Archivum Mathematicum

Vítězslav Novák

Universal cyclic relations

Archivum Mathematicum, Vol. 22 (1986), No. 3, 125--128

Persistent URL: http://dml.cz/dmlcz/107254

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 22, No. 3 (1986) 125–128

UNIVERSAL CYCLIC RELATIONS

VÍTĚZSLAV NOVÁK

(Received March 14, 1985)

Abstract. Let n be a positive integer, m a cardinal. Denote by $\boldsymbol{n}$ an n-element cycle, i.e. an n-element set with a cyclic n-ary relation. Then for any cyclic n-ary structure $\boldsymbol{G}$ there exists in a structure of type $(2\boldsymbol{n})^m$ a substructure $\boldsymbol{G}'$ such that $\boldsymbol{G}$ is a strong homomorphic image of $\boldsymbol{G}'$.

Key words. Cyclic n-ary structure, power of structures, strong homomorphism.

MS Classification. 04 A 05.

1. Introduction. In [6] there is constructed, for any cardinal m, an m-universal cyclically ordered set. A cyclically ordered set ([4]) is a pair (G, C) where G is a nonempty set and C a ternary relation on G which is asymmetric ($(x, y, z) \in C$ implies $(z, y, x) \bar{\in} C$), cyclic ($(x, y, z) \in C$ implies $(y, z, x) \in C$) and transitive ($(x, y, z) \in C$ and $(x, z, u) \in C$ imply $(x, y, u) \in C$). By an m-universality the following property is meant: For any cyclically ordered set (G, C), where card $G = m$, there exists a substructure in the constructed m-universal set such that (G, C) is its strong homomorphic image (the definition of a strong homomorphism follows below). The aim of this note is to construct an m-universal set for sets with a cyclic relation of arbitrary arity.

2. Basic notions. Let $G \neq \emptyset$ be a set, let n be a positive integer. An arbitrary subset C of the n-th cartesian power G^n of the set G is called an *n-ary relation* on G. If G is a set and C is an n-ary relation on G, then the pair $\boldsymbol{G} = (G, C)$ will be called an *n-ary structure*. An n-ary structure $\boldsymbol{G} = (G, C)$ is *cyclic*, iff the relation C is cyclic, i.e. it has the property

$$(x_1, x_2, \ldots, x_n) \in C \Rightarrow x_i \neq x_j \text{ for } i \neq j \text{ and } (x_2, x_3, \ldots, x_n, x_1) \in C.$$

Let $\boldsymbol{G} = (G, C)$ be a cyclic n-ary structure and $x \in G$. The element x is *nonisolated* iff there exist elements $x_2, \ldots, x_n \in G$ such that $(x, x_2, \ldots, x_n) \in C$; otherwise it is *isolated*.

Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be n-ary structures. A mapping $f: G \to H$ is called a *homomorphism* of $\boldsymbol{G}$ into $\boldsymbol{H}$ iff it has a property

$$x_1, x_2, \ldots, x_n \in G, (x_1, x_2, \ldots, x_n) \in C \Rightarrow (f(x_1), f(x_2), \ldots, f(x_n)) \in D.$$

We denote by Hom $(\boldsymbol{G}, \boldsymbol{H})$ the set of all homomorphisms of $\boldsymbol{G}$ into $\boldsymbol{H}$. A bijective homomorphism f of $\boldsymbol{G}$ onto $\boldsymbol{H}$ such that f^{-1} is a homomorphism of $\boldsymbol{H}$ onto $\boldsymbol{G}$ is an *isomorphism* of $\boldsymbol{G}$ onto $\boldsymbol{H}$. A homomorphism f of $\boldsymbol{G}$ into $\boldsymbol{H}$ is called *strong* iff it is surjective and has a property

$$y_1, y_2, \ldots, y_n \in H, (y_1, y_2, \ldots, y_n) \in D \Rightarrow \text{there exist } x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2), \ldots, x_n \in f^{-1}(y_n) \text{ such that } (x_1, x_2, \ldots, x_n) \in C.$$

