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Abstract. Different approaches to fuzzification of mathematical objects, especially automata, have been unified by Arbib and Manes using monads. A concrete version of monads of fuzzy species is introduced–standard monads- and proceeding from monad axioms in clone form, their underlying algebraic structure, that is the range of characteristic functions of the corresponding fuzzification is characterized to be completely lattice ordered semigroups. Homomorphisms between these structure are related to monad transformations of standard monads and vice versa, to every such transformation a homomorphism is attributed. In this the power set monad is proved to be initial object of standard monads.

Key words. Monad, monad transformation, Kleisli construction, brouwerian lattice, fuzzification, power set monad, monad L-Fuzz, complete lattice ordered semigroup.

MS Classification. 18 C 20, 18 A 23.

INTRODUCTION

Monads over a category **K** has been used in fuzzifying mathematical objects, especially automata [1, 2, 3]. The manner of fuzzification depends upon the algebraic structure of the ranges of generalized characteristic functions, often lattices but also other structures, determining the monad in the individual case, one of which will be explicated in section 1. There are several equivalent notions of a monad [7], their socalled clone form is most suitable in this direction, and proceeding from this clone form of monad axioms in some cases of monads of fuzzy species, they are summarized under the notion of a standard monad in section 2. Then the problem is arising, what an algebraic structure is imposed upon the ranges of corresponding characteristic functions, which will be solved also in section 2, leading to complete lattice ordered semigroups in the sense of Goguen [4], already used in the earlies of fuzzy theory. In section 3 relations between monad transformations of standard monads and homomorphisms of their underlying ranges will be investigated.

Basic facts on monads just needed in this paper are put together in section 1 and at the beginning of 3. Categorical framework reduces to elementary one and may be found in [6]. Composition of morphisms is left before right and consequently, functions symbols will be written to the right of the argument, but functors are standing to the left of the object.

1. MONADS, ESPECIALLY L-FUZZ

A monad (T, η, μ) in monoid form over a category $\mathbf{K}$ consists of an endofunctor $T : \mathbf{K} \to \mathbf{K}$, natural transformations $\eta : Id \dot{\to} T$, where Id is the identity functor of $\mathbf{K}$, and $\mu : T^2 \dot{\to} T$, such that

$$\eta_{TA}\mu_A = 1_{TA} = T_{\eta_A \mu_A}$$
$$T\mu_A\mu_A = \mu_{TA}\mu_A$$

for every object $A \in |\mathbf{K}|$.

By Kleisli's construction [5] every monad gives rise to a category $\mathbf{K}_T$ with the same object class as $\mathbf{K}$ and morphism classes $\mathbf{K}_T(A, B) = \mathbf{K}(A, TB)$, where morphism composition is

$$\alpha \bigcirc \beta = \alpha T\beta\mu_C,$$

$\alpha \in \mathbf{K}_T(A, B)$, $\beta \in \mathbf{K}_T(B, C)$. There exists a pair of adjoint functors $(\Delta, \#)$ between $\mathbf{K}$ and $\mathbf{K}_T$, given by

$$A^\Delta = A, \qquad f^\Delta = f\eta_B$$
$$A^\# = TA, \qquad \alpha^\# = T\alpha\mu_B$$

if $f \in \mathbf{K}(A, B)$ and $\alpha \in \mathbf{K}_T(A, B)$. As Kleisli has shown, $(\Delta, \#)$ generates the given monad: $TA = A^{\Delta\#}$, $Tf = f^{\Delta\#}$, η unit of the adjunction and μ the natural transformation associated with the counit ε.

The Kleisli composition $\bigcirc$ can be used for an equivalent definition of monads.

$(T, \eta, \bigcirc)$ is a monad in clone form over $\mathbf{K}$ iff T is an object map of $\mathbf{K}$, $\eta = (\eta_A)_{A \in |\mathbf{K}|}$ a family of morphisms $\eta_A : A \to TA$ and $\bigcirc$ a family $(\bigcirc_{ABC})_{A,B,C \in |\mathbf{K}|}$ of mappings

$$\bigcirc_{ABC} : \mathbf{K}(A, TB) \times \mathbf{K}(B, TC) \to \mathbf{K}(A, TC)$$

such that (object indices in the composition sign will be omitted)

$$(\alpha \bigcirc \beta) \bigcirc \gamma = \alpha \bigcirc (\beta \bigcirc \gamma)$$
$$\alpha \bigcirc \eta_B = \alpha$$
$$(f\eta_B) \bigcirc \beta = f\beta$$

for all composable morphisms α, β, γ, f.

