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ASYMPTOTIC FORMULAS FOR SOLUTIONS OF THE DIFFERENTIAL EQUATION WITH ADVANCED ARGUMENT $(x'(t)/r(t))' + q(t)f(x(g(t))) = 0$*

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Abstract. Asymptotic formulas for solutions of the differential equation with advanced argument $(x'(t)/r(t))' + q(t)f(x(g(t))) = 0$ are derived in terms of so-called $(1, p)$-integral equivalence to the equation $(y'(t)/r(t))' = 0$.

Key words. Differential equation with advanced argument, (ψ, p)-integral equivalence, restricted (ψ, p)-integral equivalence.

MS Classification: 34 K 15, 34 C 11.

INTRODUCTION

In the papers [1], [2] the integral equivalence of two systems

$$x' = A(t, x), \tag{1}$$

$$y' = B(t, y) \tag{2}$$

in the following sence was studied. Let $p \in R$, $p \geqq 1$ and let ψ be a nonnegative continuous function defined on $\langle 0, \infty)$. We say that the systems (1), (2) are (ψ, p)-integral equivalent if to every solution x of (1) defined for $t \to \infty$ there exists a solution y of (2) such that

$$\psi^{-1}(t) \mid x(t) - y(t) \mid \in \mathscr{L}_p \tag{3}$$

and conversely, if to every solution y of (2) defined for $t \to \infty$ there exists a solution x of (1) with the property (3). By restricted (ψ, p)-integral equivalence between (1) and (2) is meant that the relation (3) is satisfied for some subsets of solutions of (1) and (2) e.g. for bounded solutions. The same concepts appeared in (3) in connection with systems of functional differential equations $x'(t) = f(t, x_t)$, $y'(t) =$

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$= f(t, y_t) + g(t, y_t)$, where the asymptotic behaviour of the difference $x(t) - y(t)$ was studied.

In this paper asymptotic properties of the differential equation with advanced argument

$$(4) \qquad (x'(t)/r(t))' + q(t) f(x(g(t))) = 0$$

are derived in terms of $(1, p)$-integral equivalence to the equation

$$(5) \qquad (y'(t)/r(t))' = 0.$$

Some of the usual assumptions concerning the absolute convergence of certain integrals are replaced by relative one.

In the case $r(t) = 1$, $g(t) = t$ the asymptotic formulas (10) and (11) have been proved by Naito in (4).

MAIN RESULT

Lemma 1. ([2], p. 394) *Let $b(t) \geqq 0$ be a continuous function on $\langle 0, \infty)$ and such that*

$$(6) \qquad \int_0^\infty s^{1/p} b(s)\, ds < \infty \qquad \text{for some} \qquad p \geqq 1.$$

Then

$$\int_t^\infty b(s)\, ds \in \mathscr{L}_{p'},\ p' \geqq p.$$

Lemma 2. *Let $r(t)$, $h(t)$ be continuous functions on $\langle 0, \infty)$, $r(t) > 0$. If*

$$\lim_{t \to \infty} \int_0^t r(s) \int_0^s h(u)\, du\, ds$$

exists and is finite, then the equation (5) *and*

$$(7) \qquad (x'(t)/r(t))' = h(t)$$

are $(1, p)$-integral equivalent if and only if

$$(8) \qquad \int_t^\infty r(s) \int_\sigma^s h(u)\, du\, ds \in \mathscr{L}_p$$

for some $\sigma > 0$.

Proof. Any solution of (7) is of the form

$$(9) \qquad x(t) = y(t) - \int_t^\infty r(s) \int_\sigma^s h(u)\, du\, ds,$$

where $y(t) = \alpha + \beta \int_0^t r(s)\, ds$, $\alpha, \beta \in R$, is a suitable solution of (5) and conversely if y is any solution of (5) then (9) is a solution of (7).

It is evident that $x(t) - y(t) \in \mathscr{L}_p$ if and only if (8) holds.

Remark. The condition (8) is satisfied if

$$\int_0^\infty s^{1/p} r(s) \left| \int_\sigma^s h(u)\,\mathrm{d}u \right| \mathrm{d}s < \infty .$$

Let R^+ be a set of all positive numbers. Let $I \subseteq R$ be an open interval, $J_0 = \langle 0, \infty)$.