Let $\boldsymbol{G} = (G, C), \boldsymbol{H} = (H, D)$ be n-ary structures. A *power* $\boldsymbol{G}^{\boldsymbol{H}}$ is an n-ary structure (K, E) where $K = \text{Hom}(\boldsymbol{H}, \boldsymbol{G})$ and the relation E is defined pointwise, i.e. for $f_1, f_2, \ldots, f_n \in \text{Hom}(\boldsymbol{H}, \boldsymbol{G})$ there is

$$(f_1, f_2, \ldots, f_n) \in E \Leftrightarrow (f_1(x), f_2(x), \ldots, f_n(x)) \in C \qquad \text{for any } x \in H.$$

By the symbol $\boldsymbol{n}$ we denote an n-element *cycle*

$$(\{1, 2, \ldots, n\}, \{(1, 2, \ldots, n), (2, 3, \ldots, n, 1), \ldots, (n, 1, 2, \ldots, n-1)\})$$

and also a type of this structure; $2\boldsymbol{n}$ denotes a structure which is a direct sum of two n-element cycles, i.e.

$$2\boldsymbol{n} = (\{1, 2, \ldots, n, 1', 2', \ldots, n'\}, \{(1, 2, \ldots, n), (2, 3, \ldots, n, 1), \ldots \ldots, (n, 1, 2, \ldots, n-1), (1', 2', \ldots, n'), (2', 3', \ldots, n', 1'), \ldots \ldots, (n', 1', 2', \ldots, (n-1)')\}),$$

and a type of this structure.

An n-ary structure $\boldsymbol{G} = (G, C)$ is *discrete*, iff $C = \emptyset$; a type of such a structure is denoted by $\boldsymbol{m}$ where m is a cardinality of G.

3. Theorem. *Let m be a cardinal. For any cyclic n-ary structure $\boldsymbol{G} = (G, C)$ with card $G = m$ there exists a substructure $\boldsymbol{G}'$ of a structure of type $(2\boldsymbol{n})^{\boldsymbol{m}}$ such that $\boldsymbol{G}$ is a strong homomorphic image of $\boldsymbol{G}'$.*

Proof. Let $\boldsymbol{G} = (G, C)$ be a cyclic n-ary structure with card $G = m$. Let M be any set with card $M = m$ and $\boldsymbol{M} = (M, \emptyset)$ a discrete n-ary structure with carrier M. A power $(2\boldsymbol{n})^{\boldsymbol{M}}$ has a type $(2\boldsymbol{n})^{\boldsymbol{m}}$ and its carrier, i.e. the set Hom $(\boldsymbol{M}, 2\boldsymbol{n})$, contains all mappings $f : M \to 2\boldsymbol{n}$. Denote by E the n-ary relation of this power. Let $i : G \to M$ be a bijection. Let us define a mapping T of G into the set of all subsets of Hom $(\boldsymbol{M}, 2\boldsymbol{n})$ in the following manner:

(1) Let $x \in G$ be nonisolated. We denote by $S(x)$ the set of all $(n-1)$-sequences $(x_2, x_3, \ldots, x_n)$ of elements in G such that $(x, x_2, \ldots, x_n) \in C$. For any $s = (x_2, x_3, \ldots, x_n) \in S(x)$ let $T(s)$ be the set of all mappings $f \in \text{Hom}(\boldsymbol{M}, 2\boldsymbol{n})$ with the properties

$$f(i(x)) = 1, \; f(i(x_2)) = n, \; f(i(x_3)) = n-1, \ldots, f(i(x_n)) = 2,$$

f is a constant mapping on M-$\{i(x), i(x_2), \ldots, i(x_n)\}$ with a value in the set $\{1', 2', \ldots, n'\}$. Finally, we put $T(x) = \bigcup_{s \in S(x)} T(s)$.

