Archivum Mathematicum

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Archivum Mathematicum, Vol. 23 (1987), No. 2, 63--68

Persistent URL: http://dml.cz/dmlcz/107280

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 23, No. 2 (1987), 63–68

ON LINEAR METHODS OF SUMMATIONS OF A FOURIER SERIES*

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(Received December 27, 1984)

Abstract. In this paper the authors have discussed the necessary and sufficient conditions for the Fourier series of the function $f(t)$ to be summable (A) at a point $t = x$.

Key words. Triangular matrix, Fourier Series, Sequence transformation, (A) summable and Abel's transformation.

MS Classification. 40 C 05.

Introduction. Let $A = (\alpha_{n,k})$, $(n = 0, 1, 2, \dots;\ k = 0, 1, \dots, n;\ \alpha_{n,o} = 1)$, be a triangular matrix of real numbers. For a given infinite series Σu_n, let the series to sequence transformation σ_n be defined by

$$\sigma_n = \sum_{k=o}^{n} \alpha_{n,k} u_k. \tag{1.1}$$

If $\sigma_n \to s$ as $n \to \infty$, then series Σu_n is said to be summable (A) to s.

The following notations will be frequently used:

$$\Phi(t) \equiv \Phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\};$$

$$D_k(t) = \frac{\sin(k + 1/2)t}{2\sin t/2};$$

$$M_k(t) \equiv \sum_{i=o}^{k} D_i(t) = \frac{\sin^2(k+1)t/2}{2\sin^2 t/2},$$

and

$$N_k(t) \equiv N_{k,n}(t) = M_n(t) - M_k(t) = \frac{\sin(n+k+2)t/2 \sin(n-k)t/2}{2\sin^2 t/2}.$$

Let A be a triangular matrix satisfying

$$\sum_{k=o}^{n} \frac{(k+1)(n-k)}{n+1} |\Delta^2 \alpha_{n,k}| \leqq A. \tag{1.2}$$

* Research supported by University Grants Commission, New Delhi (INDIA).

Then Efimov [1] has shown that

$$\sum_{k=o}^{\nu}(k+1)\,|\,\Delta^2\alpha_{n,k}\,|\leqq A, \tag{1.3}$$

$$\sum_{k=\nu+1}^{n-1}(n-k)\,|\,\Delta^2\alpha_{n,k}\,|\leqq A, \tag{1.4}$$

$$|\,\alpha_{n,k}\,|\leqq A, \qquad k\in N \text{ (natural numbers)}, \tag{1.5}$$

$$k\,|\,\Delta\alpha_{n,k}\,|\leqq A \qquad \text{for } 1\leqq k<\nu, \tag{1.6}$$

$$(n-k+1)\,|\,\Delta\alpha_{n,k}\,|\leqq A \qquad \text{for } \nu<k\leqq n, \tag{1.7}$$

and

$$\sum_{k=o}^{n}|\,\Delta\alpha_{n,k}\,|\leqq A, \tag{1.8}$$

where $\Delta^2\alpha_{n,k}=\alpha_{n,k}-2\alpha_{n,k+1}+\alpha_{n,k+2}$, A is an absolute constant and $\nu = [n/2]$, $[x]$ represents integral part of x.

The following well known relations will also be used in the sequel:

$$\frac{1}{k+1}\int_o^{\pi}M_k(t)\,\mathrm{d}t=\frac{\pi}{2}, \qquad k\in N, \tag{1.9}$$

and

$$\int_o^{\pi}|\,N_{n-m-1}(t)\,|\,\mathrm{d}t=O(n) \tag{1.10}$$

for $m=\nu-1,\ \nu,\ \nu+1$; and following relations hold uniformly with respect to n and $k=0,1,2,\ldots,n$ as

$$\frac{1}{k+1}\int_{2\pi/(k+2)}^{\pi}|\,N_{n-k-1}(t)\,|\,\mathrm{d}t=O(1), \tag{1.11}$$

$$\frac{N_k(t)}{2n-k}\leqq A, \tag{1.12}$$

$$\int_o^{2\pi/(k+2)}|\,N_{n-k-1}(t)-N_{n-k}(t)\,|=O\left(\frac{1}{k+1}\right), \tag{1.13}$$

$$\int_{2\pi/(k+3)}^{2\pi/(k+2)}|\,N_{n-k-1}(t)\,|\,\mathrm{d}t=O(1), \tag{1.14}$$

and

$$\frac{\sin(2n-k+1)\,t/2\,\sin(k+1)\,t/2}{(k+1)\,t^2}-\frac{\sin(2n-k+2)\,t/2\,\sin kt/2}{kt^2}=O(1). \tag{1.15}$$

2. We now prove

Theorem. *For a triangular matrix* $(\alpha_{n,k})$ *with positive elements such that* $\alpha_{n,k} \to 1$ *as* $n \to \infty$, *if* (1.2) *is satisfied and* $\varphi(t) = o(1/\log t^{-1})$ *as* $t \to o$, *then*

$$\sum_{k=o}^{n-1} \frac{\alpha_{n,k}}{(n-k+1)\log(n-k+1)} \leqq A, \tag{2.1}$$

if and only if the Fourier series of $f(t)$ *is summable* (Λ) *to* $f(x)$ *at* $t = x$.