Theorem. *Suppose that:*

(i) *$r\colon J_0 \to R^+$, $q\colon J_0 \to R$ are continuous, $g\colon J_0 \to J_0$, $f\colon I \to R$ have continuous derivatives of the first order, $g(t) \geqq t$ on J_0;*

(ii) $$\int_0^\infty r(s)\,\mathrm{d}s = \infty ;$$

(iii) $$\int_0^t q(s)\,\mathrm{d}s \quad \textit{has a finite limit as } t \to \infty \textit{ so that}$$

$$Q(t) = \int_t^\infty q(s)\,\mathrm{d}s \to 0 \qquad \textit{as} \qquad t \to \infty ;$$

(iv) $$\int_0^\infty s^{1/p} R(s)\, H(s)\,\mathrm{d}s < \infty \qquad \textit{for some } p, \qquad 1 \leqq p \leqq \infty ,$$

where $R(s) = \operatorname{Max}\,(r(s), r(g(s)))$, $H(s) = \operatorname{Max}\,(|\,Q(s)\,|, |\,Q(s)\,g'(s)\,|, |\,Q(g(s))\,|)$;

(v) $$\int_0^\infty s^{1/p} R(s)\, G(s)\,\mathrm{d}s < \infty \qquad \text{where} \qquad G(s) = \int_s^\infty R(u)\, H^2(u)\,\mathrm{d}u .$$

Then for any $\alpha \in I$ there exists at least one solution x of the equation (4) *with the ollowing propert ies*

(10) $$x(t) = \alpha + 0(\varepsilon(t)), \qquad \varepsilon(t) = \operatorname{Max}\left(\int_t^\infty R(s)\, H(s)\,\mathrm{d}s, \quad \int_t^\infty R(s)\, G(s)\,\mathrm{d}s\right);$$

(11) $$x'(t)/r(t) = 0(\delta(t)), \qquad \delta(t) = \operatorname{Max}\,(H(t), G(t));$$

(12) $$x(t) - \alpha \in \mathscr{L}_p .$$

Conversely any bounded solution of (4) *approaches to a finite limit as $t \to \infty$ so that the relations* (10), (11), (12) *are satisfied.*

P r o o f. Let us note that conditions (iv) and (v) imply the convergence of the integrals

$$\int_0^\infty R(s)\, H(s)\,\mathrm{d}s \qquad \text{and} \qquad \int_0^\infty R(s)\, G(s)\,\mathrm{d}s .$$

Let $\alpha \in I$ and let $\varepsilon > 0$ be such that $\langle \alpha - \varepsilon, \alpha + \varepsilon \rangle \subset I$. Denote

$$\beta = \operatorname*{Max}_{|x-\alpha| \leqq \varepsilon} |\,f(x)\,| , \qquad \gamma = \operatorname*{Max}_{|x-\alpha| \leqq \varepsilon} |\,f'(x)\,| .$$

Put $\eta = 2\beta\gamma$ and choose a $t_0 > 0$ such that

$$2\gamma \int_{t_0}^{\infty} R(s) H(s) \, ds \leqq 1 \tag{13}$$

and

$$\beta \int_{t_0}^{\infty} R(s) H(s) \, ds + \eta \int_{t_0}^{\infty} R(s) G(s) \, ds \leqq \varepsilon. \tag{14}$$

The condition (13) implies

$$\gamma\Big(\beta + \eta \int_{t_0}^{\infty} R(s) H(s) \, ds\Big) \leqq \eta. \tag{15}$$

Let $\mathcal{F}$ be the Fréchet space of all continuously differentiable functions on $\langle t_0, \infty)$ with the family of seminorms defined by

$$\| x \|_h = \sup_{t_0 \leqq t \leqq t_0 + h} \{| x(t) | + | x'(t)/r(t) |\}.$$

We have the convergence $x_n \to x$ in the topology of $\mathcal{F}$ if and only if $x_n(t) \to x(t)$ and $x_n'(t) \to x'(t)$ almost uniformly on $\langle t_0, \infty)$. Let $X \subset \mathcal{F}$ be a system of functions x such that

$$| x(t) - \alpha | \leqq \varepsilon, \ \mathrm{Max} \{| x'(t)/r(t) |, | x'(g(t))/r(g(t)) |\} \leqq \beta H(t) + \eta G(t).$$

Define the operator T on X by

$$(Tx)(t) = \alpha - \int_t^{\infty} r(s) Q(s) f(x(g(s))) \, ds -$$
$$- \int_t^{\infty} r(s) \int_s^{\infty} Q(u) f'(x(g(u))) \, x'(g(u)) \, g'(u) \, du \, ds.$$

Any fixed point of the operator T is a solution of (4) with the properties (10), (11), (12). First, let us prove its existence.

The space X is evidently a nonempty closed convex subset in $\mathcal{F}$. If $x \in X$ then

$$| (Tx)(t) - \alpha | \leqq \beta \int_t^{\infty} r(s) | Q(s) | \, ds + \gamma \int_t^{\infty} r(s) \int_s^{\infty} | Q(u) \, x'(g(u)) \, g'(u) | \, du \, ds \leqq$$
$$\leqq \beta \int_t^{\infty} R(s) H(s) \, ds + \gamma \int_t^{\infty} R(s) \int_s^{\infty} R(u) H(u) [\beta H(u) + \eta G(u)] \, du \, ds$$

for $t \geqq t_0$.

