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LEPAGEAN 2-FORMS IN HIGHER ORDER HAMILTONIAN MECHANICS I. REGULARITY*)

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Abstract. In this paper the notion of the Lepagean equivalent of a locally variational form (= Euler-Lagrange form) is introduced. Applied to the higher order Hamiltonian mechanics it enables one to reformulate the Hamilton theory for the whole set of equivalent lagrangians. Consequently, a generalization of the standard regularity condition is obtained, and a general Legendre transformation is proposed and investigated. These concepts carry over all main properties of the classical first order theory.

Key words. Lepagean form, locally variational form, Hamilton form, Hamilton extremal, regular variational problem, Legendre transformation, Hamilton equations.

MS classification. 58 F 05, 70 H 05.

1. NOTATION

Throughout this paper we will consider smooth finite-dimensional manifolds and smooth mappings, and use the standard summation convention (unless otherwise explicitly stated). Our underlying structure will be a fibered manifold $\pi : Y \to X$, $\dim X = 1$, $\dim Y = m + 1$. We denote by d the exterior derivative of forms, i_ξ the contraction by a vector ξ, * the pull-back. The s-jet prolongation of Y (resp. the natural projection of jet spaces, resp. the s-jet prolongation of $j^r Y$) is denoted by $\pi_s : j^s Y \to X$ (resp. $\pi_{r,s} : j^r Y \to j^s Y$, where $0 \leqq s < r$, resp. $(\pi_r)_s : j^s(j^r Y) \to \to X$). The fiber chart on Y (resp. the associated chart on $j^r Y$, resp. on $j^s(j^r Y)$) is denoted by (V, ψ), $\psi = (t, q^\sigma)$ (resp. (V_r, ψ_r), $\psi_r = (t, q^\sigma_i)$, resp. $((V_r)_s, (\psi_r)_s)$ $(\psi_r)_s = (t, q^\sigma_{i,k})$), where $1 \leqq \sigma \leqq m$, $0 \leqq i \leqq r$, $0 \leqq k \leqq s$; in particular, $q^\sigma_{0,0} = = q^\sigma$, $q^\sigma_{i,0} = q^\sigma_i$, $q^\sigma_{0,k} = q^\sigma_{,k}$. The set of (local) sections of π is denoted by $\Gamma(\pi)$. If $\gamma \in \Gamma(\pi)$ is a section then the s-jet prolongation of γ is denoted by $j^s\gamma$. For the module of p-forms (resp. π_s-horizontal p-forms, resp. $\pi_{s,r}$-horizontal p-forms,

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$0 \leqq r < s$, resp. k-contact p-forms, $k \geqq 1$) on j^sY the notation $\Omega^p(j^sY)$ (resp. $\Omega^p_X(j^sY)$, resp. $\Omega^p_{j^rY}(j^sY)$, resp. $\Omega^{p-k,k}(j^sY)$) is used. The π-horizontalization, π-contactization, π_{s-1}-horizontalization, and π_{s-1}-contactization are denoted by $h, p, \bar{h}$, and $\bar{p}$, respectively. Recall that in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y it holds $h(f) = f \circ \pi_{r+1,r}$, resp. $\bar{h}(f) = f \circ (\pi_{s-1})_{r+1,r}$ for a function f on $V_r \subset j^rY$, resp. on $(V_{s-1})_r \subset j^r(j^{s-1}Y)$, and

$$(1.1) \qquad h(dt) = dt, \qquad h(dq_i^\sigma) = q_{i+1}^\sigma \, dt, \qquad 0 \leqq i \leqq r-1,$$

$$\bar{h}(dt) = dt, \qquad \bar{h}(dq_{i,k}^\sigma) = q_{i,k+1}^\sigma \, dt, \qquad 0 \leqq i \leqq s-1, \qquad 0 \leqq k \leqq r-1.$$

The forms $p(dq_i^\sigma)$ (resp. $\bar{p}(dq_{i,k}^\sigma)$) on j^rY (resp. on $j^r(j^{s-1}Y)$) are denoted by ω_i^σ (resp. $\bar{\omega}_{i,k}^\sigma$); we note that

$$(1.2) \qquad \begin{aligned} \omega_i^\sigma &= dq_i^\sigma - q_{i+1}^\sigma \, dt, \qquad 0 \leqq i \leqq r-1, \\ \bar{\omega}_{i,k}^\sigma &= dq_{i,k}^\sigma - q_{i,k+1}^\sigma \, dt, \qquad 0 \leqq i \leqq s-1, \qquad 0 \leqq k \leqq r-1. \end{aligned}$$

The formal derivative operator with respect to t relative to h (resp. $\bar{h}$) is denoted by d/dt (resp. $\bar{d}/dt$).

Recall that each form $\eta \in \Omega^p_{j^{s-1}Y}(j^sY)$ admits a unique decomposition

$$(1.3) \qquad \eta = h(\eta) + \eta_1 + \eta_2 + \dots + \eta_p,$$

where $\eta_k \in \Omega^{p-k,k}(j^sY)$, $1 \leqq k \leqq p$.

2. INTRODUCTORY REMARKS

In this section we recall some facts concerning the theory of Lepagean forms [6], [9] and locally variational forms [7], [8], adapted to fibered manifolds with one-dimensional bases, and we recapitulate the main ideas of the theory of Hamilton extremals contained in [10], [11] since we shall follow them in this paper; we shortly comment our approach to the theory.

Let $s \geqq 2$ be an integer. A form $\varrho \in \Omega^1_{j^{s-2}Y}(j^{s-1}Y)$ is called *Lepagean* if $d\varrho$ admits a decomposition $\pi^*_{s,s-1} \, d\varrho = E + F$ where $E \in \Omega^{1,1}_Y(j^sY)$ and $F \in \Omega^{0,2}(j^sY)$. If ϱ is Lepagean then there exists an integer r such that $h(\varrho) \in \Omega^1_X(j^rY)$ (up to $\pi_{r,s-1}$ or $\pi_{s-1,r}$). The form $h(\varrho)$ is denoted by λ and called a *lagrangian of order r* for π, and the corresponding Lepagean form ϱ is said to be the *Lepagean equivalent of* λ. To each lagrangian λ the Lepagean equivalent exists and is unique; hereinafter it will be denoted by Θ_λ. (We note that for $\dim X > 1$ this generally is not the case [9]). In a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y, where $\lambda = L\, dt$, we get

$$(2.1) \qquad \Theta_\lambda = L\, dt + \sum_{i=0}^{r-1} f_\sigma^{i+1} \omega_i^\sigma,$$

where

$$(2.2)\qquad f^i_\sigma = \sum_{k=0}^{r-i} (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q^\sigma_{i+k}}\right), \qquad 1 \leqq i \leqq r.$$