Proof. For given $\varepsilon > o$, we choose $\delta > o$ such that

$$|\varphi(t)| \log(1/t) < \varepsilon, \qquad o < |t| < \delta. \tag{2.2}$$

Taking (1.1) into account and using Riemann–Lebesgue theorem and regularity of (Λ), we have

$$\sigma_n - f(x) = \frac{2}{\pi} \sum_{k=o}^{n} \Delta\alpha_{n,k} \int_o^\pi \varphi(t) D_k(t)\, dt =$$

$$= \frac{2}{\pi} \{ \sum_{k=o}^{v} \Delta\alpha_{n,k} \int_o^\delta \varphi(t) D_k(t)\, dt + \sum_{k=v+1}^{n} \Delta\alpha_{n,k} \int_o^\delta \varphi(t) D_k(t)\, dt +$$

$$+ \sum_{k=o}^{n} \Delta\alpha_{n,k} \int_\delta^\pi \varphi(t) D_k(t)\, dt \} = I_1 + I_2 + o(1) \qquad \text{(say)}. \tag{2.3}$$

On applying Abel's transformation and (1.3), (1.9), (1.6) and (2.2), we observe that

$$|I_1| = \left| \frac{2}{\pi} \int_o^\delta \varphi(t) \{ \sum_{k=o}^{v-1} M_k(t)\, \Delta^2\alpha_{n,k} + M_v(t)\, \Delta\alpha_{n,k} \}\, dt \right| \leqq$$

$$\leqq \frac{2}{\pi} \int_o^\delta |\varphi(t)| \{ \sum_{k=o}^{v-1} M_k(t) |\Delta^2\alpha_{n,k}| + M_v(t) |\Delta\alpha_{n,k}| \}\, dt \leqq$$

$$\leqq \varepsilon \sum_{k=o}^{v-1} (k+1) |\Delta^2\alpha_{n,k}| + \varepsilon(v+1) |\Delta\alpha_{n,k}| < A\varepsilon, \tag{2.4}$$

and in view of (1.10), we get

$$|I_2| = \left| \frac{2}{\pi} \int_o^\delta \varphi(t) \{ \sum_{k=v+1}^{n} \Delta\alpha_{n,k}(N_{k-1}(t) - N_k(t)\}\, dt \right| =$$

$$= \frac{2}{\pi} \left| \int_o^\delta \varphi(t) \{ \Delta\alpha_{n,v+1} N_v(t) - \sum_{k=v+1}^{n-1} \Delta^2\alpha_{n,k} N_k(t) \}\, dt \right| \leqq$$

$$\leqq \frac{2}{\pi} \int_o^\delta |\varphi(t)| \left| \sum_{k=v+1}^{n-1} N_k(t)\, \Delta^2\alpha_{n,k} \right| dt + \frac{2}{\pi} |\Delta\alpha_{n,v+1}| \int_o^\delta |N_v(t)\, \varphi(t)|\, dt \leqq$$

$$\leqq \frac{2}{\pi} \int_o^\delta |\varphi(t)| \left| \sum_{k=v+1}^{n-1} N_k(t)\, \Delta^2\alpha_{n,k} \right| dt + A\varepsilon(n-v) |\Delta\alpha_{n,v+1}|. \tag{2.5}$$

Let

$$N_k(t) = V_k(t) + W_k(t),$$

where

$$V_k(t) = \begin{cases} N_k(t) & o \leqq t \leqq 2\pi/(n-k+2) \\ o & 2\pi/(n-k+2) \leqq t \leqq \pi \end{cases}$$

and

$$W_k(t) = \begin{cases} o & o \leqq t \leqq 2\pi/(n-k+2) \\ N_k(t) & 2\pi/(n-k+2) < t \leqq \pi. \end{cases}$$

Applying (1.7) in (2.5) and further use of above notations leads to

$$|I_2| \leqq \frac{2}{\pi} \int_o^{2\pi/(n-k+2)} |\varphi(t)| \, \Big|\sum_{k=v+1}^{n-1} V_k(t)\Delta^2\alpha_{n,k}\Big| \, dt +$$

$$+ \int_{2\pi/(n-k+2)}^{\delta} |\varphi(t)| \, \Big|\sum_{k=v+1}^{n-1} W_k(t)\Delta^2\alpha_{n,k}\Big| \, dt + A\varepsilon. \tag{2.6}$$