It should be noticed the advantage of avoiding functor property of T and naturality of η in the last definition. The return from clone to monoid form is given by

$$Tf = 1_{TA} \circ (f\eta_B)$$
$$\mu_A = 1_{T^2A} \circ 1_{TA}$$

for $f \in \mathbf{K}(A, B)$.

To motivate standard monads and give an example the monad L-Fuzz will be now introduced. In what follows $\mathbf{K}$ always means the category **Set.**

Let L be a brouwerian lattice, that is a complete lattice with the additional property, that intersection distributes over suprema

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i).$$

Defining now for $A \in |\mathbf{K}|$

$$TA := L^A,$$

the set of all maps from A to L, one gets a fuzzification based upon L, interpreting for $a \in A$, $p \in TA$ the lattice element $(a)p$ as the grade of membership of the element a in the fuzzy set p.

By $\alpha : A \to TB$ a fuzzy set $(a)\alpha$ on B is attributed to every $a \in A$ and in notational similarity to conditional probability we set

$$\alpha(b/a) := (b)((a)\alpha)$$

$a \in A$, $b \in B$. Defining now $\eta_A : A \to TA$ by

$$\eta_A(a'/a) := \begin{cases} 1, & a' = a, \\ 0, & a' \neq a, \end{cases}$$

where 1 denotes the greatest and 0 the smallest element of L, we have a family $\eta = (\eta_A)_{A \in |\mathbf{K}|}$ of object maps in the category $\mathbf{K} =$ **Set.**

Further, $\alpha : A \to TB$ and $\beta : B \to TC$ are composed to $\alpha \circ \beta : A \to TC$ by

$$(\alpha \circ \beta)(c/a) := \bigvee_{b \in B} (\alpha(b/a) \wedge \beta(c/b))$$

$a \in A$, $c \in C$.

One verifies without difficulty that $(T, \eta, \circ)$ is a monad in clone form over **Set.** In this the algebraic structure of L plays an important role of course.

Special cases are attained by choosing $L = \{0, 1\}$ (power set monad, see also 3.3 below), or $L = [0, 1]$, the real intervall with respect to inf, sup, which corresponds to the original form of fuzzification by Zadeh [9].

Taking $L = [0, 1]$ and replacing the binary operation minimum in the Kleisli composition by the ordinasy product a further monad of fuzzy species is resulting, which may be conceived to be the algebraic background of max-product-machines, introduced by Santos [8].

Of course a dual variant of L-Fuzz can be built up by postulating

$$x \vee \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \vee y_i)$$

and dualizing the definiens of η_A and $\alpha \circ \beta$.

2. THE NOTION OF A STANDARD MONAD AND THE MAIN PROBLEM

2.0 All the monads just mentioned are based upon an algebraic structure L equipped with an infinitary operation Q, e.g. supremum, a binary operation $*$ and two nullary operations 0,1. In every case TA equals the set of maps from A to L and Kleisli-composition has been defined by the same term in $Q, *$ and variables for functions. Finally, η is fixed by the nullary operations. The properties of the algebra $(L; Q, *, 0, 1)$ then guarantee validity of the monad axioms.

Monads of fuzzy species will be comprehended now by the notion of a standard monad, formally defined by clone postulates w.r.t. the algebra $(L; Q, *, 0, 1)$. According to the choice of η and $Q(\emptyset)$ (either $Q(\emptyset) = 0$ or $Q(\emptyset) = 1$) they are divided into two types. We put into practice the type which L-Fuzz is belonging to.