In this way there arises a mapping $Lep : \Omega^1_X(j^r Y) \ni \lambda \to \Theta_\lambda \in \Omega^1_{j^{r-1}Y}(j^{2r-1}Y)$ such that

$$(2.3)\qquad \pi^*_{2r,2r-1}\, d\Theta_\lambda = E_\lambda + F_\lambda,$$

where

$$(2.4)\qquad E_\lambda = E_\sigma(L)\, dq^\sigma \wedge dt, \qquad E_\sigma(L) = \sum_{k=0}^{r} (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q^\sigma_k}\right),$$

and

$$(2.5)\qquad F_\lambda = \sum_{i=0}^{r-1} \sum_{k=0}^{2r-1-i} \frac{\partial f^{i+1}_\sigma}{\partial q^\nu_k}\, \omega^\nu_k \wedge \omega^\sigma_i.$$

The form $E \in \Omega^{1,1}_Y(j^{2r}Y)$ is called the *Euler-Lagrange form* of λ, and the mapping $\varepsilon : \Omega^1_X(j^r Y) \ni \lambda \to E_\lambda \in \Omega^{1,1}_Y(j^{2r}Y)$ is called the *Euler-Lagrange mapping*. Two lagrangians $\lambda_1 \in \Omega^1_X(j^k Y)$, $\lambda_2 \in \Omega^1_X(j^l Y)$, $k \geqq l$, are said to be *equivalent*, $\lambda_1 \sim \lambda_2$, if $E_{\lambda_1} = E_{\lambda_2}$ (up to a projection).

Let $E \in \Omega^{1,1}_Y(j^s Y)$ be a form. E is called *variational* if there exist an integer r and a lagrangian $\lambda \in \Omega^1_X(j^r Y)$ such that (up to a projection) $E = E_\lambda$; E is called *locally variational* if $j^s Y$ can be covered by open sets in such a way that E restricted to each of these sets is variational. We recall a method for constructing a lagrangian to a locally variational form (see e.g. [8]). Let $\alpha \in \Omega^2_{j^{s-1}Y}(j^s Y)$ be a form, suppose that $d\alpha = 0$. There exists a covering of $j^s Y$ by open sets W such that on each W $\alpha = d\varrho$ for a form ϱ defined on W. Let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y satisfying $V_s \cap W \neq \emptyset$, let

$$(2.6)\qquad \alpha = \sum_{i=0}^{s-1} E^i_\sigma \omega^\sigma_i \wedge dt + \sum_{i,k=0}^{s-1} F^{ik}_{\sigma\nu} \omega^\sigma_i \wedge \omega^\nu_k, \qquad F^{ik}_{\sigma\nu} = -F^{kl}_{\nu\sigma},$$

be the chart expression of α. Define a mapping $\chi_s : [0, 1] \times U \to U$ where $U = V_s \cap W$ by

$$(2.7)\qquad \chi_s(u, (t, q^\sigma, \ldots, q^\sigma_s)) = (t, uq^\sigma, \ldots, uq^\sigma_s),$$

and put

$$(2.8)\qquad A\alpha = \sum_{i=0}^{s-1} q^\sigma_i \int_0^1 (E^i_\sigma \circ \chi_s)\, du)\, dt + \sum_{k=0}^{s-1} \left(\sum_{i=0}^{s-1} 2q^\sigma_i \int_0^1 (F^{ik}_{\sigma\nu} \circ \chi_s)\, u\, du\right) \omega^\nu_k.$$

It holds $dA\alpha + A\, d\alpha = \alpha$; hence $\varrho = A\alpha$. Obviously, if $E \in \Omega^{1,1}_Y(j^s Y)$ is a locally variational form then $\lambda = AE$ is a local lagrangian of order s for π such that (up to a projection) $E = E_\lambda$ on U.

Now, let us turn to the theory of Hamilton extremals of a lagrangian, as it is investigated in [10] and [11]. Let $\lambda \in \Omega^1_X(j^r Y)$ be a lagrangian, Θ_λ its Lepagean equivalent. A section $\delta \in \Gamma(\pi_{2r-1})$ is called a *Hamilton extremal of the lagrangian λ*

if for each π_{2r-1}-vertical vector field ξ on $j^{2r-1}Y$ it satisfies the equation

$$\delta^* i_\xi \, d\Theta_\lambda = 0 \tag{2.9}$$

(see also [4]), or, equivalently,

$$H_\lambda \circ j^1\delta = 0, \tag{2.10}$$

where H_λ is the *Hamilton form of* λ (see [11]). If a section $\gamma \in \Gamma(\pi)$ is an extremal (= Euler-Lagrange extremal) of λ then $\delta = j^{2r-1}\gamma$ is a Hamilton extremal of λ. Now, a question is studied under which conditions there exists a one-to-one correspondence between extremals and Hamilton extremals of λ, or, which is the same, under which conditions each Hamilton extremal δ is regular, i.e. satisfies $\delta = j^{2r-1}\gamma$ where γ is an extremal of λ. Then, assuming that an (in that way obtained) regularity condition is satisfied, a Legendre transformation is constructed as a transformation of local coordinates on $j^{2r-1}Y$ which transforms the form Θ_λ to a "canonical form"; in these coordinates the equations for Hamilton extremals of λ (2.10) take the form of the Hamilton canonical equations.

This approach has lead not only to a better understanding of the geometrical meaning of the Hamilton theory but it has also provided a method to study regularity conditions, and to introduce Legendre-transformation formulas more general than the "standard" ones

$$\det\left(\frac{\partial^2 L}{\partial q_r^\sigma \, \partial q_r^\nu}\right) \neq 0, \tag{2.11}$$

and

$$p_\sigma^i = f_\sigma^{i+1}, \qquad 0 \leqq i \leqq r-1, \qquad 1 \leqq \sigma \leqq m, \tag{2.12}$$

respectively – see e.g. [2], [3], [10]; (here $\lambda = L\,dt$ is the chart expression of a lagrangian $\lambda \in \Omega_X^1(j^rY)$ in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y, and f_σ^{i+1} are defined by (2.2)). Let us recall some results of [11] adapted to fibered manifolds with one-dimensional bases. In that paper a class of lagrangians $\lambda \in \Omega_X^1(j^2Y)$ is studied which in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y can be expressed in the form $\lambda = L\,dt$ where

$$L = L_0 + g_\sigma q_2^\sigma, \tag{2.13}$$

and the function L_0 (resp. g_σ, $1 \leqq \sigma \leqq m$) on V_2 depends on t, q^ν, q_1^ν (resp. t, q^ν) only. For such lagrangians the regularity condition has been found to be of the form

$$\det\left(\frac{\partial g_\sigma}{\partial q^\nu} + \frac{\partial g_\nu}{\partial q^\sigma} - \frac{\partial^2 L_0}{\partial q_1^\sigma \, \partial q_1^\nu}\right) \neq 0, \tag{2.14}$$

and "momenta" have been defined by

$$p_\sigma = \frac{\partial L_0}{\partial q_1^\sigma} - \frac{\partial g_\sigma}{\partial t} - \left(\frac{\partial g_\sigma}{\partial q^\nu} + \frac{\partial g_\nu}{\partial q^\sigma}\right) q_1^\nu, \qquad 1 \leqq \sigma \leqq m. \tag{2.15}$$

