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# ON A THEOREM OF J. PEETRE

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**Abstract.** It is well known that over the domain of $C^\infty$-smooth functions, all linear operators which do not enlarge the supports can be expressed by finite linear combinations of partial derivatives. We find that the assumptions of linearity and $C^\infty$-smoothness may be essentially weakened.

**Key words.** Jets, differential operators.

**MS Classification.** 5 8 G 9 9.

**0. Introduction.** An operator $A$ sending an arbitrary $C^\infty$-smooth cross-section $p$ of a vector bundle to certain $C^\infty$-smooth cross-section $Ap$ of another vector bundle over the same base space is called a *differential operator* if supp $Ap \subset$ supp $p$, for every mentioned cross-section $p$. The familiar *Peetre's theorem* states that all $\mathbf{R}$-linear differential operators are locally of finite order, that means, they are expressible by finite linear combinations of partial derivatives with $C^\infty$-coefficients in local coordinate systems, see [1]. We are interested in the question whether the assumptions of $C^\infty$-smoothness and linearity are indeed necessary.

**1. Notation.** Since the problem is of local nature, the trivial vector bundle $\pi : E \to B$ with the total space $E = \mathbf{R}^{n+m}$ (coordinates $x^1, \dots, x^n, y^1, \dots, y^m$), base $B = \mathbf{R}^n$ (coordinates $x^1, \dots, x^n$), and the projection defined by $\pi^* x^i \equiv x^i$ will be sufficient for our aims. Moreover, we may consider only the linear spaces noted $\mathbf{C}^\varkappa$ ($\varkappa = 0, 1, \dots, \infty$) of all $C^\varkappa$-smooth cross-sections $p, q, \dots : B \to E$ with *compact supports*, i. e., $p(x) \equiv 0$ if the argument $x \in B$ is far enough from the origin. Besides these cross-sections, various functions $f, g, \dots$ with compact supports on the base $B$ will appear. Partial derivatives are denoted by $\partial^{|I|}p/\partial x^I$, $\partial^{|I|}f/\partial x^I$, etc., with non-decreasing multiindices $I = i_1 \dots i_s$, $i_1 \leqq \dots \leqq i_s$, $|I| = s$. Following a naive variant of the common terminology, the families of all derivatives of order $|I| \leqq \varkappa$ (if $\varkappa < \infty$) or $|I| < \varkappa$ (if $\varkappa = \infty$) are identified with the familiar $\varkappa$-jets. They are denoted by $j^\varkappa p, j^\varkappa f$, etc., and the values at a point $a \in B$ are $j^\varkappa_a p, j^\varkappa_a f$, so that $j^0_a p = p(a)$, $j^0_a f = f(a)$ as a particular case.

We shall employ the max-norm. That means, $\| a \| = \max | x^i(a) |$ for a point $a \in B$, and $\| j^k p \| = \sup | \partial^{|I|} p / \partial x^I |$, $\| j^k f \| = \sup | \partial^{|I|} f / \partial x^I |$ (where $|I| \leqq k$ and we suppose $k = \varkappa < \infty$ here) for the spaces of $k$-jets. Quite analogous are the norms of jets at a fixed point $a \in B$. As an example, the relation $\lim_{l \to \infty} j_a^k f^l = j_a^k f$ means that $\lim_{l \to \infty} \| j_a^k (f^l - f) \| = 0$, that is, $\lim_{l \to \infty} (\partial^{|I|} (f^l - f) / \partial x^I)(a) = 0$, whenever $|I| \leqq k$.

Various constants are denoted by the same letter $c$.

Let $\chi \in C^\infty(B)$ be a fixed function satisfying $\chi(x) \equiv 1$ ($\| x \| \leqq 1/2$), $\chi(x) \equiv 0$ ($\| x \| \geqq 1$). Let $\chi(x, a, \varepsilon) = \chi((x - a)/\varepsilon)$, where the point $a \in B$ and the positive constant $\varepsilon$ are viewed as mere parameters. One can easily verify the estimate $| \partial^{|I|} \chi(x, a, \varepsilon) / \partial x^I | \leqq c \varepsilon^{-|I|}$ and it follows that $\lim_{\varepsilon \to 0} \| j^k \chi(., a(\varepsilon), \varepsilon) p \| = 0$ provided $j^k p(a) = 0$, $\| a(\varepsilon) - a \| \leqq c\varepsilon$.

