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OSCILLATORY BEHAVIOUR OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS*

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Abstract. Some oscillation criteria for $L_n x(t) = f(t, x[g_1(t)], \ldots, x[g_m(t)])$, $n \geqq 2$ are established. Here $L_0 x(t)$, $L_k x(t) = a_k(t)(L_{k-1}x(t))^{\cdot}$, $\left(\cdot = \frac{d}{dt}\right)$, $k = 1, 2, \ldots, n$, $a_0 = a_n = 1$. The results generalize those of Werbowski [Funkcial Ekvac, 25 (1982)]. However, they are not valid for the corresponding ordinary differential equations, which is due to the fact that deviations g_i can destroy oscillations, and also can generate oscillations depending on the "size" of the deviations.

Key words. Oscillatory solutions, differential equations, deviating arguments, non-oscillatory solutions.

1. INTRODUCTION

The purpose of this paper is to establish some results concerning the oscillatory behavior of the equation

$$L_n x(t) = f(t, x[g_1(t)], \ldots, x[g_m(t)]), \qquad n \geqq 2, \tag{1}$$

where

$$L_0 x(t) = x(t), \qquad L_k x(t) = a_k(t)(L_{k-1}x(t))^{\cdot}, \qquad \left(\cdot = \frac{d}{dt}\right),$$
$$k = 1, 2, \ldots, n, \qquad a_n = a_0 = 1.$$

Here we study the nonlinear oscillations generated by general deviating arguments g_k. These results are not valid for the corresponding ordinary differential equations. For examples we refer the reader to the papers of the present authors [1–3], Kitamura and Kusano [4], Naito [5], Philos [6], Sficas and Staikos [7, 8] and Werbowski [9].

In what follows we are primarily interested in the situation in which equation (1) is oscillatory. We have been motivated by the observation that there are very few

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criteria for equation (1), with a general operator L_n, to be oscillatory, though equation (1) and its nonlinear analogue have been the object of intensive investigation in recent years. Our main results in the form of oscillation criteria are given in Sec. 2. These results generalize oscillation theorems of Werbowski [9] for the particular equation

$$x^{(n)} = f(t, x[g_1(t)], \ldots, x[g_m(t)]).$$

2. MAIN RESULTS

Consider the equation

$$L_n x(t) = f(t, x[g_1(t)], \ldots, x[g_m(t)]), \tag{1}$$

where $L_0 x(t) = x(t)$, $L_k x(t) = a_k(t)\,(L_{k-1}x(t))'$, $k = 1, 2, \ldots, n$, $a_0 = a_n = 1$, $a_i : R_+ = [0, \infty) \to (0, \infty)$ $(i = 1, 2, \ldots, n-1)$, $g_k : R_+ \to R = (-\infty, \infty)$ with $g_k(t) \to \infty$ as $t \to \infty$ $(k = 1, 2, \ldots, m)$ and $f: R_+ \times R^m \to R$ as continuous. We assume that:

$$\int^{\infty} \frac{1}{a_i(s)}\,ds = \infty \qquad (i = 1, 2, \ldots, n-1). \tag{2}$$

We further assume that there exist continuous functions $q : R_+ \to (0, \infty)$ and $F_k : R_+ \to R_+$ $(k = 1, 2, \ldots, m)$ such that

$$(-1)^n f(t, x_1, \ldots, x_m)\,\mathrm{sgn}\, x_1 \geqq q(t) \prod_{i=1}^{m} F_k(|x_k|) > 0 \tag{3}$$

for $\quad t \in R_+ \quad$ and $\quad x_k \neq 0\ (k = 1, 2, \ldots, m)$;

(4) $\quad F_k\ (k = 1, 2, \ldots, m)$ are non-decreasing on R_+;

$$F_k(uv) \geqq F_k(u)\,F_k(v) \quad (k = 1, 2, \ldots, m) \quad \text{for } u, v \in R_+; \tag{5}$$

$$\int_0^{\varepsilon} \frac{du}{F(u)} < \infty \quad \text{for some} \quad \varepsilon > 0, \quad \text{where} \quad F(u) = \prod_{i=1}^{m} F_k(u). \tag{6}$$

The domain $D(L_n)$ for L_n is defined to be the set of all functions $x: R_+ \to R$ such that $L_j x(t)$, $0 \leqq j \leqq n$, exist and are continuous on R_+. By a solution of equation (1) we mean a function $x \in D(L_n)$ which satisfies (1) on R_+. A nontrivial solution of (1) is called oscillatory if the set of its zeros is unbounded and it is called nonoscillatory otherwise. The following three lemmas will be needed in the Sequel. The first lemma can be found in [6] and the other two appear in [2]