One may ask the question whether there are any other regular lagrangians than (2.11) and (2.14) and, if the answer is positive, how do they look like and how the corresponding Legendre transformations should be defined.

In this paper we develop a Hamilton theory directly from variational equations; this theory is independent of the choice of equivalent lagrangians for the corresponding Euler-Lagrange expressions. We introduce the notion of a Lepagean 2-form ($2 = \dim X + 1$) in analogy with Lepagean (1-)forms and associate Hamilton extremals with the Lepagean equivalent of a locally variational form. We obtain a general regularity condition (which contains (2.11) and (2.14) as special cases), find general Legendre transformation formulas and derive the corresponding Hamilton canonical equations. The paper contains several examples showing the relation of our approach to the known results (lagrangians of type (2.11) and (2.14)), and demonstrating the proposed ideas and methods explicitly.

3. LEPAGEAN 2-FORMS

Our aim is to extend the mapping *Lep* to 2-forms in such a way that the following diagam commutes:

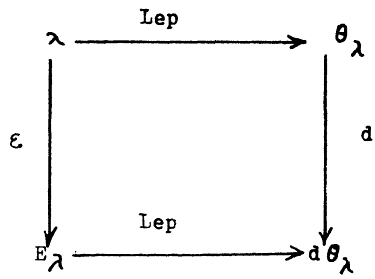

Fig. 1

Let $p \geqq 1$ be an integer. A form $\alpha \in \Omega^2_{j^{p-1}Y}(j^pY)$ is called *Lepagean* if (1) it admits a decomposition $\alpha = E + F$, where $E \in \Omega^{1,1}_Y(j^pY)$ and $F \in \Omega^{0,2}(j^pY)$, and (2) $d\alpha = 0$. Obviously, the one contact part E of a Lepagean 2-form is a locally variational form. We shall show that each Lepagean 2-form is uniquelly determined by its one-contact part.

Theorem 1. *The following conditions are equivalent:*

(1) *a form* $\alpha \in \Omega^2_{j^{p-1}Y}(j^pY)$ *is Lepagean,*

(2) *in each fiber chart* (V, ψ), $\psi = (t, q^\sigma)$ *on* Y α *is expressed in the form*

$$(3.1) \qquad \alpha = E_\sigma\, dq^\sigma \wedge dt + \sum_{j,k=0}^{p-1} F^{jk}_{\sigma\nu}\omega^\sigma_j \wedge \omega^\nu_k, \qquad F^{jk}_{\sigma\nu} = -F^{kj}_{\nu\sigma},$$

where the functions E_σ satisfy for $0 \leqq l \leqq p$

$$\frac{\partial E_\sigma}{\partial q_l^\nu} - (-1)^l \frac{\partial E_\nu}{\partial q_l^\sigma} - \sum_{k=l+1}^{p} (-1)^k \binom{k}{l} \frac{d^{k-l}}{dt^{k-l}} \left(\frac{\partial E_\nu}{\partial q_k^\sigma} \right) = 0 \tag{3.2}$$

and

$$F_{\sigma\nu}^{jk} = \frac{1}{2} \sum_{l=0}^{p-j-k-1} (-1)^{j+l} \binom{j+l}{l} \frac{d^l}{dt^l} \left(\frac{\partial E_\sigma}{\partial q_{j+k+l+1}^\nu} \right), \qquad 0 \leqq j + k \leqq p - 1, \tag{3.3}$$

$$F_{\sigma\nu}^{jk} = 0, \qquad p \leqq j + k \leqq 2p - 2. \tag{3.4}$$

Proof. Let $\alpha \in \Omega_{j^{p-1}Y}(j^p Y)$ be a Lepagean form. In each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y α can be expressed in the form (3.1). Computing $d\alpha$ we get

$$\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma} + 2 \frac{d}{dt}(F_{\nu\sigma}) = 0, \tag{3.5}$$

$$\frac{\partial E_\sigma}{\partial q_k^\nu} - 2 \frac{d}{dt}(F_{\sigma\nu}^{0k}) - 2F_{\sigma\nu}^{0,k-1} = 0, \qquad 1 \leqq k \leqq p - 1, \tag{3.6}$$

$$\frac{\partial E_\sigma}{\partial q_p^\nu} - 2F_{\sigma\nu}^{0,p-1} = 0, \tag{3.7}$$

$$\frac{d}{dt}(F_{\sigma\nu}^{jk}) + F_{\sigma\nu}^{j-1,k} + F_{\sigma\nu}^{j,k-1} = 0, \qquad 1 \leqq j, k \leqq p - 1, \tag{3.8}$$

$$F_{\sigma\nu}^{p-1,k} = 0, \qquad 1 \leqq k \leqq p - 1, \tag{3.9}$$

$$\frac{\partial F_{\sigma\nu}^{jk}}{\partial q_l^\varrho} + \frac{\partial F_{\varrho\sigma}^{lj}}{\partial q_k^\nu} + \frac{\partial F_{\nu\varrho}^{kl}}{\partial q_j^\sigma} = 0, \qquad 0 \leqq j, k, l \leqq p. \tag{3.10}$$

The relations (3.7) and (3.6) enable one to express the functions $F_{\sigma\nu}^{0k}$, $0 \leqq k \leqq$ $\leqq p - 1$, by means of E_σ: one easily obtains

$$F_{\sigma\nu}^{0k} = \frac{1}{2} \sum_{l=0}^{p-k-1} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial E_\sigma}{\partial q_{k+l+1}^\nu} \right), \qquad 0 \leqq k \leqq p - 1. \tag{3.11}$$