2.1 Definition. *Let be* $(T, \eta, \circ)$ *a monad over* $\mathbf{K} = \mathbf{Set}$.
$(T, \eta, \circ)$ is a standard monad w.r.t. $(L; Q, *, 0, 1)$ iff

(1) L set, $\{0, 1\} \subseteq L$, $Q : PL \to L$, ($PL :=$ power set of L) $* : L \times L \to L$ and

$$Q(\emptyset) = 0$$
$$\forall x \in L: \qquad x * 1 = x$$

(2) $\forall A : TA = L^A$

$$\forall A\, \forall a, a' \in A : \qquad \eta_A(a'/a) = \begin{cases} 1 & \text{if } a' = a \\ 0 & \text{else} \end{cases}$$

$\forall A, B, C \;\; \forall a \in A \;\; \forall c \in C \;\; \forall \alpha \in \mathbf{K}(A, TB) \;\; \forall \beta \in \mathbf{K}(B, TC)$:

$$(\alpha \circ \beta)(c/a) = \underset{b \in B}{Q} (\alpha(b/a) * \beta(c/b))$$

$(L; Q, *, 0, 1)$ *is called the range of the standard monad.*

As is easily seen an example, apart from the monads of section 1, arises by a nonvoid set X and $L = P(X \times X)$, Q settheoretical union, $*$ relational product, $0 = \emptyset$, $1 = 1\times$ (identical relation). It should be noticed, that $Q(L) \neq 1$ in this case. The main question is stated now: what an algebraic structure is enjoined on the range L of a standard monad? We start with algebraic identities arising from monad postulates.

2.2 Proposition. *If* $(T, \eta, \circ)$ *is a standard monad w.r.t.* $(L; Q, *, 0, 1)$, *then the following identities are valid in L*

(1) $\underset{k \in K}{Q} (\underset{i \in I}{Q} (x_i * y_{ik}) * z_k) = \underset{i \in I}{Q} (x_i * \underset{k \in K}{Q} (y_{ik} * z_k))$

(2) $Q(\{x\} \cup \{y_i * 0 / i \in I\}) = x$

(3) $Q(\{1 * x\} \cup \{0 * y_i / i \in I\}) = x$

Proof. Choose

$$\alpha : \{1\} \to L^I, \qquad \alpha(i/1) = x_i \quad (i \in I)$$
$$\beta : I \to L^K, \qquad \beta(k/i) = y_{ik} \quad (i \in I, k \in K)$$
$$\gamma : K \to L^{\{1\}}, \qquad \gamma^{(1/k)} = z_k \quad (k \in K).$$

Then

$$((\alpha \circ \beta) \circ \gamma)(1/1) = \underset{k \in K}{Q}\,(\underset{i \in I}{Q}\,(\alpha(i/1) * \beta(k/i)) * \gamma(1/k)) =$$
$$= \underset{k \in K}{Q}\,(\underset{i \in I}{Q}\,(x_i * y_{ik}) * z_k)$$

Analogously

$$(\alpha \circ (\beta \circ \gamma))(1/1) = \underset{i \in I}{Q}\,(x_i * \underset{k \in K}{Q}\,(y_{ik} * z_k)),$$

from which follows (1).
If $B = I \cup \{j\}$, $j \notin I$ and $\alpha : \{1\} \to L^B$ with $\alpha(j/1) = x$, $\alpha(i/1) = y_i$ $(i \in I)$, then the second monad postulate yields

$$x = \alpha(j/1) = (\alpha \circ \eta_B)(j/1) = \underset{b \in B}{Q}\,(\alpha(b/1) * \eta_B(j/b)) =$$
$$= Q(\{x * 1\} \cup \{y_i * 0/i \in I\}).$$

Because of $x * 1 = x$ also (2) holds.
The last postulate is transformed into (3) with the same B and $\beta : B \to L^{\{1\}}$, $\beta(1/j) = x$, $\beta(1/i) = y_i$ $(i \in I)$ and $f = 1_B$:

$$x = (f\beta)(1/j) = (f\eta_B \circ \beta)(1/j) = \underset{b \in B}{Q}\,(\eta_B(b/(j)f) * \beta(1/b)) =$$
$$= Q(\{1 * x\} \cup \{0 * y_i/i \in I\}).$$

2.3. Surely, proposition 2.2 can be reversed, precisely the following statement is true:

Proposition. *Let* $(L; Q, *, 0, 1)$ *be an algebraic structure with the properties from* 2.1.1), *such that identities* (1), (2), (3) *are valid. If* $(T, \eta, \circ)$ *is defined according to* 2.1.2), *then* $(T, \eta, \circ)$ *is a standard monad w.r.t.* $(L; Q, *, 0, 1)$.