(2) Let $x \in G$ be isolated. Then we put $T(x) = \{f\}$ where $f(i(x)) = 1$ and $f(t) = 1'$ for any $t \in M - \{i(x)\}$. Note that $f \in T(x)$ implies $f(i(x)) = 1$ and $f(t) \neq 1$ for any $t \in M - \{i(x)\}$. From this there follows $x, y \in G$, $x \neq y \Rightarrow T(x) \cap T(y) = \emptyset$, for $f \in T(x) \cap T(y)$ implies $f(i(x)) = 1 = f(i(y))$, thus $i(x) = i(y)$ and $x = y$. The mapping T is therefore injective. We define an n-ary relation D on the set $\{T(x); x \in G\}$ by

$$(T(x_1), T(x_2), \dots, T(x_n)) \in D \Leftrightarrow \text{there exist } f_1 \in T(x_1), f_2 \in T(x_2), \dots, f_n \in T(x_n) \text{ with } (f_1, f_2, \dots, f_n) \in E.$$

We show that T is an isomorphism of $\mathbf{G}$ onto $(\{T(x); x \in G\}, D)$. Clearly, $T : G \to \{T(x); x \in G\}$ is bijective. Let $x_1, x_2, \dots, x_n \in G$, $(x_1, x_2, \dots, x_n) \in C$. Let us define mappings $f_1, f_2, \dots, f_n : M \to \mathbf{2n}$ so:

$$f_1(i(x_1)) = 1,\ f_1(i(x_2)) = n,\ f_1(i(x_3)) = n - 1, \dots, f_1(i(x_n)) = 2,\ f_1(t) = 1' \quad \text{for } t \in M - \{i(x_1), \dots, i(x_n)\}$$

$$f_2(i(x_1)) = 2,\ f_2(i(x_2)) = 1,\ f_2(i(x_3)) = n, \dots, f_2(i(x_n)) = 3,\ f_2(t) = 2' \quad \text{for } t \in M - \{i(x_1), \dots, i(x_n)\}$$

$$\vdots$$

$$f_n(i(x_1)) = n,\ f_n(i(x_2)) = n - 1,\ f_n(i(x_3)) = n - 2, \dots, f_n(i(x_n)) = 1,\ f_n(t) = n' \quad \text{for } t \in M - \{i(x_1), \dots, i(x_n)\}$$

Then $\mathbf{s}_1 = (x_2, x_3, \dots, x_n) \in S(x_1)$, $f_1 \in T(\mathbf{s}_1)$, $\mathbf{s}_2 = (x_3, x_4, \dots, x_n, x_1) \in S(x_2)$, $f_2 \in T(\mathbf{s}_2)$, …, $\mathbf{s}_n = (x_1, x_2, \dots, x_{n-1}) \in S(x_n)$, $f_n \in T(\mathbf{s}_n)$. From this $f_1 \in T(x_1)$, $f_2 \in T(x_2)$, …, $f_n \in T(x_n)$. Further, we see easily $(f_1, f_2, \dots, f_n) \in E$. Thus $(T(x_1), T(x_2), \dots, T(x_n)) \in D$.

On the other hand, let $x_1, x_2, \dots, x_n \in G$ and $(T(x_1), T(x_2), \dots, T(x_n)) \in D$. Then there exist $f_1 \in T(x_1)$, $f_2 \in T(x_2)$, …, $f_n \in T(x_n)$ such that $(f_1, f_2, \dots, f_n) \in E$. Thus

$f_1(i(x_1)) = 1$, $f_2(i(x_2)) = 1$, …, $f_n(i(x_n)) = 1$ and hence it must be

$$f_1(i(x_1)) = 1,\ f_2(i(x_1)) = 2,\ f_3(i(x_1)) = 3, \dots, f_n(i(x_1)) = n$$
$$f_1(i(x_2)) = n,\ f_2(i(x_2)) = 1,\ f_3(i(x_2)) = 2, \dots, f_n(i(x_2)) = n - 1$$
$$\vdots$$
$$f_1(i(x_n)) = 2,\ f_2(i(x_n)) = 3,\ f_3(i(x_n)) = 4, \dots, f_n(i(x_n)) = 1.$$

By definition of the set $T(x_1)$ the element x_1 is nonisolated in G and for the sequence $\mathbf{s} = (x_2, x_3, \dots, x_n)$ it holds $\mathbf{s} \in S(x_1)$ and $f_1 \in T(\mathbf{s})$. Thus $(x_1, x_2, \dots, x_n) \in C$.