The relations (3.9) and (3.8) lead immediately to (3.4), and, after some labour, to the formulas

$$F_{\sigma\nu}^{jk} = \sum_{l=0}^{p-j-k-1} (-1)^{k+l} \binom{k+l-1}{l} \frac{d^l}{dt^l} (F_{\sigma\nu}^{j+k+l,0}), \qquad 2 \leqq j + k \leqq p - 1, \quad j, k \neq 0 \tag{3.12}$$

and

$$F_{\sigma\nu}^{0k} = \sum_{l=0}^{p-k-1} (-1)^{k+l} \binom{k+l-1}{l} \frac{d^l}{dt^l} (F_{\sigma\nu}^{k+l,0}), \qquad 1 \leqq k \leqq p - 1. \tag{3.13}$$

Applying the antisymmetry condition $F_{\sigma\nu}^{i0} = -F_{\nu\sigma}^{0i}$ and the relation (3.11) to (3.12) (resp. to (3.13)) we can express all the (nonvanishing) functions $F_{\sigma\nu}^{jk}$, $j, k \neq 0$, by means of E_σ (resp. we obtain some restrictive relations for the functions E_σ), namely, using the formula

$$\sum_{i=0}^{n} \binom{s+i}{i} = \binom{s+n+1}{n}; \tag{3.14}$$

we get

$$F_{\nu\sigma}^{kj} = \frac{1}{2} \sum_{l=0}^{p-j-k-1} (-1)^{k+l} \binom{k+l}{l} \frac{d^l}{dt^l} \left(\frac{\partial E_\nu}{\partial q_{j+k+l+1}^\sigma} \right), \quad 1 \leqq j + k \leqq p - 1, \quad k \neq 0. \tag{3.15}$$

This formula for $j \neq 0$ together with (3.11) is precisely the relation (3.3). Let $j = 0$ in (3.15). Substituting (3.11) into it we obtain after some straightforward calculation the equations (3.2) for $2 \leqq l \leqq p$. The equations (3.2) for $l = 1$ (resp. $l = 0$) are obtained from the antisymmetry relation $F_{\sigma\nu} = -F_{\nu\sigma}$ (resp. from (3.5)). Finally we shall show that the relations (3.10) are fulfilled identically. Put

$$G_{\sigma\nu\varrho}^{jkl} = \frac{\partial F_{\sigma\nu}^{jk}}{\partial q_l^\varrho} + \frac{\partial F_{\varrho\sigma}^{lj}}{\partial q_k^\nu} + \frac{\partial F_{\nu\varrho}^{kl}}{\partial q_j^\sigma}, \quad 0 \leqq j, k, l \leqq p, \quad 1 \leqq \sigma, \nu, \varrho \leqq m. \tag{3.16}$$

Differentiating the relations (3.5) − (3.8) with respect to q_{p+1}^ϱ we obtain

$$G_{\sigma\nu\varrho}^{jkp} = \frac{\partial F_{\sigma\nu}^{jk}}{\partial q_p^\varrho} = 0, \quad 0 \leqq j, k \leqq p, \quad 1 \leqq \sigma, \nu, \varrho \leqq m. \tag{3.17}$$

One can prove using (3.8) that the functions (3.16) satisfy

$$\frac{d}{dt}(G_{\sigma\nu\varrho}^{jkl}) + G_{\sigma\nu\varrho}^{j-1,k,l} + G_{\sigma\nu\varrho}^{j,k-1,l} + G_{\sigma\nu\varrho}^{j,k,l-1} = 0, \tag{3.18}$$

$$1 \leqq j, k, l \leqq p, \quad 1 \leqq \sigma, \nu, \varrho \leqq m.$$

Now, proceeding by induction starting from (3.17) we get $G_{\sigma\nu\varrho}^{jkl} = 0$ for $0 \leqq j, k, l \leqq \leqq p$, $1 \leqq \sigma, \nu, \varrho \leqq m$.

The converse is proved in an obvious way.

We note that a close assertion is due to [5].

Let $E \in \Omega_Y^{1,1}(j^s Y)$ be a locally variational form. According to Theorem 1 there exists a unique Lepagean 2-form α_E such that $h(i_\xi \alpha_E) = i_\xi E$ for each π_s-vertical vector field ξ on $j^s Y$. The form α_E will be called the *Lepagean equivalent of E*.

Corollary. *Let $E \in \Omega_Y^{1,1}(j^s Y)$, i.e. in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y let $E = E_\sigma \omega^\sigma \wedge dt$, where E_σ, $1 \leqq \sigma \leqq m$ are functions on V_s. The form E is locally variational iff E_σ, $1 \leqq \sigma \leqq m$, satisfy* (3.2).

Notice that the relations (3.2) were firstly proved in [12], and that they are a particular case (for dim $X = 1$) of the *ADK-conditions* [1], [7].

As a simple consequence of the ADK-conditions we obtain the following result, which however, is of fundamental importance for the theory of Hamilton extremals.

Theorem 2. *Let $E \in \Omega_Y^{1,1}(j^s Y)$ be a locally variational form, let α_E be its Lepagean equivalent. Then $\alpha_E \in \Omega^2(j^{s-1}Y)$.*

Proof. Let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y, let $E = E_\sigma \omega^\sigma \wedge dt$, where E_σ, $1 \leqq \sigma \leqq m$ are functions on V_s, be the chart expression of E. According to Theorem 1

$$\alpha_E = E_\sigma \omega^\sigma \wedge dt + \sum_{i=0}^{s-1} \sum_{k=0}^{s-1-i} F_{\sigma\nu}^{ik} \omega_i^\sigma \wedge \omega_k^\nu, \tag{3.19}$$

where

$$F_{\sigma\nu}^{ik} = \frac{1}{2} \sum_{l=0}^{s-i-k-1} (-1)^{i+l} \binom{i+l}{l} \frac{d^l}{dt^l} \left(\frac{\partial E_\sigma}{\partial q_{i+k+l+1}^\nu} \right), \qquad 0 \leqq i + k \leqq s - 1. \tag{3.20}$$

Obviously, $F_{\sigma\nu}^{ik}$ are defined on $V_{2s-1-(i+k)}$, i.e. $\alpha_E \in \Omega^2(j^{2s-1}Y)$. We shall show that α_E is projectable onto $j^{s-1}Y$. We can write (3.19) equivalently in the form (of a non-invariant decomposition)

$$\begin{aligned} \alpha_E = {} & (E_\sigma - \sum_{k=0}^{s-1} 2F_{\sigma\nu}^{0k} q_{k+1}^\nu)\, dq^\sigma \wedge dt - \\ & - \sum_{i=1}^{s-1} \sum_{k=0}^{s-1-i} 2F_{\sigma\nu}^{ik} q_{k+1}^\nu \, dq_i^\sigma \wedge dt + \sum_{i=0}^{s-1} \sum_{k=0}^{s-1-i} F_{\sigma\nu}^{ik} \, dq_i^\sigma \wedge dq_k^\nu. \end{aligned} \tag{3.21}$$

It is enough to show that (a) $F_{\sigma\nu}^{ik}$, $0 \leqq i + k \leqq s - 1$, and (b) $E_\sigma - \sum_{k=0}^{s-1} 2F_{\sigma\nu}^{0k} q_{k+1}^\nu$ are defined on V_{s-1}.