Since

$$\sum_{k=v+1}^{n-1} \Delta^2\alpha_{n,k}V_k(t) = \sum_{k=v+2}^{n-1} (n-k)\Delta\alpha_{n,k}\left\{\frac{V_k(t)}{n-k} - \frac{V_{k-1}(t)}{n-k+1}\right\} +$$

$$+ \Delta\alpha_{n,v+1}V_{v+1}(t) - \sum_{k=v+2}^{n} \Delta\alpha_{n,k}\frac{V_{k-1}(t)}{n-k+1} =$$

$$= \sum_{k=v+2}^{n-1} (n-k)\Delta\alpha_{n,k}\left\{\frac{V_k(t)}{n-k} - \frac{V_{k-1}(t)}{n-k+1}\right\} + \Delta\alpha_{n,v+1}V_{v+1}(t) -$$

$$- \alpha_{n,v+2}\frac{V_{v+1}(t)}{n-k+1} - \sum_{k=v+3}^{n} \alpha_{n,k}\left\{\frac{V_{k-1}(t)}{n-k+1} - \frac{V_{k-2}(t)}{n-k+2}\right\},$$

on using (2.2), (1.4), (1.10) and (1.12) it follows that

$$\int_o^{\delta} |\varphi(t)| \, \Big|\sum_{k=v+1}^{n-1} V_k(t)\Delta^2\alpha_{n,k}\Big| \, dt \leqq \sum_{k=v+2}^{n-1} (n-k)|\Delta\alpha_{n,k}| \times$$

$$\times \left\{\int_o^{2\pi/(n-k+2)} |\varphi(t)| \left|\frac{V_k(t)}{n-k} - \frac{V_{k-1}(t)}{n-k+1}\right| dt + \int_{2\pi/(n-k+2)}^{2\pi/(n-k+1)} |\varphi(t)| \left|\frac{V_k(t)}{n-k}\right| dt\right\} +$$

$$+ |\Delta\alpha_{n,v+1}| \int_o^{2\pi/(n-v+1)} |\varphi(t)| \, |V_{v+1}(t)\, dt + |\alpha_{n,v+2}| \times$$

$$\times \int_o^{2\pi/(n-v+1)} |\varphi(t)| \left|\frac{V_k(t)}{n-v+1}\right| dt + \sum_{k=v+3}^{n} |\alpha_{n,k}| \times$$

$$\times \left\{\int_o^{2\pi/(n-k+3)} \left|\frac{V_{k-1}(t)}{n-k+1} - \frac{V_{k-2}(t)}{n-k+2}\right| |\varphi(t)|\, dt + \int_{2\pi/(n-k+3)}^{2\pi/(n-k+2)} |\varphi(t)| \left|\frac{V_{k-1}(t)}{n-k+3}\right| dt\right\} \leqq$$

$$\leqq A\varepsilon\left[\sum_{k=v+2}^{n-1} |\Delta\alpha_{n,k}| \left|\frac{n-k}{n-k+1}\right| + \sum_{k=v+3}^{n} \frac{\alpha_{n,k}}{(n-k+1)\log(n-k+1)}\right] +$$

$$+ A\varepsilon[(n-v-1)|\Delta\alpha_{n,v+1}| + |\alpha_{n,v+2}|] < A\varepsilon. \tag{2.7}$$

Now, for the expression involving W_k in (2.6) we proceed as (in view of (1.11) and (1.4))

$$\int_0^\delta |\varphi(t)|\,|\sum_{k=\nu+1}^{n-1} \Delta^2\alpha_{n,k}W_k(t)|\,dt \leqq \sum_{k=\nu+1}^{n-1} |\Delta^2\alpha_{n,k}| \int_0^\delta |N_k(t)|\,|\varphi(t)|\,dt \leqq A\varepsilon.$$

Using (2.7) and above in (2.6), we are able to show that $|I_2| < \varepsilon$. This fact coupled with (2.4) and taking (2.6) into account, gives

$$|\sigma_n(x) - f(x)| < A\varepsilon + o(1).$$

To prove necessary part, it is enough to show that $I_2 = o(1)$ as $n \to \infty$ implies the condition (2.1).

