \qquad \int_c^t v(\tau)\,\mathrm{d}\tilde{P}(\tau) = \int_c^t \tilde{v}(\tau)\,\mathrm{d}\tilde{P}(\tau),$$

for every $c, t \in (\alpha, \beta)$.

For $t \in (\alpha, \beta)$ and every $\delta > 0$, $\delta < t - \alpha$ we have $\tilde{v}(t) = v(t) = v(\alpha) + \int_\alpha^t \tilde{z}(\tau)\,\mathrm{d}R(\tau) = v(\alpha) + \int_\alpha^{\alpha+\delta} \tilde{z}(\tau)\,\mathrm{d}R(\tau) + \int_{\alpha+\delta}^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau) = v(\alpha) + \int_\alpha^{\alpha+\delta} \tilde{z}(\tau)\,\mathrm{d}[R(\tau) - \tilde{R}(\tau)] + \int_\alpha^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau)$. Since $\int_\alpha^{\alpha+\delta} \tilde{z}(\tau)\,\mathrm{d}[R(\tau) - \tilde{R}(\tau)] = \lim_{\delta\to 0+} \int_\alpha^{\alpha+\delta} \ldots = \tilde{z}(\alpha)\,[\Delta^+ R(\alpha) - \Delta^+ \tilde{R}(\alpha)]$ (see [7], 1.4.12 Theorem) we have

$$\tilde{v}(t) = v(\alpha) + \Delta^+ R(\alpha)\, z(\alpha) - \Delta^+ \tilde{R}(\alpha)\, z(\alpha) + \int_\alpha^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau) =$$
$$= \tilde{v}(\alpha) + \int_\alpha^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau)$$

for $t \in (\alpha, \beta)$ and

$$\tilde{z}(t) = z(t) = z(\alpha) + \int_\alpha^t z(\tau)\,\mathrm{d}P(\tau) = \tilde{z}(\alpha) + \int_\alpha^t \tilde{z}(\tau)\,\mathrm{d}\tilde{P}(\tau)$$

since $\tilde{P}(\alpha) = P(\alpha)$ by (9).

Similarly we can verify in the remaining cases that for every $c, t \in [\alpha, \beta]$ we have

$$\tilde{v}(t) = \tilde{v}(c) + \int_c^t \tilde{z}(\tau)\,\mathrm{d}\tilde{R}(\tau),$$

$$\tilde{z}(t) = \tilde{z}(c) + \int_c^t \tilde{v}(\tau)\,\mathrm{d}\tilde{P}(\tau),$$

i.e. that the assertion of the Lemma holds.

Corollary 1. *Assume that $R, P : (a, b) \to R$ satisfy (5), (6) and (9) and that the couple of functions (v, z) is a solution of (10) on the interval $[\alpha, b)$, $\alpha \in (a, b)$ such that $z(\alpha) \neq 0$ and $z(\beta) \neq 0$ for some $\beta \in (\alpha, b)$. Then there exists $\tilde{R} : (a, b) \to R$ satisfying (5), (6) and (9) such that $\tilde{R}(t) = R(t)$ for $t \in (\alpha, \beta]$ and that the solution $\tilde{v}$, $\tilde{z}$ of*

$$\mathrm{d}v = z\,\mathrm{d}\tilde{R},$$
$$\mathrm{d}z = v\,\mathrm{d}P$$

on the interval $[\alpha, b)$ which coincides with v, z on (α, β) satisfies

$$\tilde{v}(\alpha) = A, \qquad \tilde{v}(\beta +) = B,$$

where $A \leqq v(\alpha+)$ *if* $z(\alpha) > 0$, $A \geqq v(\alpha+)$ *if* $z(\alpha) < 0$, $B \leqq v(\beta)$ *if* $z(\beta) < 0$ *and* $B \geqq v(\beta)$ *if* $z(\beta) > 0$.

Proof. Let us consider the case when $z(\alpha) > 0$, $z(\beta) < 0$, and take $A \leqq v(\alpha+)$, $B \leqq v(\beta)$ arbitrary. For the other cases mentioned in the assertion of the lemma a similar proof can be given.