2.4. Of course the identities (1), (2), (3) merely represent a reformulation of the monad identities, from which no "well-known" algebraic structure regarding L can be read off. In the sequel (2.6 – 2.8) a series of simpler ones will be deduced from them, ordered into groups corresponding to classical structures in any case. Throughout these sections and also in the next $(T, \eta, \circ)$ is supposed to be a standard monad w.r.t. $(L; Q, *, 0, 1)$, But next to a lemma will be made available.

2.5. *For every* $x \in L$ *and* $M \subseteq L$

$$Q(\{x\}) = x$$
$$Q(M) = Q(M \cup \{0\}).$$

Proof. (2) with $I = \varnothing$ gives the first assertion. Suppose $z \notin M$ and $f: M \to \{z\}$ the constant map (resp. $f = \varnothing$ if $M = \varnothing$). Then f factors over the inclusion *in*: $M \to M \cup \{z\}$ and the constant map $g: M \cup \{z\} \to \{z\}$. Since monoid and clone foım are equivalent T is a functor and therefore

$$Tf = T\,in\,Tg.$$

Let $p \in TM = L^M$ with $(a)\,p = a$, $a \in M$. Then

$$Tf(z/p) = (1_{TM} \circ f\eta_{\{z\}})\,(z/p) = \underset{a \in M}{Q}\,(1_{TM}(a/p) * \eta_{\{z\}}(z/(a)\,f)) =$$
$$= Q(\{(a)\,p * 1/a \in M\}) = Q(M).$$

Setting $q = (p)\,T\,in$, one gets analogously

$$(a)\,q = T \quad in \quad (a/p) = \begin{cases} Q(\{a\}) = a, & a \in M \\ Q(\varnothing) = 0, & a = z \end{cases}$$

and

$$Tg(z/q) = Q(\{(a)\,q/a \in M \cup \{z\}, (a)\,g = z\}) = Q(M \cup \{0\}).$$

2.6.

$$\begin{aligned} &1.\ 1 * x = x \\ &2.\ (x * y) * z = x * (y * z) \\ &3.\ 0 * x = 0 = x * 0 \end{aligned}$$

1. follows from (3) with $I = \varnothing$ and the first assertion of 2.5.
2. is a consequence of (1) taking $|I| = |K| = 1$ and the same of 2.5.

The first equation of 3. arises by (3) within $|I| = 1$, $x = 0$, $y_i = x$ and using 1. and 2.5. The second with the same specialization by (2) and 2.5. Considering $x * 1 = x$ by supposition, we have: $(L; *, 1)$ is a monoid with zero 0.

2.7.

$$1.\ Q(\{Q(\{x, y\}), z\}) = Q(\{x, Q(\{y, z\})\})$$
$$2.\ \underset{k \in K}{Q}\,(Q(M_k)) = Q(\bigcup_{k \in K} M_k).$$

From (1) follows with the specification $|I| = |K| = 2$, $x_i = z_k = 1$, $y_{11} = x$, $y_{21} = y$, $y_{12} = 0$, $y_{22} = z$

$$Q(\{Q(\{x, y\}), Q(\{0, z\})\}) = Q(\{Q(\{x, 0\}), Q(\{y, z\})\}).$$

Applying now 2.5 yields 1.

As for the statement 2., identity (1) is used. Set $I := L \times K$ and

$$\begin{aligned} x_{(a,k)} &:= a, && \text{if } a \in M_k \text{ else } 0 \\ y_{(a,k')k} &:= 1, && \text{if } k = k' \text{ and } a \in M_k \text{ else } 0 \\ z_k &:= 1. \end{aligned}$$

Firstly, for every $k \in K$

$$\underset{(a,k')\in I}{Q} (x_{(a,k')} * y_{(a,k')k}) = Q(\{x_{(a,k)} * 1/(a, k) \in I\} \cup \{0\}) =$$
$$= Q(\{x_{(a,k)}/(a, k) \in I, a \in M_k\} \cup \{0\}) = Q(M_k \cup \{0\}) = Q(M_k).$$

With this in mind we get by (1)

$$\underset{k\in K}{Q} (Q(M_k)) = \underset{(a,k')\in I}{Q} (x_{(a,k')} * \underset{k\in K}{Q} (y_{(a,k')k} * 1)) =$$
$$= \underset{(a,k')\in I}{Q} (x_{(a,k')} * Q(\{y_{(a,k')k'}\})) =$$
$$= \underset{(a,k')\in I}{Q} (x_{(a,k')} * y_{(a,k')k'}) = \underset{k\in K, a\in M_k}{Q} (a * 1) = Q(\bigcup_{k\in K} M_k).$$

These two identities point out Q to be a supremum operation, as will be shown at once.