(a) Differentiating the relations (3.5) and (3.6) consecutively with respect to $q_{2s}^\varrho, q_{2s-1}^\varrho, \ldots, q_{s+1}^\varrho$ and taking into account that the functions E_σ, $1 \leqq \sigma \leqq m$ depend on $t, q^\nu, \ldots, q_s^\nu$ only, we obtain $\partial F_{\sigma\nu}^{0k}/\partial q_l^\varrho = 0$, $s \leqq l \leqq 2s - 1$, $0 \leqq k \leqq$ $\leqq s - 1$. In a similar way the relations (3.8) imply $\partial F_{\sigma\nu}^{ik}/\partial q_l^\varrho = 0$, $s \leqq l \leqq 2s - 1$, $1 \leqq i, k \leqq s - 1$. Thence

$$\frac{\partial F_{\sigma\nu}^{ik}}{\partial q_l^\varrho} = 0, \qquad s \leqq l \leqq 2s - 1 - (i + k). \tag{3.22}$$

(b) Taking into account that E_σ are defined on V_s we conclude that the coefficients at q_{s+1}^ϱ in (3.2) for $l = s - 1$ have to vanish, i.e. $\partial^2 E_\sigma / \partial q_s^\nu \, \partial q_s^\varrho = 0$, $1 \leqq \sigma, \nu, \varrho \leqq m$. Hence

$$E_\sigma = A_\sigma + B_{\sigma\nu} q_s^\nu, \qquad 1 \leqq \sigma \leqq m, \tag{3.23}$$

where A_σ, $B_{\sigma\nu}$ depend on $t, q^\varrho, \ldots, q_{s-1}^\varrho$ only. Consequently, we obtain

$$E_\sigma - \sum_{k=0}^{s-2} 2F_{\sigma\nu}^{0k} q_{k+1}^\nu - 2F_{\sigma\nu}^{0,s-1} q_s^\nu = A_\sigma - \sum_{k=0}^{s-2} 2F_{\sigma\nu}^{0k} q_{k+1}^\nu, \tag{3.24}$$

which, according to (3.22), is a function defined on V_{s-1}.

This completes the proof.

Let $\lambda_1 \in \Omega^1_X(j^k Y)$, $\lambda_2 \in \Omega^1_X(j^r Y)$, $k \geqq r$, be two equivalent lagrangians, Θ_{λ_1}, Θ_{λ_2} their Lepagean equivalents. Using Theorems 1 and 2 one can see easily that there exists an integer $s \leqq 2r$ such that $d\Theta_{\lambda_2} \in \Omega^2(j^{s-1}Y)$, and $d\Theta_{\lambda_1} = d\Theta_{\lambda_2}$ (up to a projection). Hence the form $d\Theta_\lambda$ is the same for all lagrangians, equivalent with a given lagrangian λ. Notice that this implies the well known result $\lambda_1 \sim \lambda_2$ iff $\lambda_1 = \lambda_2 + h(df)$ for a form df on $j^{k-1}Y$.

4. HAMILTON EXTREMALS

In what follows let $\mathscr{E} \in \Omega^{1,1}_Y(j^s Y)$ denote a *locally variational* form which is supposed to be *not* $\pi_{s,k}$*-projectable* for any $k < s$; notice that this means that in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y $\mathscr{E}$ is of the form $\mathscr{E} = E_\sigma \, dq^\sigma \wedge dt$ where the functions E_σ (defined on V_s) satisfy (3.2), and, for some σ and ν

$$\frac{\partial E_\sigma}{\partial q^\nu_s} \neq 0. \tag{4.1}$$

Denote by $\{\lambda\}$ the set of all equivalent lagrangians associated with $\mathscr{E}$ and let $\alpha_{\mathscr{E}} \in \Omega^2(j^{s-1}Y)$ be the Lepagean equivalent of $\mathscr{E}$. To each form $\mathscr{E} \in \Omega^{1,1}_Y(j^s Y)$ there exists a unique form $\mathscr{H} \in \Omega^2(j^1(j^{s-1}Y))$ such that for each π_{s-1}-vertical vector field ξ on $j^{s-1}Y$

$$i_{j^1\xi}\mathscr{H} = \bar{h}(i_\xi \alpha_{\mathscr{E}}) \tag{4.2}$$

(compare with [11]). According to Theorem 1 $\mathscr{H}$ is uniquelly determined by $\mathscr{E}$; we shall call $\mathscr{H}$ the *Hamilton form* of the set $\{\lambda\}$ of all equivalent lagrangians *associated with* $\mathscr{E}$. In a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y where $\alpha_{\mathscr{E}}$ is given by (3.19) and (3.20) one obtains from (4.2)

$$\mathscr{H} = \sum_{i=0}^{s-1} H^i_\sigma \, dq^\sigma_i \wedge dt, \tag{4.3}$$

where

$$H_\sigma = E_\sigma + \sum_{k=0}^{s-1} 2F^{0k}_{\sigma\nu}(q^\nu_{k,1} - q^\nu_{k+1}), \qquad 1 \leqq \sigma \leqq m, \tag{4.4}$$

$$H^i_\sigma = \sum_{k=0}^{s-1-i} 2F^{ik}_{\sigma\nu}(q^\nu_{k,1} - q^\nu_{k+1}), \qquad 1 \leqq i \leqq s-1, \qquad 1 \leqq \sigma \leqq m. \tag{4.5}$$

