Jiří Kaďourek

Multiplicity types of algebras - an example

Archivum Mathematicum, Vol. 22 (1986), No. 3, 129--133

Persistent URL: http://dml.cz/dmlcz/107255

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 22, No. 3 (1986) 129–134

MULTIPLICITY TYPES OF ALGEBRAS—AN EXAMPLE

JIŘÍ KAĎOUREK

(Received March 13, 1985)

Abstract. This note is devoted to the question of possible "generative power" of ternary operations on finite sets. To any natural number $n \geqq 2$ an example of a finite set B and a ternary operation t on B is given, such that the clone on B generated by the operation t contains no unary operation except the identity on B and it can be generated by a set of n binary operations on B, but it is not generated by any set of binary operations of cardinality less than n.

Key words. Clone, polynomial, multiplicity type, minimal type.

1. PRELIMINARIES AND NOTATIONS

The following considerations deal with finite algebras without nullary operations.

Let A be a finite set. Given any k-ary operation f on A and any l-ary operations $f_1, \dots, f_k$ on A, composing them one gets an l-ary operation on A which is denoted by $f(f_1, \dots, f_k)$. For any positive integer k and any $i \in \{1, \dots, k\}$, the i-th k-ary projection on A is denoted by ${}^A e_i^k$ or briefly by e_i^k.

By a clone on A any set of operations on A is meant, containing all projections on A and closed under composition of operations. Any finite set Φ of operations on A generates the least clone $\mathscr{F}$ on A containing Φ as a subset. It is just the set of all polynomials of the finite algebra $\mathfrak{A} = (A, \Phi)$.

By a finite multiplicity type any sequence $\mu = (\mu_0, \mu_1, \dots)$ of non-negative integers is meant, having only a finite number of positive members. The algebra $\mathfrak{A}$ is of type μ if for any non-negative integer k there exist k-ary polynomials g_ξ^k of $\mathfrak{A}$ where $\xi \in \{1, \dots, \mu_k\}$, such that the set of polynomials $\{g_\xi^k \mid k = 0, 1, \dots; \xi \in \{1, \dots, \mu_k\}\}$ generates the same clone $\mathscr{F}$ on A as the set of operations Φ does.

On the set T of all finite multiplicity types μ with $\mu_0 = 0$ an order may be introduced putting $\mu \leqq \nu$ iff $\mu_k + \mu_{k+1} + \dots \leqq \nu_k + \nu_{k+1} + \dots$ for any positive integer k. In [2], it is shown that if Φ contains no nullary operations then the set $T\mathfrak{A}$ of all finite types of the algebra $\mathfrak{A}$ is an order-filter in the ordered set $(T, \leqq)$. Any order-filter in $(T, \leqq)$ is determined by the set of all its minimal elements, and this set is finite. Therefore, the set of all finite types of $\mathfrak{A}$ is determined by the finite set $T_0\mathfrak{A}$ of all minimal types of the algebra $\mathfrak{A}$.

For more detailed information on the mentioned notions see e.g. [1], [2].

The question arises, which finite sets of pairwise incomparable finite types can be obtained as the sets of all minimal types of suitable finite algebras. One class of examples demonstrating the possible richness of these sets is given in [2]. Another special case is considered in what follows.

2. THE EXAMPLE

To any natural number $n \geqq 2$ a finite algebra $\mathfrak{B}$ is found such that

$$T_0\mathfrak{B} = \{(0, 0, n, 0, 0, \ldots), (0, 0, 0, 1, 0, \ldots)\}.$$

Let $B = \{+, 0, 1, 2, \ldots, n-1, \bar{1}, \bar{2}, \ldots, \overline{n-1}, *\}$, where all elements are pairwise different.

Let $b, b_1, b_2, \ldots, b_{n-1}$ be the binary operations on B defined as follows:

$$\begin{aligned} &b(+, 0) = 1, b(+, 1) = 2, \ldots, b(+, n-2) = n-1, b(+, n-1) = 0;\\ &b_1(+, 0) = \bar{1};\\ &b_2(+, 0) = \bar{2};\\ &\vdots\\ &b_{n-1}(+, 0) = \overline{n-1};\end{aligned}$$

and the other values are given by the scheme

$$\left.\begin{aligned} &f(\bar{r}, p) = f(p, \bar{r}) = \begin{cases} \bar{r}, \text{ if } r = p,\\ p, \text{ if } r \neq p,\end{cases}\\ &\qquad\qquad\qquad \text{for } r \in \{1, 2, \ldots, n-1\},\\ &\qquad\qquad\qquad\quad p \in \{0, 1, 2, \ldots, n-1\},\\ &f(\bar{r}, \bar{s}) = \bar{s} \quad \text{for } r, s \in \{1, 2, \ldots, n-1\},\\ &f(x, x) = x \quad \text{for } x \in \{+, 0, 1, 2, \ldots, n-1\}, \text{ and}\\ &f(x, y) = * \quad \text{in the remaining cases,}\end{aligned}\right\} \qquad (\Delta)$$

where f stands for any of the operations $b, b_1, b_2, \ldots, b_{n-1}$.