At last remind that we shall investigate a mapping (operator) $A : \mathbf{C}^\infty \to \mathbf{C}^\varkappa$ ($\varkappa$ will be specified at the place) satisfying the following requirement: *If $a \in B$, $p, q \in \mathbf{C}^\infty$ and $p(x) \equiv q(x)$ for all $x$ lying in a neighbourhood of $a$, then $Ap(a) = Aq(a)$.* (Clearly, $Ap(x) \equiv Aq(x)$ for all mentioned $x$, too.)

**2. Lemma.** *Let $A : \mathbf{C}^\infty \to \mathbf{C}^0$, $p, p^1, p^2, \ldots \in \mathbf{C}^\infty$ and assume $j_a^{k^l}(p^l - p) \equiv 0$ where $\lim_{l \to \infty} k^l = \infty$. Then $\lim_{l \to \infty} Ap^l(a) = Ap(a)$.*

Proof. We shall consider sequences of points $b^1, b^2, \ldots \in B$ and positive constants $\varepsilon^1, \varepsilon^2, \ldots$ satisfying

(1) $$\| b^{l+1} - a \| < \| b^l - a \|/3, \varepsilon^l < \| b^l - a \|/3.$$

(One can observe that the supports of different functions $\chi(., b^l, \varepsilon^l)$ are disjoint.) It is not difficult to see that an appropriate choice of these sequences ensures in addition the limits

(2) $$\lim_{l \to \infty} \| j^{k^l} \chi(., b^l, \varepsilon^l)(p^l - p) \| = 0$$

and $\lim_{l \to \infty} \| Ap^l(b^l) - Ap^l(a) \| = 0$. The relation (2) clearly implies

(3) $$r = p + \Sigma \chi(., b^l, \varepsilon^l)(p^l - p) \in \mathbf{C}^\infty.$$

It is plain that $r = p^l$ (hence $Ar = Ap^l$) near every point $b^l$, but $r(x) \equiv p(x)$ (hence $Ar(x) = Ap(x)$) for $x$ satisfying the inequalities $\| x - b^l \| \geqq \varepsilon^l$. So we have

$$\lim Ap^l(a) = \lim Ap^l(b^l) = \lim Ar(b^l) = \lim_{x \to a} Ar(x) = \lim_{x \to a} Ap(x) = Ap(a),$$

and the proof is done.

**3. Corollary.** If $j_a^\infty p = j_a^\infty q$, then $Ap(a) = Aq(a)$. (Proof: Choose $p^1 = p^2 = \ldots \equiv q$.)

**4. Theorem.** *Let $A : \mathbf{C}^\infty \to \mathbf{C}^0$, $a, a^1, a^2, \ldots \in B$, $p, p^1, p^2, \ldots \in \mathbf{C}^\infty$ and assume $j_{a^l}^{k^l}(p^l - p) \equiv 0$ where $\lim_{l \to \infty} k^l = \infty$, $\lim_{l \to \infty} a^l = a$. Then $\lim_{l \to \infty} Ap^l(a^l) = Ap(a)$.*

Proof. Turning to subsequences, the general case can be reduced either to the subcase $a^l \equiv a$, or to the subcase $a^l \neq a$ and even $\| a^{l+1} - a \| < \| a^l - a \|/3$.

The first possibility was already discussed. The second one is quite analogous and even easier so that the details may be omitted.

**5. Corollary.** *Let* $A : \mathbf{C}^\infty \to \mathbf{C}^0$, $p \in C^\infty$. *For every constant* $\delta > 0$ *there exists an integer* $K = K(p, \delta)$ *such that* $\| Ap(a) - Aq(a) \| < \delta$ *whenever* $j_a^K(q - p) = 0$.

Proof by contradiction. Assume existence of $a^1, a^2, \ldots \in B, p^1, p^2, \ldots \in \mathbf{C}^\infty$ with $j_{a^l}^l(p^l - p) \equiv 0$ and $\| Ap^l(a^l) - Ap(a^l) \| \geqq \delta$. Clearly $\| a^l \| \leqq c$ and turning to subsequences, we may assume the existence of a limit $\lim a^l = a$. Then $\lim Ap(a^l) = Ap(a)$ and Theorem 3 yields $\lim Ap^l(a^l) = Ap(a) = \lim Ap(a^l)$ which is impossible.

**6. Corollary** (a variant of *Peetre's theorem*): *Let* $A : \mathbf{C}^\infty \to \mathbf{C}^0$ *be* **R**-*linear. There exists an integer* $K$ *such that* $Aq(a) = 0$ *whenever* $j_a^K q = 0$.