Since

$$\int_t^{\infty} R(s) \int_s^{\infty} R(u) H(u) [\beta H(u) + \eta G(u)] \, du \, ds \leqq$$
$$\leqq \beta \int_t^{\infty} R(s) G(s) \, ds + \eta \int_t^{\infty} R(s) \int_s^{\infty} R(u) H(u) G(u) \, du \, ds \leqq$$

$$\leqq \int_t^\infty R(s)\,G(s)\,ds + \eta \int_t^\infty R(s)\,G(s) \int_s^\infty R(u)\,H(u)\,du\,ds \leqq$$

$$\leqq \int_t^\infty R(s)\,G(s)\,ds[\beta + \eta \int_t^\infty R(s)\,H(s)\,ds],$$

we have with respect to (15) and (14)

$$(16)\quad \begin{cases} |(Tx)(t) - \alpha| \leqq \beta \int_t^\infty R(s)\,H(s)\,ds + \gamma(\beta + \eta \int_t^\infty R(s)\,H(s)\,ds) \int_t^\infty R(s)\,G(s)\,ds \leqq \\ \leqq \beta \int_t^\infty R(s)\,H(s)\,ds + \eta \int_t^\infty R(s)\,G(s)\,ds \leqq \varepsilon \quad \text{on} \quad \langle t_0, \infty). \end{cases}$$

Moreover

$$(17)\quad \begin{cases} |(Tx)'(t)/r(t)| \leqq \beta\,|Q(t)| + \gamma \int_t^\infty |Q(s)\,x'(g(s))\,g'(s)|\,ds \leqq \\ \leqq \beta\,|Q(t)| + \gamma \int_t^\infty R(s)\,H(s)\,[\beta H(s) + \eta G(s)]\,ds \leqq \\ \leqq \beta\,|Q(t)| + \beta\gamma G(t) + \gamma\eta \int_t^\infty R(s)\,H(s)\,G(s)\,ds. \end{cases}$$

Using (15) we have

$$|(Tx)'(t)/r(t)| \leqq \beta H(t) + \eta G(t).$$

Moreover, since $g(t) \geqq t$ and $G(t)$ is nonincreasing, we receive writing $g(t)$ instead of t in (17)

$$|(Tx)'(g(t))/r(g(t))| \leqq \beta\,|Q(g(t))| + \beta\gamma \int_{g(t)}^\infty R(s)\,H^2(s)\,ds +$$

$$+ \gamma\eta \int_{g(t)}^\infty R(s)\,H(s)\,G(s)\,ds \leqq \beta H(t) + \beta\gamma G(g(t)) +$$

$$+ \gamma\eta G(g(t)) \int_{g(t)}^\infty R(s)\,H(s)\,ds \leqq \beta H(t) + G(t)\,\gamma[\beta + \eta \int_t^\infty R(s)\,H(s)\,ds].$$

Taking (15) into account we have

$$|(Tx)'(g(t))/r(g(t))| \leqq \beta H(t) + \eta G(t).$$

Thus T is well defined and maps X into itself.

Now, let $x_n \to x$, $x_n' \to x'$ almost uniformly on $\langle t_0, \infty)$. Then we have

$$r^{-1}(t)\,|(Tx_n)'(t) - (Tx)'(t)| \leqq H(t)\,|f(x_n(g(t))) - f(x(g(t)))| +$$

$$+ \int_t^\infty R(s)\,H(s)\,|f'(x_n(g(s))\,x_n'(g(s))/r(g(s)) -$$

$$- f'(x(g(s)))\,x'(g(s))/r(g(s))|\ |g'(s)|\,ds.$$

Let us note that $f(x_n(g(t)))$ converges almost uniformly to $f(g(t))$ on $\langle t_0, \infty)$ since f and g are continuous. The sequence

$$(H(s)\, R(s) \mid f'(x_n(g(s)))\, x_n'(g(s)) - f'(x(g(s)))\, x'(g(s)) \mid \times$$
$$\times \mid r^{-1}(g(s))\, g'(s) \mid)_{n=1}^{\infty}$$

tends to zero almost uniformly on $\langle t_0, \infty)$ and is bounded above by the integrable function $2\gamma H(s)\, R(s)\, [\beta H(s) + \eta G(s)]$. Applying the Lebesgue dominated convergence theorem, we find that $(Tx_n)'(t) \to (Tx)'(t)$ almost uniformly on $\langle t_0, \infty)$. Moreover since $(Tx_n)'(t) - (Tx)'(t)$ is bounded above by the integrable function $2R(t) \mid \beta H(t) + \eta G(t) \mid$ and $\mid (Tx_n)(t) - (Tx)(t) \mid \leqq \int_t^\infty \mid (Tx_n)'(s) - (Tx)'(s) \mid \mathrm{d}s$ on $\langle t_0, \infty)$ we conclude $(Tx_n)(t)$ to be convergent almost uniformly on $\langle t_0, \infty)$ to $(Tx)(t)$. Thus T is continuous on X.