Lemma 1. *Suppose that condition* (2) *is satisfied. Let* $y \in D(L_n)$ *be a positive bounded function on the interval* $[T, \infty)$, $T \geqq t_0$ *such that*

$$(-1)^n L_n y(t) \geqq 0 \qquad \textit{for every } t > T.$$

Then

(α) $\quad (-1)^i L_i y(t) > 0$ *for every* $t \geqq T$ $\quad (i = 1, 2, \ldots, n-1)$,

(β) *For every* u *and* v *with* $v \geqq u \geqq T$,

$$y(u) \geqq (-1)^{n-1}\left[\int_{u=s_0}^{v} \frac{1}{a_1(s_1)} \int_{s_1}^{v} \frac{1}{a_2(s_2)} \cdots \int_{s_{n-2}}^{v} \frac{1}{a_{n-1}(s_{n-1})} ds_{n-1} \ldots ds_1\right] L_{n-1}y(v).$$

Lemma 2. *Let condition* (2) *hold and let* $x \in D(L_n)$. *If* $x(t)\,L_n x(t)$ *is of constant sign and not identically zero for all large* t, *then there exist* $t_x \geqq t_0$ *and an integer* l, $0 \leqq l \leqq n$ *with* $n + l$ *even for* $x(t)\,L_n x(t) \geqq 0$ *or* $n + l$ *odd for* $x(t)\,L_n x(t) \leqq 0$ *and such that for every* $t \geqq t_x$

$$l > 0 \text{ implies } x(t)\,L_k x(t) > 0 \qquad (k = 0, 1, \ldots, l-1)$$

and

$$l \leqq n-1 \text{ implies } (-1)^{l+k}\, x(t)\,L_k x(t) > 0 \qquad (k = l, l+1, \ldots, n-1).$$

In the following lemma we let

$$\mu_i(t) = \max_{0 \leqq t_0 \leqq s \leqq t} a_i(s) \qquad (i = 1, \ldots, n-1)$$

and

$$\int^{\infty} \frac{1}{\mu_i(s)} ds = \infty, \tag{7}$$

Lemma 3. *Let* $x \in D(L_n)$ *be a positive function on* $[t_0, \infty)$. *If* $\lim_{t\to\infty} x(t) \neq 0$, *and* $L_{n-1}x(t)\,L_n x(t) < 0$ *for* $t \geqq t_1 \geqq t_0$, t_1 *sufficiently large and* $L_n x(t)$ *not identically zero for all large* t, *then there exist a* $T \geqq t_1$ *and a positive constant* M *such that*

(i) $$|L_{n-1}x(t)| > 0$$

and

(ii) $$x(t) \geqq M\left[\int_T^t \frac{1}{\mu_1(s_1)} \int_T^{s_1} \cdots \int_T^{s_{n-2}} \frac{1}{\mu_{n-1}(s_{n-1})} ds_{n-1} \ldots ds_1\right] |L_{n-1}x(t)|.$$

For convenience we use the following notations in the sequel. For any $T \geqq t_0 \geqq 0$ and all $t \geqq T$ we let

$$D_k = \{t \in R_+ : g_k(t) < t\}, \qquad D = D_1 \cap D_2 \cap \ldots \cap D_m, \qquad D_T = D \cap [T, \infty).$$

$$w(t, T) = \int_T^t \frac{1}{\mu_1(s_1)} \int_T^{s_1} \cdots \int_T^{s_{n-2}} \frac{1}{\mu_{n-1}(s_{n-1})} ds_{n-1} \ldots ds_1$$

and

$$\alpha(u, v) = \int_{u=s_0}^{v} \frac{1}{a_1(s_1)} \int_{s_1}^{v} \cdots \int_{s_{n-2}}^{v} \frac{1}{a_{n-1}(s_{n-1})} ds_{n-1} \ldots ds_1, \qquad \text{for } v \geqq u \geqq T.$$