Let t be the ternary operation on B given by the equality:

$$t = b(b_{n-1}(e_1^3, e_2^3), \ldots b(b_2(e_1^3, e_2^3), b(b_1(e_1^3, e_2^3), e_3^3)) \ldots).$$

Let $\mathfrak{B} = (B, \{t\})$.

Now $(0, 0, 0, 1, 0, \ldots) \in T\mathfrak{B}$ and $(0, 0, n, 0, 0, \ldots) \in T\mathfrak{B}$ as well since the set of operations $\{b, b_1, b_2, \ldots, b_{n-1}\}$ generates the same clone on B as the operation t does, as it follows from the equalities:

$$b = \underbrace{t(e_1^2, e_1^2, \dots t(e_1^2, e_1^2, t(e_1^2, e_1^2, e_2^2)) \dots)}_{n-1},$$

$$b_{n-1} = t(e_1^2, e_2^2, t(e_1^2, e_1^2, e_2^2)),$$
$$b_{n-2} = t(e_1^2, e_2^2, t(e_1^2, e_1^2, t(e_1^2, e_1^2, e_2^2))),$$
$$\vdots$$
$$b_1 = t(e_1^2, e_2^2, \underbrace{t(e_1^2, e_1^2, \dots t(e_1^2, e_1^2, t(e_1^2, e_1^2, e_2^2)) \dots)}_{n-1}).$$

The verification of these equalities is mechanic and is omitted.

Evidently, the only unary polynomial of $\mathfrak{B}$ is the identity ${}^Be_1^1$, because all the operations $b, b_1, b_2, \dots, b_{n-1}$ are idempotent.

It remains to show that for any type $\mu \in T\mathfrak{B}$ at least one of the two conditions $\mu \geqq (0, 0, n, 0, 0, \dots)$ or $\mu \geqq (0, 0, 0, 1, 0, \dots)$ holds. But the types satisfying no such condition are just of the form $(0, i, j, 0, 0, \dots)$, where i is a non-negative integer and $j \in \{0, 1, 2, \dots, n-1\}$. Since $\mathfrak{B}$ has no unary polynomials except the unary projection, it is enough to prove that $(0, 0, j, 0, 0, \dots) \in T\mathfrak{B}$ implies $j \geqq n$.

Let $b^\alpha, {}^\gamma b_\beta, {}^\gamma b^\alpha, b^*$, where $\alpha, \gamma \in \{0, 1, 2, \dots, n-1\}$ and $\beta \in \{1, 2, \dots, n-1\}$, be the binary operations on B defined as follows:

$b^\alpha(+, p)$ for any $p \in \{0, 1, 2, \dots, n-1\}$ is the number from $\{0, 1, 2, \dots, n-1\}$ congruent to $p + \alpha$ modulo n;

${}^\gamma b_\beta(+, \gamma) = \bar{\beta}$;

${}^\gamma b^\alpha(+, \gamma) = \alpha$;

and the other values are given by the scheme (Δ), where f stands for any of the operations $b^\alpha, {}^\gamma b_\beta, {}^\gamma b^\alpha, b^*$.

All the operations $b^\alpha, {}^\gamma b_\beta, {}^\gamma b^\alpha, b^*$ are binary polynomials of $\mathfrak{B}$, as it follows from the equalities:

$$b^1 = b,$$
$$b^\alpha = b(e_1^2, b^{\alpha-1}) \text{ for } \alpha \in \{2, \dots, n-1\},$$
$$b^0 = b(e_1^2, b^{n-1}),$$
$${}^0b_\beta = b_\beta,$$
$${}^\gamma b_\beta = b_\beta(e_1^2, b^{n-\gamma}) \text{ for } \beta, \gamma \in \{1, 2, \dots, n-1\},$$
$${}^0b^0 = b(b_1, b^0),$$
$${}^0b^\alpha = b^\alpha(e_1^2, {}^0b^0),$$
$${}^\gamma b^0 = b({}^\gamma b_1, b^{n-\gamma}),$$
$${}^\gamma b^\alpha = b^\alpha(e_1^2, {}^\gamma b^0) \text{ for } \alpha, \gamma \in \{1, 2, \dots, n-1\},$$
$$b^* = b(b, e_2^2).$$

Moreover, all the operations $b, b_1, b_2, \ldots, b_{n-1}$ occur among those polynomials.