Theorem 3. *The form $\mathscr{H}$ is locally variational.*

Proof. The local variationality of $\mathscr{E}$ means that $j^s Y$ can be covered by open sets W such that there exists a lagrangian λ_W on each W, satisfying $E_{\lambda_W} = \mathscr{E}|_W$

and $\Theta_{\lambda_W} \in \Omega^1(j^{s-1}Y)$ (up to a projection). Put

$$\bar{\lambda}_W = \bar{h}(\Theta_{\lambda_W}). \tag{4.6}$$

Obviously, $\bar{\lambda}_W$ is a local lagrangian of order one for π_{s-1}. Let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y such that $V_s \cap W \neq \emptyset$. Denote $\lambda_W = L\,dt$ on $V_s \cap W$. Then $\bar{\lambda}_W = \bar{L}\,dt$, where

$$\bar{L} = L + \sum_{i=0}^{s-1} f_\sigma^{i+1}(q_{i,1}^\sigma - q_{i+1}^\sigma), \tag{4.7}$$

f_σ^{i+1}, $0 \leqq i \leqq s-1$ are defined by (2.2). The Lepagean equivalent $\Theta_{\bar{\lambda}_W}$ of $\bar{\lambda}_W$ is defined on an open subset of $j^1(j^{s-1}Y)$ and

$$\Theta_{\bar{\lambda}_W} = \bar{L}\,dt + \sum_{k=0}^{s-1} \frac{\partial \bar{L}}{\partial q_{k,1}^\sigma} \bar{\omega}_k^\sigma = (L + \sum_{i=0}^{s-1} f_\sigma^{i+1}(q_{i,1}^\sigma - q_{i+1}^\sigma))\,dt + \sum_{i=0}^{s-1} f_\sigma^{i+1} \bar{\omega}_i^\sigma, \tag{4.8}$$

i.e. it is $(\pi_{s-1})_{1,0}$-projectable and (up to this projection) $\Theta_{\bar{\lambda}_W} = \Theta_{\lambda_W}$. (Notice that $\Theta_{\bar{\lambda}_W}$ is nothing but the anholonomic decomposition of $(\pi_{s-1})_{1,0}^* \Theta_{\lambda_W}$ to its horizontal and contact part on $j^1(j^{s-1}Y)$). Hence, locally, for each π_{s-1}-vertical vector field ξ on $j^{s-1}Y$

$$\begin{aligned} i_{j^1\xi}\mathscr{H} &= \bar{h}(i_\xi \alpha_{\mathscr{E}}) = \bar{h}(i_\xi\, d\Theta_{\lambda_W}) = \bar{h}(i_{j^1\xi}\, d\Theta_{\bar{\lambda}_W}) = \\ &= \bar{h}(i_{j^1\xi}(E_{\bar{\lambda}_W} + F_{\bar{\lambda}_W})) = i_{j^1\xi} E_{\bar{\lambda}_W}, \end{aligned} \tag{4.9}$$

i.e. for each W

$$\mathscr{H} = E_{\bar{\lambda}_W}. \tag{4.10}$$

This completes the proof.

Let us define the canonical embedding $\iota : j^sY \to j^1(j^{s-1}Y)$ by the equations $q_{i,1}^\sigma \circ \iota = q_{i+1}^\sigma$, $0 \leqq i \leqq s-1$. Obviously, on each W,

$$\iota^* \bar{\lambda}_W = \lambda_W. \tag{4.11}$$

$\bar{\lambda}_W$ is therefore called the *extended lagrangian* associated with λ_W [11].

According to Theorems 3 and 1 to each Hamilton form $\mathscr{H}$ its Lepagean equivalent $\alpha_{\mathscr{H}} \in \Omega^2(j^1(j^{s-1}Y))$ can be associated, which, in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y, where $\mathscr{H}$ is expressed in the form (4.3)–(4.5), has the chart expression

$$\alpha_{\mathscr{H}} = \sum_{i=0}^{s-1} H_\sigma^i\, dq_i^\sigma \wedge dt + \sum_{i,k=0}^{s-1} \frac{1}{2} \frac{\partial H_\sigma^i}{\partial q_{k,1}^\nu} \bar{\omega}_i^\sigma \wedge \bar{\omega}_k^\nu. \tag{4.12}$$

Obviously, $\alpha_{\mathscr{H}}$ is $(\pi_{s-1})_{1,0}$-projectable and (up to this projection) $\alpha_{\mathscr{H}} = \alpha_{\mathscr{E}}$.

Consider the form $\mathscr{E} \in \Omega_Y^{1,1}(j^sY)$, let $\alpha_{\mathscr{E}} \in \Omega^2(j^{s-1}Y)$ be its Lepagean equivalent, let $\delta \in \Gamma(\pi_{s-1})$ be a (local) section. δ will be called a *Hamilton extremal associated with* $\mathscr{E}$ if for each π_{s-1}-vertical vector field ξ on $j^{s-1}Y$

$$\delta^*(i_\xi \alpha_{\mathscr{E}}) = 0. \tag{4.13}$$

Obviously, δ is a Hamilton extremal associated with $\mathscr{E}$ iff for each π_{s-1}-vertical vector field ξ on $j^{s-1}Y$

$$j^1\delta^*(i_{j^1\xi}\alpha_{\mathscr{H}}) = 0, \tag{4.14}$$

resp. iff

$$\mathscr{H} \circ j^1\delta = 0, \tag{4.15}$$

where $\mathscr{H}$ (resp. $\alpha_{\mathscr{H}}$) is the Hamilton form associated with $\mathscr{E}$ (resp. the Lepagean equivalent of $\mathscr{H}$). The first order equation for δ (4.14), resp. (4.15) is called the *equation for Hamilton extremals*. Locally it is represented by a system of equations

$$H^i_\sigma \circ j^1\delta = 0, \qquad 0 \leqq i \leqq s-1, \qquad 1 \leqq \sigma \leqq m, \tag{4.16}$$

where H^i_σ are given by (4.4)–(4.5).

5. REGULARITY

Consider the form $\mathscr{E}$ (recall: $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$, is locally variational and satisfies (4.1)), let $\{\lambda\}$ denote the set of all corresponding equivalent lagrangians. It is easy to prove that if $\lambda \in \Gamma(\pi)$ is an extremal of $\lambda \in \{\lambda\}$ then $j^{s-1}\gamma$ is a Hamilton extremal associated with $\mathscr{E}$. We say that a Hamilton extremal $\delta \in \Gamma(\pi_{s-1})$ (resp. the form $\mathscr{E}$) is *regular* if $\delta = j^{s-1}\gamma$ for some extremal $\gamma \in \Gamma(\pi)$ of $\lambda \in \{\lambda\}$ (resp. the mapping $\gamma \to j^{s-1}\gamma$ of the set of extremals into the set of Hamilton extremals associated with $\mathscr{E}$ is bijective) (compare with [10], [11]).