Theorem 1. *Let conditions* (2) – (6) *hold. If*

(8) $$\int_D q(t) \prod_{k=1}^m F_k(\alpha[g_k(t), t])\, dt = \infty,$$

then every bounded solution of (1) *is oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1) and let $x(t) \neq 0$ for $t \geqq t_0$. Then there exists a $t_1 \geqq t_0$ such that $x[g_k(t)] \neq 0$ $(k = 1, \ldots, m)$ for $t \geqq t_1$. Thus, by (1) and (3) we have $(-1)^n x(t) L_n x(t) > 0$ for $t \geqq t_1$. Then from Lemma 1 it follows that

$$(-1)^i x(t) L_i x(t) > 0 \qquad (i = 0, 1, \ldots, n)$$

for $t \geqq t_2 \geqq t_1$ and

$$|x(t)| \geqq \alpha(t, s) |L_{n-1}x(s)|, \qquad \text{for } s \geqq t \geqq t_2.$$

Therefore for $t \in D_T$, $T \geqq t_2$, we obtain

(9) $$|x[g_k(t)]| \geqq \alpha[g_k(t), t] |L_{n-1}x(t)| \qquad (k = 1, 2, \ldots, m).$$

From (1), in view of (3)–(5) and (9) we get for $t \in D_T$

$$|L_n x(t)| \geqq q(t) \prod_{k=1}^m F_k(|x[g_k(t)]|) \geqq q(t) \prod_{k=1}^m F_k(\alpha[g_k(t), t] |L_{n-1}x(t)|) \geqq$$
$$\geqq q(t) \prod_{k=1}^m F_k(|L_{n-1}x(t)|) \prod_{k=1}^m F_k(\alpha[g_k(t), t]).$$

Thus

$$\int_{D_T} q(t) \prod_{k=1}^m F_k(\alpha[g_k(t), t])\, dt \leqq \int_{D_T} \frac{|L_n x(t)|}{F(|L_{n-1}x(t)|)}\, dt \leqq$$
$$\leqq \int_T^\infty \frac{|L_n x(t)|}{F(|L_{n-1}x(t)|)}\, dt \leqq \int_0^\varepsilon \frac{dy}{F(y)} < \infty,$$

where $\varepsilon = |L_{n-1}x(T)|$, which contradicts (8).

Similarly we can prove the following theorem.

Theorem 2. *Let condition* (2) *hold, and assume that there exists continuous functions* $q_k : R_+ \to (0, \infty)$ *and* $F_k : R_+ \to R_+$ $(k = 1, 2, \ldots, m)$ *such that*

(10) $$(-1)^n f(t, x_1, \ldots, x_m) \operatorname{sgn} x_1 \geqq \sum_{k=1}^m q_k(t) F_k(|x_k|) > 0$$

for $t \in R_+$ *and* $x_k \neq 0$ $(k = 1, \ldots, m)$. *If for some* $i_0 (1 \leqq i_0 \leqq m)$ *the following conditions hold:*

(11) F_{i_0} *is non-decreasing on* R_+;

(12) $$F_{i_0}(uv) \geqq F_{i_0}(u) F_{i_0}(v) \qquad \text{for } u, v \in R_+;$$

(13) $$\int_0^\varepsilon \frac{du}{F_{i_0}(u)} < \infty \qquad \text{for some } \varepsilon > 0;$$

$$\int_{D_{i_0}} q_{i_0}(t) F_{i_0}(w[g_{i_0}(t), t]) \, dt = \infty, \tag{14}$$

then all bounded solutions of (1) *are oscillatory.*

Remarks

1. If $a_i = 1$ $(i = 1, \dots, n-1)$, then our Theorems 1 and 2 and Theorems 1 and 2 in [9] are the same.

2. As in [3], if

$$\lim_{t \to \infty} \frac{1}{\alpha_1(t)} \sum_{i=0}^{k} c_i \alpha_i(t) > 0, \qquad \alpha_0(t) = 1, \tag{*}$$

for every choice of the constants c_i with $c_k > 0$ $(k = 1, 2, \dots, n-1)$, where

$$\alpha_1(t) = \int_c^t \frac{1}{a_1(s)} \, ds, \qquad \alpha_2(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^s \frac{1}{a_2(s_2)} \, ds_2 \, ds_1,$$

and

$$\alpha_k(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \dots \int_c^{s_{k-1}} \frac{1}{a_k(s_k)} \, ds_k \dots ds_1 \qquad (k = 3, \dots, n-1),$$

$t \geqq c \geqq 0$, then the conclusion of both Theorem 1 and 2 can be replaced by the statement that „every solution $x(t)$ of (1) with the property that $\frac{x(t)}{\alpha_1(t)} \to 0$ as $t \to \infty$ is oscillatory". Thus using (*) we can enlarge the class of oscillatory solutions of (1). In case $a_i = 1$ $(i = 1, \dots, n-1)$, the condition (*) is obviously verified and thus the class of bounded solutions of (1) can be replaced by the class of solutions x of (1) such that $\frac{x(t)}{t} \to 0$ as $t \to \infty$. This fact improves Theorems 1 and 2 in [9].