Define

$$\tilde{R}(t) = R(t) \qquad \text{for } t \in (\alpha, \beta],$$

$$\tilde{R}(t) = R(t) + R(\alpha+) - R(\alpha) - \frac{v(\alpha+) - A}{z(\alpha)} \qquad \text{for } t \in (a, \alpha],$$

$$\tilde{R}(t) = R(t) - R(\beta+) + R(\beta) + \frac{B - v(\beta)}{z(\beta)} \qquad \text{for } t \in (\beta, b).$$

Then

$$\Delta^+\tilde{R}(\alpha) = R(\alpha+) - \left[R(\alpha) + R(\alpha+) - R(\alpha) - \frac{v(\alpha+) - A}{z(\alpha)}\right] = \\ = \frac{v(\alpha+) - A}{z(\alpha)} \geqq 0$$

and

$$\Delta^+\tilde{R}(\beta) = R(\beta+) - R(\beta+) + R(\beta) + \frac{B - v(\beta)}{z(\beta)} - R(\beta) = \frac{B - v(\beta)}{z(\beta)} \geqq 0,$$

and $\tilde{R}$ satisfies (5), (6) and (9). By Lemma 2, for $\tilde{v}$, $\tilde{z}$ coinciding with v, z on (α, β) we have

$$\tilde{v}(\alpha) = v(\alpha+) - \Delta^+\tilde{R}(\alpha)\, z(\alpha) = v(\alpha+) - \frac{v(\alpha+) - A}{z(\alpha)}\, z(\alpha) = A$$

and

$$\tilde{v}(\beta+) = \tilde{v}(\beta) + \Delta^+\tilde{R}(\beta)\, \tilde{z}(\beta) = \tilde{v}(\beta) + \frac{B - v(\beta)}{z(\beta)}\, \tilde{z}(\beta) = B,$$

since $\tilde{v}(\beta) = \tilde{v}(\beta-) = v(\beta-) = v(\beta)$ and $\tilde{z}(\beta) = \tilde{z}(\beta-) + \Delta^-P(\beta)\, \tilde{v}(\beta) = z(\beta-) + \Delta^+P(\beta)\, v(\beta) = z(\beta)$. Thus the corollary is proved.

Remark 2. Lemma 2 and Corollary 1 enables us to make changes in the coefficients of (10) in such a way that a given solution (v, z) of (10) on a certain interval $[\alpha, \beta] \subset (a, b)$ remains unchanged except the values of $v(\alpha)$ and $z(\beta)$ which are changed.

A COMPARISON RESULT

Let us consider two systems of the form (10), i.e.

$$(10_k)\qquad \begin{aligned} dv &= z\,dR_k, \\ dz &= v\,dP_k, \qquad k = 1, 2 \end{aligned}$$

where the functions R_k, P_k satisfy the assumptions (5), (6) and (9).

Lemma 3. *If* (v_k, z_k), $k = 1, 2$ *are solutions of* (10_k) *on an interval* $[\alpha, \beta] \subset (a, b)$ *then the following identity is satisfied:*

$$\begin{aligned} &\int_\alpha^\beta v_1(t)\,v_2(t)\,d[P_2(t) - P_1(t)] + \int_\alpha^\beta z_1(t)\,z_2(t)\,d[R_1(t) - R_2(t)] = \\ (11)\qquad &= v_1(\beta)\,z_2(\beta) - v_1(\alpha)\,z_2(\alpha) - v_2(\beta)\,z_1(\beta) + v_2(\alpha)\,z_1(\alpha). \end{aligned}$$

Proof. Using the substitution theorem for Perron–Stieltjes integrals (see Theorem I.4.25 in [7]) we have by (10_2)

$$(12)\qquad \int_\alpha^\beta v_1(t)\,dz_2(t) = \int_\alpha^\beta v_1(t)\,d\Big[\int_\alpha^t v_2(\tau)\,dP_2(\tau)\Big] = \int_\alpha^\beta v_1(t)\,v_2(t)\,dP_2(t).$$

On the other hand, by (10_1) we have

$$\begin{aligned} &\int_\alpha^\beta v_1(t)\,dz_2(t) = \int_\alpha^\beta \Big(v_1(\alpha) + \int_\alpha^t z_1(\tau)\,dR_1(\tau)\Big)\,dz_2(t) = \\ (13)\qquad &= v_1(\alpha)\,(z_2(\beta) - z_2(\alpha)) - \int_\alpha^\beta \Big(\int_\alpha^t z_1(\tau)\,dR_1(\tau)\Big)\,dz_2(t). \end{aligned}$$