2.8. **Definition.** *Let be $x, y \in L$ and*

$$x \leqq y \text{ iff } Q(\{x, y\}) = y.$$

Proposition. *$(L, \leqq)$ is a partial order and for every $M \subseteq L$: $Q(M)$ is the supremum of M w.r.t. $\leqq$.*

Proof. Obviously, $\leqq$ being reflexive and antisymmetric, the transitivity is an immediate consequence of 2.7.1.

If $x \in M$ then by 2.7.2 and 2.5.1

$$Q(\{x, Q(M)\}) = Q(\{Q(\{x\}), Q(M)\}) = Q(M),$$

consequently $\forall x \in M : x \leqq Q(M)$.

Let be z an upper bound of M. Then by the same arguments

$$Q(\{Q(M), z\}) = Q(M \cup \{z\}) = Q(\bigcup_{x\in M} \{x, z\}) =$$
$$= \underset{x\in M}{Q} (Q(\{x, z\}) = z.$$

Corollary. *$(L; Q)$ is a complete sup-semilattice with the smallest element* $0 = Q(\emptyset)$.

Remark. 1 has not to be the greatest element $Q(L)$ of L w.r.t. $\leqq$ (see example from 2.1)

2.8. $*$ *is two-sided distributive over Q*:

$$x * Q(M) = \underset{y\in M}{Q} (x * y), \qquad Q(M) * z = \underset{y\in M}{Q} (y * z).$$

Applying (1) with $K = M$, $|I| = 1$, $z_k = 1$ and 2.5.1 one gets the assertion.

2.9. We are now in position of characterizing the range of a standard monad.

Theorem. $(L; Q, *, 0, 1)$ *is the range of a standard monad iff*

(1) *L set, $Q : PL \to L, * : L \times L \to L, 0, 1 \in L$*
(2) *$(L; *, 1)$ is a monoid with zero 0*
(3) *$(L; Q)$ is a complete sup-semilattice*
(4) *$*$ is two-sided distributive over Q.*

Proof. Necessity of the conditions has been shown in the foregoing.

In the opposite direction it suffices to point out the suppositions of 2.3. Directly $x * 1 = x$ is given and from (2) and (4) comes

$$0 = Q(\emptyset) * 0 = \underset{x \in \emptyset}{Q}(x * 0) = Q(\emptyset).$$

Because of associativity and distributivity of $*$ and the supremum property of Q the identities can easily be derived:

$$\underset{k \in K}{Q}(\underset{i \in I}{Q}(x_i * y_{ik}) * z_k) = \underset{k \in K}{Q}(\underset{i \in I}{Q}((x_i * y_{ik}) * z_k)) =$$
$$= \underset{k \in K}{Q}(\underset{i \in I}{Q}(x_i * (y_{ik} * z_k)) = \underset{i \in I}{Q}(\underset{k \in K}{Q}(x_i * (y_{ik} * z_k)) =$$
$$= \underset{i \in I}{Q}(x_i * \underset{k \in K}{Q}(y_{ik} * z_k)),$$
$$Q(\{x\} \cup \{y_i * 0 / i \in I\}) = Q(\{x, 0\}) = x.$$

Analogously for the resting identity.

2.10. These algebraic structures having been found out to be ranges of standard monads are exactly complete lattice ordered semigroups (=: clos) in sense of Goguen [4] and they has been used by this author already in the earlies of fuzzy theory for fuzzification of mathematical objects without any references to monads. From monad theoretic point of view now by the result 2.9 the way of Goguen's fuzzification seems to be the very natural one.