3. If $n = 1$, then condition (5) can be disregarded, and hence Theorems 1 and 2 are extensions of some of the results in [4].

For illustration we consider the following example:

Example. Consider the equation

$$(\sqrt{t} x')' = x^{2/3} \left[t - \frac{1}{t} \right] \operatorname{sgn} x \left[t - \frac{1}{t} \right], \qquad t > 1. \tag{15}$$

From Theorem 1, it follows that all bounded solutions of (15) are oscillatory, since

$$\int^{\infty} q(s) F(\alpha[g(s), s]) \, ds = \int^{\infty} \left[2\sqrt{s} \left(1 - \left[1 - \frac{1}{s^2} \right]^{1/2} \right) \right]^{2/3} ds \geqq \int^{\infty} \left(\frac{2\sqrt{s}}{s^2} \right)^{2/3} ds = \infty$$

We note that Theorems 1 and 2 in [9] are not applicable since $a_1(t) \neq 1$. Also Theorem 3 in [6] is not applicable, since condition (C_3) in [6] fails (i.e.)

$$\int^{\infty} \frac{g'(s)}{a_1[g(s)]} \int_{g(s)}^{s} q(s_1) \, ds_1 \, ds = \int^{\infty} [s^{-7/2} + s^{-3/2}] \left[1 - \frac{1}{s^2} \right]^{-1/2} ds < \infty.$$

Theorem 3. *Let the assumptions of Theorem 1 hold. In addition, let for T sufficiently large*

$$\int^{\infty} q(s) \prod_{k=1}^{m} F_k(w[h_k(s), T])\, ds = \infty, \tag{16}$$

where

$$h_k(t) = \begin{cases} g_k(t) & \text{for } n = 2, \\ \min(t, g_k(t)) & \text{for } n > 2. \end{cases}$$

Then for n odd, all solutions of (1) *are oscillatory, while for n even, every solution x of* (1) *is either oscillatory or* $\lim_{t\to\infty} |L_i x(t)| = \infty$ $(i = 0, 1, \ldots, n-1)$ *monotonically.*

Proof. Suppose that equation (1) has a nonoscillatory solution $x(t) \neq 0$ for $t \geqq t_0$. Since $\lim_{t\to\infty} g_k(t) = \infty$ $(k = 1, \ldots, m)$, there exists a $t_1 \geqq t_0$ such that $x[g_k(t)] \neq 0$ for $t \geqq t_1$. Then it follows from (1) and (3) that $(-1)^n x(t) L_n x(t) > 0$ for $t \geqq t_1$. And from Lemma 2 it follows, that there exist an even integer $l \in \{0, 1, \ldots, n\}$ and a number $t_2 \in [t_1, \infty)$ such that

$$\begin{aligned} x(t)\, L_i x(t) > 0 \qquad & (i = 0, 1, \ldots, l), \\ (-1)^{l+i} x(t)\, L_i x(t) > 0 \qquad & (i = l + 1, \ldots, n), \end{aligned} \tag{17}$$

for $t \geqq t_2$. From Theorem 1 it follows that the case $l = 0$ is impossible. Therefore (17) hold for $l \in \{2, \ldots, n\}$. Let n be odd. Then $n > 2$ and $l \in \{2, \ldots, n-1\}$. We shall prove that the case $l \in \{2, \ldots, n-1\}$ is also impossible. From Lemma 3 for $l \in \{2, \ldots, n-1\}$ we have

$$| x(t) \geqq M w[t, t_3] \, | L_{n-1} x(t) |, \qquad (M > 0),$$

for all large $t \geqq t_3 \geqq t_2$. Since $| x(t) |$ is increasing and $| L_{n-1} x(t) |$ is decreasing, from the above inequality we obtain

$$\begin{aligned} | x[g_k(t)] | \geqq | x[h_k(t)] | \geqq M w[h_k(t), t_3] \, | L_{n-1} x[h_k(t)] | \geqq \\ \geqq M w[h_k(t), t_3] \, | L_{n-1} x(t) | \qquad (k = 1, 2, \ldots, m), \end{aligned} \tag{18}$$

for $t \geqq T \geqq t_3$.