For the integral on the right hand side of this equality the Dirichlet formula for Perron–Stieltjes integrals (see Theorem I.4.32 in [7]) can be used, i.e. we get

$$\begin{aligned} \int_\alpha^\beta \Big(\int_\alpha^t z_1(\tau)\,dR_1(\tau)\Big)\,dz_2(t) &= \int_\alpha^\beta \Big(\int_t^\beta z_1(t)\,dz_2(\tau)\Big)\,dR_1(t) + \\ &+ \sum_{t \in (\alpha, \beta]} \Delta^- z_2(t)\,z_1(t)\,\Delta^- R_1(t) - \sum_{t \in [\alpha, \beta)} \Delta^+ z_2(t)\,z_1(t)\,\Delta^+ R_1(t); \end{aligned}$$

since $\Delta^+ z_2(t) = 0$ and $\Delta^- R_1(t) = 0$ for every $t \in (a, b) \cap [\alpha, \beta]$ we obtain by (13) the equality

$$\begin{aligned} \int_\alpha^\beta v_1(t)\,dz_2(t) &= v_1(\alpha)\,(z_2(\beta) - z_2(\alpha)) + \int_\alpha^\beta \Big(\int_t^\beta z_1(t)\,dz_2(\tau)\Big)\,dR_1(t) = \\ &= v_1(\alpha)\,(z_2(\beta) - z_2(\alpha)) + \int_\alpha^\beta z_1(t)\,(z_2(\beta) - z_2(t))\,dR_1(t) = \\ &= v_1(\alpha)\,(z_2(\beta) - z_2(\alpha)) + z_2(\beta)\int_\alpha^\beta z_1(t)\,dR_1(t) - \int_\alpha^\beta z_1(t)\,z_2(t)\,dR_1(t) = \\ (14)\qquad &= v_1(\alpha)\,(z_2(\beta) - z_2(\alpha)) + z_2(\beta)\,(v_1(\beta) - v_1(\alpha)) - \int_\alpha^\beta z_1(t)\,z_2(t)\,dR_1(t). \end{aligned}$$

Since the left hand sides of (12) and (14) coincide, we obtain the identity

$$\int_\alpha^\beta v_1(t)\,v_2(t)\,\mathrm{d}P_2(t) + \int_\alpha^\beta z_1(t)\,z_2(t)\,\mathrm{d}R_1(t) =$$
$$= v_1(\alpha)\,(z_2(\beta) - z_2(\alpha)) + z_2(\beta)\,(v_1(\beta) - v_1(\alpha)) =$$
$$= v_1(\alpha)\,z_2(\alpha) + v_1(\beta)\,z_2(\beta). \tag{15}$$

Using the same procedure for the integral $\int_\alpha^\beta v_2(t)\,\mathrm{d}z_1(t)$ we obtain the identity

$$\int_\alpha^\beta v_1(t)\,v_2(t)\,\mathrm{d}P_1(t) + \int_\alpha^\beta z_1(t)\,z_2(t)\,\mathrm{d}R_2(t) = -v_2(\alpha)\,z_1(\alpha) + v_2(\beta)\,z_1(\beta),$$

and subtracting this from (15) yields (11).

Definition. If (v, z) is a maximal nontrivial solution of (10) defined on (a, b) and $[\alpha, \beta] \subset (a, b)$ then we say that it *has not a zero in* (α, β) if either

$$v(\alpha) \geqq 0, \qquad v(t) > 0 \text{ for } t \in (\alpha, \beta), \qquad v(\beta <) \geqq 0$$

or

$$v(\alpha) \leqq 0, \qquad v(t) < 0 \text{ for } t \in (\alpha, \beta), \qquad v(\beta +) \leqq 0.$$

Otherwise we say that the solution (v, z) *has a zero in* (α, β).

Remark 3. Let us mention that the solution (v, z) has a zero in (α, β) if there exists $t \in (\alpha, \beta)$ such that 0 belongs to the interval with endpoints $v(t)$, $v(t +)$ or 0 is an internal point of the interval with endpoints $v(\alpha)$, $v(\alpha +)$ or 0 is an internal point of the interval with endpoints $v(\beta)$, $v(\beta +)$. For systems with "strong" impulses this concept replaces the case when the solution of the "classical" system has a zero in the open interval (α, β). In fact for classical equations the first component of a solution is continuous and all intervals mentioned above are degenerated. Hence at the points α and β these intervals cannot contain an internal point and the only possibility ist that the solution crosses the zero axis.