Let $\Psi' = \{b^\alpha, {}^\gamma b_\beta, {}^\gamma b^\alpha, b^* \mid \alpha, \gamma \in \{0, 1, 2, \ldots, n-1\}, \beta \in \{1, 2, \ldots, n-1\}\}$. Let $\Psi = \{{}^B e_1^2, {}^B e_2^2\} \cup \Psi' \cup \{f(e_2^2, e_1^2) \mid f \in \Psi'\}$. Then Ψ is just the set of all binary polynomials of the algebra $\mathfrak{B}$. To see that, it remains to check that the set Ψ is closed under composition of polynomials. This may be done simultaneously with the verification of the following assertion:

Whenever $f, g, h \in \Psi$ are such that $f(g, h) = b^\alpha$ or $f(g, h) = b^\alpha(e_2^2, e_1^2)$ for some $\alpha \in \{0, 1, 2, \ldots, n-1\}$, then there is $\alpha' \in \{0, 1, 2, \ldots, n-1\}$ such that one of the polynomials f, g, h is equal to $b^{\alpha'}$ or to $b^{\alpha'}(e_2^2, e_1^2)$. Whenever $f, g, h \in \Psi$ are such that $f(g, h) = {}^\gamma b_\beta$ or $f(g, h) = {}^\gamma b_\beta(e_2^2, e_1^2)$ for some $\beta \in \{1, 2, \ldots, n-1\}$ and $\gamma \in \{0, 1, 2, \ldots, n-1\}$, then there is $\gamma' \in \{0, 1, 2, \ldots, n-1\}$ such that one of the polynomials f, g, h is equal to ${}^{\gamma'} b_\beta$ or to ${}^{\gamma'} b_\beta(e_2^2, e_1^2)$.

To verify the last two assertions, it is enough to take $f \in \Psi'$ and to consider the following cases:

(i) $g = {}^B e_1^2, h \in \Psi'$,
(ii) $g = {}^B e_2^2, h \in \Psi'$,
(iii) $g \in \Psi', h = {}^B e_1^2$,
(iv) $g \in \Psi', h = {}^B e_2^2$.
(v) $g, h \in \Psi'$,
(vi) $g \in \Psi'$ and $h = h'(e_2^2, e_1^2)$, where $h' \in \Psi'$.

The cases (iii) and (vi) are easy. In (ii), (iv) and (v) a differentiation is to be done with respect to possible kinds of g and h. The remaining case (i) needs a full differentiation according to various kinds of f and h. The complete calculation is a routine and is omitted.

Now, let $(0, 0, j, 0, 0, \ldots) \in T\mathfrak{B}$ and let $\{f_1, \ldots, f_j\}$ be the corresponding set of binary polynomials of $\mathfrak{B}$ generating the same clone on B as the set of operations $\{b, b_1, b_2, \ldots, b_{n-1}\}$ does. Without loss of generality one may suppose $\{f_1, \ldots, f_j\} \subseteq \Psi'$. From the preceding assertion it follows that there is $\alpha \in \{0, 1, 2, \ldots, n-1\}$ such that $b^\alpha \in \{f_1, \ldots, f_j\}$ and, further, for each $\beta \in \{1, 2, \ldots, n-1\}$ there is $\gamma \in \{0, 1, 2, \ldots, n-1\}$ such that ${}^\gamma b_\beta \in \{f_1, \ldots, f_j\}$. But these operations are pairwise different. Hence $j \geqq n$.

3. ADDENDUM

For any natural number $m \geqq 4$ there exists a finite algebra $\mathfrak{D}$ such that

$$T_0\mathfrak{D} = \{(0, 0, 2, 0, \ldots), (\underbrace{0, 0, \ldots, 0}_{m}, 1, 0, \ldots)\}.$$

To any natural number $m \geqq 4$ an example of a finite set D and of two binary operations u and w on D may be provided such that the clone on D generated by the set of operations $\{u, w\}$ can be generated by a single m-ary operation d on D, but it is not generated by any single operation of the arity less than m. Moreover, the operations u, w can be chosen in such a way that $u(e_1^1, e_1^1) = w(e_1^1, e_1^1)$ is a constant map of the set D to a certain element of D, the algebra $\mathfrak{D} = (D, \{u, w\})$ has no other unary polynomials than the identity ${}^D e_1^1$ and the mentioned constant map, and if

$$d = w(u(u(e_1^m, e_m^m), \ldots u(u(e_1^m, e_5^m), u(u(e_1^m, e_4^m), u(u(e_1^m, e_3^m), e_2^m))) \ldots), e_2^m),$$

then the following equalities hold:

$$u = d(e_1^2, e_2^2, e_2^2, e_2^2, e_2^2, \ldots, e_2^2),$$

$$w = d(e_1^2, e_2^2, u, u(e_1^2, u), u(e_1^2, u(e_1^2, u)), \ldots, \underbrace{u(e_1^2, \ldots u(e_1^2, u(e_1^2,}_{m-2} u)) \ldots)).$$

Unfortunately, the description of the operations u and w remains still too much complicated and the proof of the corresponding assertion is lengthy and cannot be presented here.

Acknowledgement. The author is indebted to L. Polák who made him familiar with the related topics and suggested him the problem under consideration.

REFERENCES

[1] J. Niederle and L. Polák: *The category of non-indexed algebras and weak homomorphisms*, Colloquium Math. XLI (1979), 167–179.

[2] L. Polák: *Multiplicity types of algebras – chain algebras*, Finite Algebra and Multiple-valued Logic, Szeged, Hungary, Colloquia Math. Soc. János Bolyai 28 (1979), 573–585.

Jiří Kaďourek
Department of Mathematics, Faculty of Science
J. E. Purkyně University
Janáčkovo nám. 2a
662 95 Brno
Czechoslovakia