We have shown that $T : G \to \{T(x); x \in G\}$ is an isomorphism of $\mathbf{G}$ onto $(\{T(x); x \in G\}, D)$; as a consequence we have that the n-ary structure $(\{T(x); x \in G\}, D)$ is cyclic.

Further, we put $G' = \bigcup_{x \in G} T(x)$ and $\mathbf{G}' = (G', E \cap G'^n)$. $\mathbf{G}'$ is a substructure of $(\mathbf{2n})^M$. Let us define a mapping $\varphi : G' \to \{T(x); x \in G\}$ by $\varphi(f) = T(x)$ where $f \in T(x)$. We show that φ is a strong homomorphism of $\mathbf{G}'$ onto $(\{T(x); x \in G\}, D)$. Clearly, φ is a surjective mapping. Let $f_1, f_2, \dots, f_n \in G'$, $(f_1, f_2, \dots, f_n) \in E$ and

$f_1 \in T(x_1), f_2 \in T(x_2), \ldots, f_n \in T(x_n)$. Then $\varphi(f_1) = T(x_1), \varphi(f_2) = T(x_2), \ldots, \varphi(f_n) = T(x_n)$ and $(T(x_1), T(x_2), \ldots, T(x_n)) \in D$ by definition of the relation D. Conversely, let $(T(x_1), T(x_2), \ldots, T(x_n)) \in D$. Then, by definition of D, there exist $f_1 \in T(x_1)$ $f_2 \in T(x_2), \ldots, f_n \in T(x_n)$ such that $(f_1, f_2, \ldots, f_n) \in E$ and, clearly, $f_1 \in \varphi^{-1}(T(x_1)), f_2 \in \varphi^{-1}(T(x_2)), \ldots, f_n \in \varphi^{-1}(T(x_n))$. Thus φ is a strong homomorphism of G' onto $(\{T(x); x \in G\}, D)$. But then $T^{-1} \circ \varphi$, as a superposition of a strong homomorphism and an isomorphism, is a strong homomorphism of G' onto G and the theorem is proved.

4. Remark. A cyclic n-ary structure of type $(2n)^m$ is thus m-universal in the following sense: To obtain all cyclic n-ary structures of cardinality m up to isomorphisms, it suffices to find all substructures of $(2n)^m$ and all their strong homomorphic images.

REFERENCES

[1] G. Birkhoff: *Generalized arithmetic*, Duke Math. Journ. 9 (1942), 283–302.

[2] J. B. Johnston: *Universal infinite partially ordered sets*, Proc. Am. Math. Soc. 7 (1956), 507–514.

[3] B. Jónsson: *Universal relational systems*, Math. Scand. 4 (1956), 193–208.

[4] V. Novák: *Cyclically ordered sets*, Czech. Math. Journ. 32 (107), (1982), 460–473.

[5] V. Novák: *On a power of relational structures*, Czech. Math. Journ. 35 (110), (1985), 167–172.

[6] V. Novák, M. Novotný: *Universal cyclically ordered sets*, Czech. Math. Journ. 35 (110), (1985), 158–161.

[7] M. Novotný: *O representaci částečně uspořádaných množin posloupnostmi nul a jedniček* (*On representation of partially ordered sets by sequences of zeros and ones*), Čas. pěst. mat. 78 (1953), 61–64.

V. Novák
Department of Mathematics
Faculty of Science, J. E. Purkyně University
Janáčkovo nám. 2a
662 95 Brno
Czechoslovakia