3. TRANSFORMATIONS OF STANDARD MONADS

3.0. As is well known monads over a category **K** together with monad transformations as morphisms constitute a category **Mon** (**K**). If (T, η, μ) and (T', η', μ') are monads in monoid form, a monad transformation λ: $(T, \eta, \mu) \dot{\to} (T', \eta', \mu')$ is a natural transformation $\lambda : T \dot{\to} T'$ of their underlying functors, such that for every object $A \in |K|$

(1) $$\eta_A \lambda_A = \eta'_A,$$

(2) $$\lambda_{TA} T' \lambda_A \mu'_A = \mu_A \lambda_A.$$

With respect to our interest in standard monads the corresponding equivalent formulation of monad transformation in clone form is more convenient [7]:

$\lambda : (T, \eta, \circ) \dot{\rightarrow} (T', \eta, \circ')$ iff

$\lambda = (\lambda_A)_{A \in |K|}$ family of morphisms $\lambda_A \in \mathbf{K}(TA, T'A)$ with (1) and

$$(2') \qquad (\alpha \circ \beta)\, \lambda_C = (\alpha\lambda_B) \circ' (\beta\lambda_C),$$

$\alpha : A \to TB,\ \beta : B \to TC,\ A, B, C \in |K|$.
Obviously, standard monads constitute a full subcategory of **Mon (Set)**.

3.1. If no confusion arises a clos $(L; Q, *, 0, 1)$ will be denoted by its underlying set L. As may be conjectured, homomorphisms between closes cause a transformation between their corresponding standard monads.

Definition. *Let L, L' be closes. A map $h : L \to L'$ is called a homomorphism from L to L' iff*

$$(1)\, h = 1', \qquad (x * y)\, h = (x)\, h *' (y)\, h \qquad (x, y \in L),$$
$$(Q(M))\, h = Q'(\{(x)\, h / x \in M\}) \qquad (M \subseteq L).$$

That is, a homomorphism is a Q-preserving monoid homomorphism. Consequently $(0)\, h = 0'$.

Theorem. *Let L, L' be closes, $(T, \eta, \circ)$, $(T', \eta', \circ')$ their corresponding standard monads and $h : L \to L'$ homomorphism. There exists a monad transformation $\lambda^h : (T, \eta, \circ) \dot{\rightarrow} (T', \eta', \circ')$ and the transition $h \mapsto \lambda^h$ defines a functor from* **Clos** *to the subcategory of standard monads of* **Mon (Set)**.

Proof. Let $p \in L^A$ and

$$(p)\, \lambda^h_A := ph.$$

This defines a family $\lambda^h = (\lambda^h_A)_{A \in |K|}$ of object maps $\lambda^h_A : TA \to T'A$. For $a \in A$

$$((a)\, \eta_A)\, \lambda^h_A = (a)\, \eta_A h = (a)\, \eta'_A,$$

considering $(1)\, h = 1'$ and $(0)\, h = 0'$.
If $\alpha : A \to TB$, $\beta : B \to TC$, $a \in A$, $c \in C$, then

$$((\alpha\lambda^h_B) \circ' (\beta\lambda^h_C))\, (c/a) = \underset{b \in B}{Q'}\, ((\alpha\lambda^h_B)\, (b/c) *' (\beta\lambda^h_C)\, (c/b)) =$$
$$= \underset{b \in B}{Q'}\, ((b)\, (((a)\, \alpha)\, \lambda^h_B) *' ((c)\, (((b)\, \beta)\, \lambda^h_C)) =$$
$$= \underset{b \in B}{Q'}\, ((\alpha(b/a)\, h) *' (\beta(c/b))\, h) =$$
$$= (\underset{b \in B}{Q}\, (\alpha(b/a) * \beta(c/b)))\, h = ((\alpha \circ \beta)\, (c/a))\, h = ((\alpha \circ \beta)\, \lambda^h_C)\, (c/a).$$

Therefore by 3.0 λ^h is a monad transformation. With the identical automorphism 1_L of L one has $(p)\, \lambda^{1_L}_A = p\, 1_L = p$. If $L'' \in |\,\mathbf{Clos}\,|$ is a further clos and g hom : $L' \to L''$, then

$$(p)\, \lambda^h_A \lambda^g_A = phg = (p)\, \lambda^{hg}_A.$$

3.2. Starting from a monad transformation λ between standard monads the question arises, if λ defines a homomorphism between their ranges. The natural way for solution is resting upon λ_A for a singleton A, e.g. $A = \{1\}$, to introduce $h_\lambda : L \to L'$ by

(1) $$(x)\, h_\lambda := \lambda_{\{1\}}(1/\{(1, x)\}),$$

being correctly, since $\{(1, x)\} \in T\{1\} = L^{\{1\}}$.
While it is easy to point out h_λ as monoid homomorphism (see 3.4, below), Q-preservation of h_λ requires some preparation. For this we shall deduce some properties of the power set monad, which seem to be also of interest in itself.