Theorem 4. *Let $\delta : I \to j^{s-1}Y$ be a Hamilton extremal associated with the form $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$, defined on an open set $I \subset X$. Suppose that to each point $x \in I$ there exists a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y such that $\delta(x) \in V_{s-1}$ and*

$$\det\left(\frac{\partial E_\sigma}{\partial q^\nu_s}\right) \neq 0 \tag{5.1}$$

at $\pi^{-1}_{s,s-1}\delta(x)$, E_σ being defined by the chart expression $\mathscr{E} = E_\sigma\, dq^\sigma \wedge dt$. Then δ is regular.

Proof. We can proceed in analogy with [10]. Let $x \in I$ be a point satisfying the assumptions of Theorem 4. Since δ is a Hamilton extremal associated with $\mathscr{E}$, it satisfies the equations (4.16). For $1 \leqq i \leqq s-1$ these equations can be considered as a system of $m(s-1)$ linear homogeneous equations for the $m(s-1)$ unknowns $(q^\nu_{k,1} - q^\nu_{k+1})$, $1 \leqq \nu \leqq m$, $0 \leqq k \leqq s-2$. This system posesses a unique (trivial) solution iff the matrix $(F^{ik}_{\sigma\nu})$ where the rows (resp. columns) are labelled by i, σ (resp. k, ν), i.e. the matrix

$$\begin{pmatrix} F^{s-1,0}_{\sigma\nu} & & 0 \\ F^{s-2,0}_{\sigma\nu} & F^{s-2,1}_{\sigma\nu} & \\ \vdots & & \ddots \\ F^{1,0}_{\sigma\nu} & F^{1,1}_{\sigma\nu} & \dots F^{1,s-2}_{\sigma\nu} \end{pmatrix} \tag{5.2}$$

is regular. However, according to (3.3)

$$F^{i,s-1-i}_{\sigma\nu} = (-1)^i \frac{1}{2} \frac{\partial E_\sigma}{\partial q^\nu_s}, \qquad 0 \leqq i \leqq s-1. \tag{5.3}$$

Hence for the absolute value of the determinant of the matrix (5.2) we obtain

$$|\det(F^{ik}_{\sigma\nu})| = \left(\frac{1}{2}\right)^{s-1} \left|\det\left(\frac{\partial E_\sigma}{\partial q^\nu_s}\right)\right|^{s-1}. \tag{5.4}$$

Under the assumption (5.1) the system has the unique solution

$$q^\nu_{k,1} - q^\nu_{k+1} = 0, \qquad 0 \leqq k \leqq s-2 \tag{5.5}$$

at the point $\delta(x)$, i.e.

$$q^\nu_{k+1}(\delta(x)) = \frac{d(q^\nu_k \circ \delta)}{dt}\bigg|_x = \dots = \frac{d^{k+1}(q^\nu \circ \delta)}{dt^{k+1}}\bigg|_x; \tag{5.6}$$

in such a way we get $\delta(x) = j^{s-1}_x\gamma$ where $\gamma = \pi_{s-1,0} \circ \delta$, and, since the point is arbitrary, $\delta = j^{s-1}\gamma$. It remains to show that γ is an extremal relative to $\mathscr{E}$. Consider the equations (4.16) for $i = 0$. Using (5.5) we obtain

$$\left(E_\sigma + \frac{\partial E_\sigma}{\partial q^\nu_s}(q^\nu_{s-1,1} - q^\nu_s)\right) \circ j^1\delta = 0, \qquad 1 \leqq \sigma \leqq m. \tag{5.7}$$

Writing E_σ in the form (3.23) and substituting into (5.7) we obtain equations which along $j^s\gamma$ coincide with the Euler–Lagrange equations.

The relation (5.1) is obviously independent of the choice of the fiber chart (V, ψ); we will call it the *regularity condition*. Notice that if the regularity condition is satisfied at each point of j^sY then each Hamilton extremal is regular, i.e. the form $\mathscr{E}$ is regular. In this case (4.14), resp. (4.15) is called *Hamilton equation*; locally it represents a system of sm first order equations, equivalent with the m s-th order Euler–Lagrange equations $E_\sigma \circ j^s\gamma = 0$, $1 \leqq \sigma \leqq m$.

We note that for s odd ($s = 2c + 1$ for an integer $c \geqq 0$) (3.2) imply $\partial E_\sigma/\partial q^\nu_s = -\partial E_\nu/\partial q^\sigma_s$, i.e. the matrix $(\partial E_\sigma/\partial s^\nu_s)$ is antisymmetric. Hence a necessary condition for a form $\mathscr{E} \in \Omega^{1,1}_Y(j^{2c+1}Y)$ to be regular is that $m = \dim Y - 1$ be even.

A lagrangian λ is called *regular* if the Euler–Lagrange form E_λ of λ is regular.

Example 1. Let λ be a lagrangian of order one for π, $\lambda = L\,dt$ its chart expression in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y. λ is regular iff it satisfies one of the following two conditions:

(1) $$\det\left(\frac{\partial^2 L}{\partial q_1^\sigma \partial q_1^\nu}\right) \neq 0,$$

(2) $$\frac{\partial^2 L}{\partial q_1^\sigma \partial q_1^\nu} = 0 \quad \text{for each } \sigma, \nu, \text{ and } \quad \det\left(\frac{\partial^2 L}{\partial q^\sigma \partial q_1^\nu} - \frac{\partial^2 L}{\partial q_1^\sigma \partial q^\nu}\right) \neq 0.$$

Notice that in the case (1) (resp. (2)) $E_\lambda \in \Omega_Y^{1,1}(j^2 Y)$ (resp. $E_\lambda \in \Omega_Y^{1,1}(j^1 Y)$).

Example 2. It is worthwhile to describe all lagrangians of order $r \leqq 2$, leading to regular second order Euler–Lagrange expressions. Let $\mathscr{E} \in \Omega_Y^{1,1}(j^2 Y)$; in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y we have $\mathscr{E} = E_\sigma\, dq^\sigma \wedge dt$, where the functions E_σ defined on V_2 satisfy (3.2). Consider all lagrangians λ for $\mathscr{E}$ of order $r \leqq 2$ defined on V_2 and denote $\lambda = L\, dt$. It holds

(5.8) $$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_1^\sigma}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q_2^\sigma}\right),$$

where

(5.9) $$\frac{\partial^2 L}{\partial q_2^\sigma \partial q_2^\nu} = 0, \qquad \frac{\partial^2 L}{\partial q_1^\sigma \partial q_2^\nu} - \frac{\partial^2 L}{\partial q_2^\sigma \partial q_1^\nu} = 0.$$

Computing $\partial E_\sigma / \partial q_2^\nu$ and using (5.9) we obtain the regularity condition (5.1) in the form

(5.10) $$\det\left(-\frac{\partial^2 L}{\partial q_1^\sigma \partial q_1^\nu} + \frac{\partial^2 L}{\partial q^\nu \partial q_2^\sigma} + \frac{\partial^2 L}{\partial q_2^\nu \partial q^\sigma} + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial q_1^\nu \partial q_2^\sigma}\right)\right) \neq 0.$$

Notice that (5.10) contains (2.11) and (2.14) as special cases. In an analogous way one could rewrite the regularity condition for higher order lagrangians related to $\mathscr{E}$.