Proof. Choosing $\delta = 1$, $p = 0$ and the cross-section $tq$ ($t \in \mathbf{R}$) instead of $q$, the Corollary 5 yields $1 > \| A(tq)(a) \| = | t | \,.\, \| Aq(a) \|$ for all $t$, hence $Aq(a) = 0$.

**7. Remark.** Since the linearity appears only at the very end of the proof, the result can be easily generalized. For example, one can consider (instead of linearity) the assumption that every set of values

$$A(p + q)(a) \qquad (a \in B, p \in \mathbf{C}^\infty \text{ are fixed, } q \in \mathbf{C}^\infty \text{ is variable, } j_a^k q = 0)$$

is either unbounded, or consists of exactly one element, for every integer $k$ large enough. (Under this assumption, $Ap(a) = Aq(a)$ whenever $j_a^K(q - p) = 0$ with certain integer $K$ large enough.) Besides the linear operators, there are covered the polynomial operators and many others. We shall not however discuss these results since they may be essentially improved if the values of the operator $A$ lie in some narrower space than $\mathbf{C}^0$. In this connection, it is instructive to compare the Corollary 3 with the following result:

**8. Theorem.** *Let* $A : \mathbf{C}^\infty \to \mathbf{C}^1$, $p \in \mathbf{C}^\infty$. *There exists an integer* $K = K(p)$ *such that the relation* $j_a^K(q - p) = 0$ *always implies* $Ap(a) = Aq(a)$.

Proof by contradiction. Assume existence of sequences $a^1, a^2, \ldots \in B, p^1, p^2, \ldots \in \mathbf{C}^\infty$ such that $j_{a^l}^{k^l}(p^l - p) \equiv 0$ with $k^l$ going to infinity, but

$$Ap(a) \neq Ap^l(a^l)$$

for every $l$. Turning to the subsequences, one can see that it is sufficient to deal either with the subcase $a^l \equiv a$, or with the subcase $a^l \neq a$ and even $\| a^{l+1} - a \| < \| a^l - a \|/3$.

If the first possibility takes place, then $j_a^{k^l}(p^l - p) \equiv 0$ and $Ap(a) \neq Ap^l(a)$. Consider auxiliary sequences of points $b^1, b^2, \ldots \in B$ and positive constants $\varepsilon^1, \varepsilon^2, \ldots$ satisfying (1), (2). We may moreover suppose that

$$\| Ap^l(b^l) - Ap(a) \| \geqq (\| b^l - a \|)^{1/2}.$$

(Indeed, if a point $b \in B$ is close enough to the point $a$, then $Ap^l(b)$ is near to $Ap^l(a)$, but at the same time $Ap^l(a) \neq Ap(a)$.) Now, remind again the cross-section $r \in \mathbf{C}^\infty$ with the properties $Ar(b^l) \equiv Ap^l(b^l)$ and $Ar(a) = Ap(a)$; see (3) and the following text. It follows that

$$\| Ar(b^l) - Ar(a) \| = \| Ap^l(b^l) - Ap(a) \| \geqq (\| b^l - a \|)^{1/2},$$

but this is impossible for $Ar \in \mathbf{C}^1$.

The second subcase $\| a^{l+1} - a \| < \| a^l - a \|/3$ is even easier since the function (3) with $b^l$ replaced by $a^l$ and constants $\varepsilon^1, \varepsilon^2, \ldots$ going fast to zero directly leads to the same contradiction as above.

**9. A continuity assumption.** Choosing $p = 0$, Theorem 7 formally reduces to Corollary 6, but for the general case of nonlinear operator $A$. The meaning of the result is however different since, in virtue of the non-linearity, Theorem 7 ensures the finite order of $A$ only *along the given cross-section* $p$ and there may exist sequences $p^1, p^2, \ldots \in \mathbf{C}^\infty$, $q^1, q^2, \ldots \in \mathbf{C}^\infty$ with $j_a^l(q^l - p^l) \equiv 0$ but $Ap^l(a) \neq Aq^l(a)$ for all $l$. We shall nevertheless see that this unpleasant phenomenon can be mildly reduced if the operator $A$ fulfils some additional and rather natural continuity assumptions.