3.3. Denoting by PA the power set of A and considering $PA \sim \{0, 1\}^A$, the power set monad $(P, \eta^P, \circ^P)$ defined by

$$(a)\, \eta_A^P := \{a\},$$
$$(a)\, (\alpha \circ^P \beta) := \bigcup \{(b)\, \beta / b \in (a)\, \alpha\},$$

$\alpha : A \to PB$, $\beta : B \to PC$, is a special case of L-Fuzz, therefore standard monad w.r.t. $\{0, 1\}$.

Theorem. *The power set monad is an initial object of the category of standard monads.*

Proof. If $(T, \eta, \circ)$ denotes an arbitrary standard monad, the family

$$\lambda = (\lambda_A)_{A \in |\mathbf{Set}|} \quad \text{with } \lambda_A : PA \to TA,$$
$$\lambda_A(a/X) = 1 \quad \text{if } a \in X,\ 0 \text{ else} \quad (X \in PA)$$

defines a monad transformation. For

(1) $$(x)\, \eta_A^P \lambda_A = (\{x\})\, \lambda_A = (x)\, \eta_A$$

and if $\alpha : A \to PB$, $\beta : B \to PC$, $a \in A$, $c \in C$

$$((\alpha \circ^P \beta)\, \lambda_C)\, (c/a) = \lambda_C(c/\bigcup \{(b)\, \beta / b \in (a)\, \alpha\}) = 1$$

iff $\exists b \in (a)\, \alpha : c \in (b)\, \beta$
iff $\exists b \in B : \lambda_B(b/(a)\, \alpha) = 1 \wedge \lambda_C(c/(b)\, \beta) = 1$.
But this is equivalent with

$$\underset{b \in B}{Q}\, ((\alpha\lambda_B(b/a) * (\beta\lambda_C)\, (c/b)) = (\alpha\lambda_B \circ \beta\lambda_C)\, (c/a) = 1.$$

It remains to show uniqueness of the monad transformation $\lambda : P \dot{\to} T$. Let now be λ an arbitrary such transformation. Then (1) must hold. Taking $2 := \{0, 1\}$, $\alpha : \{1\} \to P2$, $\beta : 2 \to PA$, such that $(1)\, \alpha = 2$, $(0)\, \beta = X$, $(1)\, \beta = Y$ one gets from

$$\lambda_A(a/\bigcup \{(b)\, \beta / b \in (a)\, \alpha\}) = \underset{b \in 2}{Q}\, (\lambda_2(b/(a)\, \alpha) * \lambda_A(a/(b)\, \beta)),$$

$$\lambda_A(a/X \cup Y) = Q(\{\lambda_2(0/2) \ast \lambda_A(a/X), \lambda_2(1/2) \ast \lambda_A(a/Y)\}).$$

The first factors of the $\ast$-products in the right side are shown to be equal:

Let π be the nonidentical permutation of 2. Considering, that λ has to be a natural transformation $P \dot{\to} T$, the identity $\lambda_2 T\pi = P\pi\lambda_2$ must hold, yielding

$$\lambda_2(0/2) = (P\pi\lambda_2)(0/2) = (\lambda_2 T\pi)(0/2) =$$
$$= Q(\{\lambda_2(x/2)/x \in 2, (x)\pi = 0\}) = \lambda_2(1/2).$$

Therefore we get

$$\lambda_A(a/X \cup Y) = l \ast Q(\{\lambda_A(a/X), \lambda_A(a/Y)\}) \tag{2}$$

with $l := \lambda_2(0/2)$.

The next property of λ_A comes from the inclusion function $\mathrm{in}_X\colon X \to A$, $X \subseteq A$. Because of

$$\lambda_X T\,\mathrm{in}_X = P\,\mathrm{in}_X\,\lambda_A, \qquad \text{and for } a \in A,\ Y \subseteq X$$
$$Q(\{\lambda_X(a/Y)/x \in X, (x)\,\mathrm{in}_X = a\}) = (\lambda_X T\,\mathrm{in}_X)(a/Y) =$$
$$= (P\,\mathrm{in}_X\,\lambda_A)(a/Y) = \lambda_A(a/Y),$$

we get

$$\lambda_A(a/y) = \begin{cases} \lambda_X(a/Y) & \text{if } a \in X, \\ 0 & \text{else.} \end{cases}$$

Consequently $\lambda_A(a/X) = 0$ if $a \notin X$, especially $\lambda_A(a/\varnothing) = 0$. This allows to determine l in equation (2).