Theorem 1. *Assume that* $-\infty \leqq a < b \leqq +M$ *and that* $R, P_k \in BV_{loc}(a, b)$, $k = 1, 2$, R *is strictly increasing* $R(t -) = R(t)$, $P_k(t +) = P_k(t)$ *for* $t \in (a, b)$. *For the systems*

$$\mathrm{d}v = z\,\mathrm{d}R, \qquad \mathrm{d}z = v\,\mathrm{d}P_k, \qquad k = 1, 2 \tag{16_k}$$

assume that

$$P_2 - P_1 \textit{ is nonincreasing} \tag{17}$$

and that the couple (v_1, z_1) *is a solution of* (16_1) *on the interval* $[\alpha, b)$, $a < \alpha$ *such that*

$$v_1(\alpha) \geqq 0, \qquad v_1(t) > 0 \text{ for } t \in (\alpha, \beta), \qquad v_1(q +) \leqq 0, \qquad \alpha < \beta < b.$$

If (v_2, z_2) *is a maximal solution of* (16_2) *on* (a, b) *then one of the following cases can occur*:

A *The solution* (v_2, z_2) *has a zero in* (α, β).

B $\operatorname{var}_\alpha^{\beta'}(P_2 - P_1) = 0$ *for every* $\beta' < \beta$ *and there exists* $\lambda \in R$ *such that* $v_2(t) = \lambda v_1(t)$, $z_2(t) = \lambda z_1(t)$ *for* $t \in [\alpha, \beta]$.

Proof. Corollary 1 implies that without loss of generality we may assume that $v_1(\alpha) = 0$, $v_1(t) > 0$ for $t \in (\alpha, \beta)$ and $v_1(\beta +) = 0$ because $z_1(\alpha) > 0$ and $z_1(\beta) < 0$ by Lemma 1.

Assume that the solution (v_2, z_2) of (16_2) does not satisfy A, i.e. that we have e.g.

$$v_2(\alpha) \geqq 0, \qquad v_2(t) > 0 \text{ for } t \in (\alpha, \beta) \text{ and } v_2(\beta +) \geqq 0.$$

The other possible case with converse inequalities can be treated similarly.

By the identity (11) we have for $\gamma \in [\alpha, b)$

$$v_1(\gamma) z_2(\gamma) - v_2(\gamma) z_1(\gamma) + v_2(\alpha) z_1(\alpha) = \\ = \int_\alpha^\gamma v_1(t) v_2(t) \, \mathrm{d}[P_2(t) - P_1(t)].$$

For $\gamma \to \beta -$ we obtain

$$-v_2(\beta +) z_1(\beta) + v_2(\alpha) z_1(\alpha) = \int_\alpha^\beta v_1(t) v_2(t) \, \mathrm{d}[P_2(t) - P_1(t)],$$

because P_1, P_2 are continous from the right at β and $v_1(\beta +) = 0$. Since by Lemma 1 $z_1(\alpha) > 0$, $z_1(\beta) < 0$ the left hand side of this inequality is nonnegative and the right hand side is nonpositive since $P_2 - P_1$ is nonincreasing and $v_1(t) v_2(t) \geqq 0$ for $t \in [\alpha, \beta]$. Hence

$$\int_\alpha^\beta v_1(t) v_2(t) \, \mathrm{d}[P_2(t) - P_1(t)] = 0 \tag{18}$$

and $v_2(\alpha) z_1(\alpha) - v_2(\beta +) z_1(\beta) = 0$. The left hand side of this equality is the sum of two nonnegative terms, hence we immediately have

$$v_2(\alpha) = 0, \qquad v_2(\beta +) = 0.$$

Let us note that by Lemma 1 we have also $z_2(\alpha) > 0$ and $z_2(\beta) < 0$. Using (18) and the fact that $v_1(t) v_2(t) > 0$ for $t \in (\alpha, \beta)$ we obtain

$$\operatorname{var}_c^d (P_2 - P_1) = 0, \qquad \alpha < c < d < \beta$$

and this relation holds also for $c = \alpha$ because the functions P_1, P_2 are continuous from the right. Hence

$$\int_\alpha^t v_2(\tau) \, \mathrm{d}P_2(\tau) = \int_\alpha^t v_2(\tau) \, \mathrm{d}P_1(\tau)$$