Then from (1), (3)–(5) and (18) we get for $t \geqq T$

$$\begin{aligned} | L_n x(t) | &\geqq q(t) \prod_{k=1}^{m} F_k(M w[h_k(t), t_3] \, | L_{n-1} x(t) |) \geqq \\ &\geqq q(t) F(M \, | L_{n-1} x(t) |) \prod_{k=1}^{m} F_k(w[h_k(t), t_3), \end{aligned}$$

which gives

$$q(t) \prod_{k=1}^{m} F_k(w[h_k(t), t_3]) \leqq \frac{| L_n x(t) |}{F(M \, | L_{n-1} x(t) |)}.$$

Integrating the last inequality from T to ∞ we have

$$\int_T^\infty q(s) \prod_{k=1}^m F_k(w[h_k(s), t_3])\, ds \leqq \frac{1}{M} \int_0^\varepsilon \frac{du}{F(u)} < \infty, \qquad \varepsilon = M \,|\, L_{n-1}x(T) \,|,$$

which contradicts assumption (16). Therefore for n odd, x is oscillatory. Let n be even. Then the inequalities (17) hold for an even integer $l \in \{2, \ldots, n\}$. As in the proof of first part, we can prove that the case $l \in \{2, \ldots, n-2\}$ is impossible. Therefore (17) holds for $l = n$, i.e.

$$(19) \qquad x(t)\, L_i x(t) > 0 \qquad (i = 0, 1, \ldots, n),$$

for $t \geqq t_2$. We shall prove that $\lim_{t\to\infty} | L_i x(t) | = \infty$ $(i = 0, 1, \ldots, n-1)$. From (19), by using an argument similar to the one used in [2, Theorem 1] it follows that there exist a $T \geqq t_2$ and a positive constant c such that

$$(20) \qquad | x[g_k(t)] | \geqq c \int_T^{g_k(t)} \frac{1}{a_1(s_1)} \int_T^{s_1} \ldots \int_T^{s_{n-2}} \frac{1}{a_{n-1}(s_{n-1})} ds_{n-1} \ldots ds_1 \geqq$$
$$\geqq c w[g_k(t), T], \qquad (k = 1, 2, \ldots, m).$$

Integrating (1) from T to t, we obtain

$$| L_{n-1}x(t) | = | L_{n-1}x(T) | + \int_T^t | f(s, x[g_1(s)], \ldots, x[g_m(s)]) | \, ds \geqq$$
$$\geqq F(c) \int_T^t q(s) \prod_{k=1}^m w[g_k(s), T]\, ds.$$

From the above inequality and (16) we derive $\lim_{t\to\infty} | L_i x(t) | = \infty$ $(i = 0, 1, \ldots, n-1)$ monotonically.

In exactly the same way we can prove the following theorem.

Theorem 4. *Suppose that the assumptions of Theorem 2 are satisfied in $n \geqq 2$. In addition let for T sufficiently large*

$$\int^\infty q_{i_0}(s) F_{i_0}(w[h_{i_0}(s), T])\, ds = \infty,$$

where h_{i_0} is as in Theorem 3. Then the conclusion of Theorem 3 holds.

Remark. If $a_i = 1$ $(i = 1, \ldots, n-1)$, then our Theorems 3 and 4 and Theorems 3 and 4 of Werbowski [9] are the same.

For illustration we consider the following example:

Example. The equation

$$(21) \qquad (\sqrt{t}(\sqrt{t}(\sqrt{t}x^{\cdot})^{\cdot})^{\cdot})^{\cdot} = 3 \,.\, (2)^{-1/3} t^{-7/6} x^{1/3}[t/2], \qquad t > 0,$$

has a nonoscillatory solution $x(t) = t^2$ satisfying $| L_i x(t) | \to \infty$ as $t \to \infty$ $(i = 0, 1, 2, 3)$. i.e. the conclusion of Theorem 3 holds. We may note that Theorem 3 in [9] is not applicable to (21) since $a_i(t) \neq 1$ $(i = 1, 2, 3)$.

Remark. The analogous results as these obtained in this paper for the case of superlinear equations seem impossible. To see this we consider the equation

$$(e^{-t}x^{\cdot})^{\cdot} = 2x^3\left[\frac{2}{3}t\right]. \tag{22}$$

It is easy to check that all conditions of Theorem 1 are satisfied $\left(\text{condition (6) is replaced by } \int_{\varepsilon}^{\infty} \frac{du}{F(u)} < \infty\right)$. However (22) has a bounded nonoscillatory solution $x = e^{-t}$.

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