$X = \{x\}$, $Y = \varnothing$ gives

$$\lambda_A(a/\{x\}) = l \ast Q(\{\lambda_A(a/\{x\}), 0\}) = l \ast \lambda_A(a/\{x\})$$

and finally $a = x$ and $\lambda_A(x/\{x\}) = \lambda_{\{x\}}(x/\{x\}) = 1$ yields $1 = l \ast 1 = l$.

The proof will be finished by showing $\lambda_A(x/X) = 1$ for $x \in X$. Namely in this case λ_A agrees with the monad transformation at the beginning of it. Now from (2) arises for $x \in X$

$$\lambda_A(x/\{x\} \cup X - \{x\}) = Q(\{\lambda_A(x/\{x\}), 0\}) = 1.$$

Conclusion. If $\lambda : T \dot{\to} T'$ is a monad transformation between standard monads, then λ preserves constant-1-maps. That is, if for $p \in TA$, $(a)p = 1$ for every $a \in A$, also

$$\lambda_A(a/p) = 1' \qquad (a \in A).$$

For, denoting by ω^T the initial morphism $P \dot{\to} T$ in the category of standard monads,

$$\lambda_A(a/p) = (\omega_A^T \lambda_A)(a/A) = \omega_A^{T'}(a/A) = 1'$$

for every $a \in A$, $A \in |\,\mathbf{Set}\,|$.

3.4. **Theorem.** *If* $\lambda : (T, \eta, \circ) \dot{\to} (T', \eta', \circ')$ *is a monad transformation of standard monads, then* $h_\lambda : L \to L'$, *defined* by 3.2. (1), *is a clos homomorphism between their ranges. The assignment* $\lambda \mapsto h_\lambda$ *is functorial.*

Proof. Clearly, $(1)\, h_\lambda = 1'$, because

$$1' = \eta'_{\{1\}}(1/1) = (\eta_{\{1\}}\lambda_{\{1\}})\,(1/1) = \lambda_{\{1\}}(1/\{(1, 1)\}) = (1)\, h_\lambda.$$

Let $x, y \in L$, $\alpha : \{1\} \to L^{\{1\}}$, $\alpha(1/1) = x$, $\beta : \{1\} \to L^{\{1\}}$, $\beta(1/1) = y$. Then $(\alpha \circ \beta)\,(1/1) = x * y$ and

$$(x * y)\, h_\lambda = \lambda_{\{1\}}(1/\{(1, x * y)\}) = ((\alpha \circ \beta)\, \lambda_{\{1\}})\,(1/1) = (\alpha\lambda_{\{1\}} \circ' \beta\lambda_{\{1\}})\,(1/1) =$$
$$= Q'(\{(x)\, h_\lambda *' (y)\, h_\lambda\}) = (x)\, h_\lambda *' (y)\, h_\lambda.$$

Taking $\alpha : \{1\} \to L^\varnothing$, $\beta : \varnothing \to L^{\{1\}}$, $(\alpha \circ \beta)(1/1)$ has to be 0 and a similar computation shows $(0)\, h_\lambda = 0'$. Now let $\varnothing \neq M \subseteq L$, $\alpha : \{1\} \to L^M$ with $\alpha(x/1) = 1$ for $x \in M$ and $\beta : M \to L^{\{1\}}$ with $\beta(1/x) = x$ for $x \in M$. It follows

$$(\alpha \circ \beta)\,(1/1) = \underset{x \in M}{Q}\,(1 * x) = Q(M),$$
$$(Q(M))\, h_\lambda = (\alpha \circ \beta)\, \lambda_{\{1\}}(1/1) = \underset{x \in M}{Q'}\,((\alpha\lambda_M)\,(x/1) *' \beta\lambda_{\{1\}}(1/x))) =$$
$$= \underset{x \in M}{Q'}\,(\lambda_M(x/(1)\,\alpha) *' \lambda_{\{1\}}(1/\{(1, x)\})).$$

By 3.3 $\lambda_M(x/(1)\,\alpha) = 1'$ for every $x \in M$, such that

$$(Q(M))\, h_\lambda = Q'(\{(x)\, h_\lambda / x \in M\}).$$

The remaining assertion holds obviously.

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