It remains to prove that TX is precompact. The functions $(Tx)(t)$, $(Tx)'(t)$ are evidently local uniformly bounded for all $x \in X$. Moreover for all $x \in X$ and for all $t', t'', t_0 \leqq t' \leqq t''$ we have

$$\mid (Tx)(t') - (Tx)(t'') \mid \leqq \int_{t'}^{t''} r(s) \mid Q(s)\, f(x(g(s))) \mid \mathrm{d}s +$$
$$+ \int_{t'}^{t''} r(s) \int_s^\infty \mid Q(u) \mid \; \mid f'(x(g(u)))\, x'(g(u))\, g'(u) \mid \mathrm{d}u\, \mathrm{d}s \leqq$$
$$\beta \int_{t'}^{t''} R(s)\, H(s)\, \mathrm{d}s + \eta \int_{t'}^{t''} R(s)\, G(s)\, \mathrm{d}s.$$

Thus the functions $(Tx)(t)$ are equicontinuous on every compact subinterval of $\langle t_0, \infty)$.

Because $z = Tx$ is a solution of the equation $(z'(t)/r(t))' = -q(t)\, f(x(g(t)))$ for any $x \in X$, the functions $((Tx)'(t)/r(t))'$ are uniformly bounded for all $x \in X$ on every compact subinterval of $\langle t_0, \infty)$. Thus the functions $(Tx)'(t)$ are equicontinuous on every compact subinterval of $\langle t_0, \infty)$ and TX is precompact according to Ascoli theorem.

Now applying Schauder–Tychonoff theorem to the operator T we receive at least one fixed point x of T which satisfies (10) and (11). To prove that $x(t) - \alpha \in \mathscr{L}_p$ it is sufficient to prove that the integrals

$$\int_t^\infty R(s)\, H(s)\, \mathrm{d}s, \qquad \int_t^\infty R(s)\, G(s)\, \mathrm{d}s$$

belong $\mathscr{L}_p$. But this follows from (iv) and (v) and Lemma 1.

Conversely let x be any bounded solution of (4). Define

$$y(t) = x(t) + \int_t^\infty r(s)\, Q(s)\, f(x(g(s)))\, \mathrm{d}s +$$
$$+ \int_t^\infty r(s) \int_s^\infty Q(u)\, f'(x(g(u)))\, x'(g(u))\, g'(u)\, \mathrm{d}u\, \mathrm{d}s.$$

Then y is a bounded solution of (5). Since any solution of (5) is of the form $y(t) = \alpha + \beta \int_{t_0}^{t} r(s)\,ds$, it is $y(t) \equiv \alpha$ in view of (ii). Thus $x(t)$ tends to α for $t \to \infty$ and has the properties (10), (11), (12). The proof is complete.

Remark. If $I = R$, Theorem implies $(1, p)$-integral equivalence of the equations (4) and (5) with respect to the set of bounded solutions.

As an example consider the generalized Emden–Fowler equation

$$(18) \qquad (t^{-\mu}x')' + t^{\lambda} \sin t \,|\, x\,|^{\gamma} \operatorname{sign} x = 0, \qquad t > 0, \gamma > 0, \mu \geqq -1, \lambda < 0.$$

Applying preceding theorem to the case $f(x) = |\,x\,|^{\gamma} \operatorname{sign} x$, $r(t) = t^{\mu}$, $q(t) = t^{\lambda} \sin t$, $|\,Q(t)\,| = |\int_t^{\infty} s^{\lambda} \sin s\,ds\,| \geqq 2t^{\lambda}$, we find that if $\mu + \lambda + 1 < -\frac{1}{p}$, then for any $\alpha \in R$, $\alpha \neq 0$ there exists at least one solution x of the equation (18) such that

$$(19) \qquad x(t) = \alpha + 0(t^{\mu+\lambda+1}), \qquad x'(t) = 0(t^{\mu+\lambda}), \qquad x(t) - \alpha \in \mathscr{L}_p$$

and any bounded solution x of (18) approaches to finite limit as $t \to \infty$, so that the formulas (19) hold.

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