We observe that

$$I_2 = \frac{2}{\pi}\int_0^\delta \varphi(t) \sum_{k=o}^{n-\nu-1} (\alpha_{n,n-k} - \alpha_{n,n-k+1}) D_{n-k}(t)\,dt =$$

$$= \frac{2}{\pi}\int_0^\delta \varphi(t) \sum_{k=o}^{n-\nu-1} (\alpha_{n,n-k} - \alpha_{n,n-k+1}) \frac{\sin(n+1)t\cos(k+1/2)t}{2\sin t/2}\,dt -$$

$$- \frac{2}{\pi}\int_0^\delta \varphi(t) \sum_{k=o}^{n-\nu-1} (\alpha_{n,n-k} - \alpha_{n,n-k+1}) \frac{\cos(n+1)t\sin(k+1/2)t}{2\sin t/2}\,dt =$$

$$= P + Q \text{ (let)}. \tag{2.8}$$

Further, by using (1.7), (1.4) and (1.9), we treat with Q as

$$|Q| = \left|\frac{2}{\pi}\int_0^\delta \varphi(t)\{\sum_{k=o}^{n-\nu-2} M_k(t)\,\Delta^2\alpha_{n,n-k} + M_{n-\nu-1}(t)(\alpha_{n,\nu+1} - \alpha_{n,\nu+2})\}\,dt\right| =$$

$$= \left|\frac{2}{\pi}\int_0^\delta \varphi(t)\{\Delta\alpha_{n,\nu+1}M_{n-\nu-1}(t) - \sum_{k=\nu+2}^{n} M_{n-k}(t)\,\Delta^2\alpha_{n,k-1}\}\,dt\right| \leqq$$

$$\leqq A\varepsilon\{(n-\nu)|\Delta\alpha_{n,\nu+1}| + \sum_{k=\nu+2}^{n} (n-k+1)|\Delta^2\alpha_{n,k-1}|\} < A\varepsilon. \tag{2.9}$$

Again, in view of (2.8) and (2.9), it is enough to varify that (2.1) is implied by

$$\int_0^\pi \frac{|\sum_{k=o}^{n-\nu-1} (\alpha_{n,n-k} - \alpha_{n,n-k+1})\{\sin(n+1)t\cos(k+1/2)t\}|}{2\sin\frac{t}{2}\log\frac{\pi e^2}{t}}\,dt = o(1).$$

On applying Abel's transformation and noting the fact $|\sin x| \geqq \sin^2 x$, we find that the above relation implies

$$\int_0^\pi \frac{\sin^2(n+1)t}{\log \pi e^2 t^{-1}}\left\{\alpha_{n,\nu+1}\frac{\cos(n-\nu-1/2)t}{2\sin t/2} + \sum_{k=o}^{n-\nu-2} \alpha_{n,n-k}\sin(k+1)t\right\}dt = o(1). \tag{2.10}$$

We know the well known results:

(2.11) $$\int_0^\pi \frac{\sin^2(n+1)t\cos(n-v-1/2)t}{2\sin t/2\log \pi e^2 t^{-1}}\,dt = O(1)$$

and

(2.12) $$\int_0^\pi \frac{\sin(k+1)t\cos 2(n+1)t}{\log \pi e^2 t^{-1}}\,dt = O\left(\frac{1}{n}\right), \qquad 1 \leqq k \leqq n.$$

Thus (2.11), (2.12) and (2.10) yield

(2.13) $$\int_0^\pi \left\{\sum_{k=0}^{n-v-2} \alpha_{n,n-k} \frac{\sin(k+1)t}{\log \pi e^2 t^{-1}}\right\} dt = O(1).$$

Further,

$$\int_0^\pi \frac{\sin(k+1)t}{\log \pi e^2 t^{-1}}\,dt = \left\{\int_0^{\pi/2(k+1)} + \int_{\pi/2(k+1)}^{\pi}\right\}\left\{\frac{\sin(k+1)t}{\log \pi e^2 t^{-1}}\,dt\right\} = R + L.$$

We treat R and L separately as

$$R = \int_0^{\pi/2} \frac{\sin t}{(k+1)\log \pi e^2(k+1)t^{-1}}\,dt \geqq \int_{\pi/3}^{\pi/2} \frac{\sin t}{(k+1)\log \pi e^2(k+1)t^{-1}}\,dt \geqq$$

(2.14) $$\geqq [2(k+1)\log\{3e^2(k+1)\}]^{-1},$$

and

$$L = \left[-\frac{\cos(k+1)t}{(k+1)\log \pi e^2 t^{-1}}\right]_{\pi/2(k+1)}^{\pi} + \frac{1}{k+1}\int_{\pi/2(k+1)}^{\pi} \frac{\cos(k+1)t}{t(\log \pi e^2 t^{-1})^2}\,dt =$$

(2.15) $$= \frac{(-1)^k}{2(k+1)} + O\left\{\frac{1}{(k+1)\log^2(k+1)}\right\}.$$

$\left|\sum_{\kappa=0}^{n-v+2} \{(-1)^k/(k+1)\}\,\alpha_{n,n-k}\right|$ is bounded. Therefore (2.13) implies (2.1) in view of (2.14) and (2.15).

This completes the proof.

ACKNOWLEDGEMENTS

The authors are extremely thankful to the referee for making the paper more complete and intelligible.

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