for every $t \in [\alpha, \beta)$ and this yields by the definition of the solution for $t \in [\alpha, \beta)$

$$v_2(t) = \int_\alpha^t z_2(\tau)\,dR(\tau),$$

$$z_2(t) = z_2(\alpha) + \int_\alpha^t v_2(\tau)\,dP_1(\tau) = \lambda z_1(\alpha) + \int_\alpha^t v_2(\tau)\,dP_1(\tau),$$

where $\lambda = z_2(\alpha):z_1(\alpha) > 0$. Hence (v_2, z_2) is a solution of (16_1) such that $v_2(\alpha) = \lambda v_1(\alpha) = 0$, $z_2(\alpha) = yz_1(\alpha)$ and consequently by unicity and by the linear structure of the solutions of (16_1) we get

$$v_2(t) = \lambda v_1(t), \qquad z_2(t) = \lambda z_1(t) \text{ for all } t \in [\alpha, \beta).$$

Since v_1, v_2 are continuous from the left we have also $v_2(\beta) = \lambda v_1(\beta)$ and $z_2(\beta -) = \lambda z_1(\beta -)$. Using the equalities $v_1(\beta +) = 0$, $v_2(\beta +) = 0$ we have further

$$v_1(\beta +) = v_1(\beta) + \Delta^+ R(\beta)\, z_1(\beta) = 0, \qquad \text{i.e.} \qquad v_1(\beta) = -\Delta^+ R(\beta)\, z_1(\beta)$$

and similarly also

$$v_2(\beta) = -\Delta^+ R(\beta)\, z_2(\beta) = -\lambda \Delta^+ R(\beta)\, z_1(\beta).$$

Therefore if $\Delta^+ R(\beta) > 0$ then we can simply conclude that $z_2(\beta) = \lambda z_1(\beta)$. If $\Delta^+ R(\beta) = 0$ then $v_1(\beta) = v_2(\beta) = 0$ and we have for $k = 1, 2$

$$z_k(\beta) = \lim_{s \to \beta -} \Big(z_k(s) + \int_s^\beta v_k(\tau)\,dP_k(\tau)\Big) = z_k(\beta -) + \lim_{s \to \beta -} v_k(\beta)\,[P_k(\beta) - P_k(s)] = $$
$$= z_k(\beta -).$$

Hence we obtain again $z_2(\beta) = z_2(\beta -) = \lambda z_1(\beta -) = \lambda z_1(\beta)$.

Remark 4. Let us shortly reconsider the case of the classical Sturm—Liouville equation (1). Let us set $R(t) = \int_c^t \frac{1}{m(\tau)}\,d\tau$, $P_k(t) = \int_c^t p_k(\tau)\,d\tau$, $t \in (a, b)$, $c \in (a, b)$ is fixed, $k = 1, 2$ where $m \in L^1_{loc}(a, b)$, $m(\tau) > 0$, $\tau \in (a, b)$, $p_k \in L^1_{loc}(a, b)$, $k = 1, 2$. Then the equations (16_k) are equivalent to

$$(19_k) \qquad -(m(t)\,v')' + p_k(t)\,v = 0.$$

(17) from Theorem 1 is equivalent to the requirement $p_2(t) \leqq p_1(t)$, $t \in (a, b)$. The assumption on (v_1, z_1) from the Theorem 1 reduces to

$$v_1(\alpha) = 0, \qquad v_1(t) > 0 \text{ for } t \in (\alpha, \beta), \qquad v_1(\beta) = 0$$

for the solution v_1 of (19_1) on $[\alpha, b]$.

A from the conclusion has the form

A' There exists $s \in (\alpha, \beta]$ such that $v_2(s) = 0$

and instead of B we have

B′ $p_1(t) = p_2(t)$ for $t \in [\alpha, \beta]$ and there exists $y \in R$ such that $v_2(t) = \lambda v_1(t)$ for $t \in [\alpha, \beta]$.

Hence for the case of classical equations Theorem 1 gives the classical and well-known result (see e.g. [6] or any classical textbook on ordinary differential equations).

It should be also mentioned that in the proof of Theorem 1 only integrations are involved without using any derivative. The proof can be transfered to the case of classical equations (19_k) also solely in terms of integrations. Hence this result for differential equations can be reached without differentiation.

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