For this aim, consider one-parameter families $q(t) \in \mathbf{C}^\infty$ $(-\infty < t < \infty)$, where we suppose that $q(t)(x) \equiv 0$ whenever $|t| > c$ (and $\| x \| \geqq c$, as usual), for technical reasons. Every such a family may be identified with certain (not necessarily continuous) cross-section $\mathbf{q} : \mathbf{B} = \mathbf{R} \times B \to \mathbf{E} = \mathbf{R} \times E$ of the obvious product bundle, $\mathbf{q}$ being defined by $\mathbf{q}(t, x) \equiv q(t)(x)$ $(t \in \mathbf{R}, x \in B)$, of course, and we also obtain the product operator $\mathbf{A}$ defined by $\mathbf{Aq}(t, x) \equiv Aq(t)(x)$. Now, when the above $\mathbf{q} : \mathbf{B} \to \mathbf{E}$ is a continuous cross-section, then $\mathbf{Aq}$ need not be of such an art. *Assume*, however, *in addition, that we deal with such an operator $A$ that every $\mathbf{C}^\infty$-smooth cross-section* $\mathbf{q}$ *yields* $\mathbf{C}^1$*-smooth cross-section* $\mathbf{Aq}$. Theorem 7 then applies to the operator $\mathbf{A}$ (instead of $A$) with the following result:

**10. Lemma.** *Let $A : \mathbf{C}^\infty \to \mathbf{C}^1$ be an operator satisfying the above continuity assumption and $q(t) \in \mathbf{C}^\infty$ $(|t| < c)$ be a $C^\infty$-smooth one-parameter family of cross-sections. There exists an integer $K = K(\mathbf{q})$ such that any relation $j_a^K(q(t) - p) = 0$ $(t, a, p$ are fixed here, but arbitrary$)$ implies $Aq(t)(a) = Ap(a)$.*

The proof is immediate and may be omitted.

**11. Theorem.** *Let $A$ be the same operator as in Lemma* 10 *and $p \in \mathbf{C}^\infty$ be fixed. Let $\delta^1, \delta^2, \ldots$ be a sequence of positive constants and* $\lim_{l\to\infty} \delta^l = 0$. *Denote by* $\mathbf{S}$ *the set of all cross-sections $q \in \mathbf{C}^\infty$ satisfying the estimates* $\sum_{|I|=l} \| \partial^l(p - q)/\partial x^I \| \leqq \delta^l$. *Then there exists a constant $K = K(\mathbf{S}, c)$ such that the conditions $\| a \| < c$, $q \in \mathbf{S}$, $j_a^K(q - p) = 0$ imply $Ap(a) = Aq(a)$.*

Proof by contradiction. Let $q^1, q^2, \ldots \in \mathbf{S}$ satisfy $j_a^l(q^l - p) = 0$, $Aq^l(a) \neq$

$\neq Ap(a)$. There exists a one-parameter family $q(t) \in \mathbf{C}^\infty$ and a sequence $\varepsilon^1, \varepsilon^2, \ldots \in \in \mathbf{R}$, $\lim_{l \to \infty} \varepsilon^l = 0$, such that $q(\varepsilon^l) = q^l$. (More exactly, we suppose the last identity satisfied in the ball $\| x \| \leqq 2c$.) Then, according to Lemma 10, there exists a constant $K$ for which the relation $j_a^K(\mathbf{q}(\varepsilon^l, .) - p) = j_a^K(q^l - p) = 0$ implies $Aq^l(a) = = Aq(\varepsilon^l (a) = Ap(a)$, contrary to the assumption.

**12. Corollary.** *If the composed operator* $j^k \circ A$ *satisfies analogous conditions as* $A$ *did in Theorem* 11 (*or Lemma* 10), *then there exists an integer* $K = K(\mathbf{S}, c)$ *such that the conditions* $\| a \| < c$, $q \in \mathbf{S}$, $j_a^K(q - p) = 0$ *imply* $j_a^k p = j_a^k q$.

**13. Concluding remarks.** According to Theorem 11, every coordinate of the points $Aq(x)$ ($x \in \mathbf{B}$, $q \in \mathbf{S}$) is determined already by the $K$-jet of the cross-section and consequently, it may be expressed by a classical differential operator $F(x, \ldots, (\partial^{|I|} q / \partial x^I)(x), \ldots)$ with $F$ a definite function on the $K$-jet space. If the Corollary 12 may be applied, these coordinates $F$ are differentiable functions and we may calculate with our opeıators $A$ quite analogously as in the common classical case.

For a linear operator $A$, the above coordinate functions $F$ are linear functions of the derivatives and yield the true values $Aq$ for all cross-sections $q \in \mathbf{C}^\infty$, not only for $q \in \mathbf{S}$ (easy). So we have the third in order (and the most complicated) proof of the *Peetre's theorem.* This proof may be, however, most easily adapted to cover a very wide class of non-linear operators $A$ of the property that they are uniquely extendable from a subset $\mathbf{S}$ to the whole space $\mathbf{C}^\infty$ (cf. Remark 7).

## REFERENCES

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