6. LEGENDRE TRANSFORMATION

Consider the form $\mathscr{E} \in \Omega_Y^{1,1}(j^s Y)$ and its Lepagean equivalent $\alpha_\mathscr{E} \in \Omega^2(j^{s-1} Y)$. Let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y, (3.19)–(3.20) the chart expression of $\alpha_\mathscr{E}$ in this chart. Denote by c the integer defined by the relation $(s/2) - 1 < c \leqq \leqq (s/2)$.

Theorem 5. *$j^{s-1} Y$ can be covered by open sets W such that the restriction of $\alpha_\mathscr{E}$ to $V_{s-1} \cap W$ can be expressed in the form*

(6.1) $$\alpha_\mathscr{E} = -dH \wedge dt + \sum_{k=0}^{s-c-1} dp_\nu^k \wedge dq_k^\nu$$

for some functions H, p_ν^k, $0 \leqq k \leqq s - c - 1$, $1 \leqq \nu \leqq m$ defined on $V_{s-1} \cap W$.

Proof. The form $\alpha_\mathscr{E}$ is closed, i.e. there exists a covering of $j^{s-1} Y$ by open sets W

such that on each W $\alpha_{\mathscr{E}} = d\varrho$ for a form $\varrho \in \Omega^1(W)$. Using (2.6)–(2.8) we obtain (up to a projection)

$$\varrho = A\alpha_{\mathscr{E}} = (q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du)\, dt + \sum_{k=0}^{s-1} (\sum_{i=0}^{s-1} 2q^\sigma_{ij} \int_0^1 (F^{ik}_{\sigma\nu} \circ \chi_{s-1})\, u\, du)\, \omega^\nu_k \tag{6.2}$$

on $U = V_{s-1} \cap W$. We shall show that there exist functions f, H, and p^k_ν, $0 \leqq k \leqq s - c - 1$, $1 \leqq \nu \leqq m$ on U such that (6.2) can be equivalently expressed in the form

$$\varrho = -H\, dt + \sum_{k=0}^{s-c-1} p^k_\nu\, dq^\nu_k + df. \tag{6.3}$$

We define a mapping $\chi_{s-1,s-c} : [0, 1] \times U \to U$ by

$$\chi_{s-1,s-c}(v, (t, q^\sigma, \dots, q^\sigma_{s-c-1}, q^\sigma_{s-c}, \dots, q^\sigma_{s-1})) = = (t, q^\sigma, \dots, q^\sigma_{s-c-1}, vq^\sigma_{s-c}, \dots, vq^\sigma_{s-1}). \tag{6.4}$$

Put

$$f = \sum_{k=s-c}^{s-1} \sum_{i=0}^{s-1-k} 2q^\nu_k q^\sigma_i \int_0^1 \int_0^1 (F^{ik}_{\sigma\nu} \circ \chi_{s-1} \circ \chi_{s-1,s-c})\, u\, du\, dv + + \varphi(t, q^\varrho, \dots, q^\varrho_{s-c-1}), \tag{6.5}$$

where φ is an arbitrary function, and define

$$p^k_\nu = \sum_{i=0}^{s-k-1} 2q^\sigma_i \int_0^1 (F^{ik}_{\sigma\nu} \circ \chi_{s-1})\, u\, du - \frac{\partial f}{\partial q^\nu_k}, \qquad 0 \leqq k \leqq s - c - 1, \tag{6.6}$$

$$-H = q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du - \sum_{k=0}^{s-1} \sum_{i=0}^{s-k-1} 2q^\sigma_i q^\nu_{k+1} \int_0^1 (F^{ik}_{\sigma\nu} \circ \chi_{s-1})\, u\, du - \frac{\partial f}{\partial t}. \tag{6.7}$$

Substituting (6.6), (6.7) and the relation

$$\frac{\partial f}{\partial q^\nu_k} = \sum_{i=0}^{s-k-1} 2q^\sigma_i \int_0^1 (F^{ik}_{\sigma\nu} \circ \chi_{s-1})\, u\, du, \qquad s - c \leqq k \leqq s - 1 \tag{6.8}$$

into (6.2) we obtain (6.3). Computing $\alpha_{\mathscr{E}} = d\varrho$, the proof is completed.

The expression (6.1) will be called the *canonical form of* $\alpha_{\mathscr{E}}$ on U, and each of the functions H (resp. p^k_ν, $0 \leqq k \leqq s - c - 1$, $1 \leqq \nu \leqq m$) defined on U by (6.7) (resp. (6.6)) is called *Hamilton function* (resp. *momentum*) of $\mathscr{E}$.

We shall study the question under which conditions momenta may become a part of coordinates on U.

Theorem 6. *Let the functions* p^k_ν, $0 \leqq k \leqq s - c - 1$, $1 \leqq \nu \leqq m$ *be defined by* (6.6).

(1) *Let s be even* ($s = 2c$). *The following two conditions are equivalent*:

(i) *for each function* $\varphi = \varphi(t, q^{\varrho}, \ldots, q^{\varrho}_{c-1})$, (U, φ_{2c-1}), *where* $\varphi_{2c-1} = (t, q^{\sigma}, \ldots, \ldots, q^{\sigma}_{c-1}, p_{\sigma}, \ldots, p^{c-1}_{\sigma})$, $1 \leqq \sigma \leqq m$ *is a coordinate chart on* $j^{2c-1}Y$,

(ii) $\mathscr{E}$ *is regular.*

(2) *Let* s *be odd* $(s = 2c + 1)$. *Choose a function* $\varphi = \varphi(t, q^{\varrho}, \ldots, q^{\varrho}_{c})$ *in such a way that on an open set* $\bar{U} \subset U$

$$\det\left(\frac{\partial p^{c}_{\nu}}{\partial q^{\sigma}_{c}}\right) \neq 0. \tag{6.9}$$

Then the following two conditions are equivalent:

(i) *for each function* $\varphi = \varphi(t, q^{\varrho}, \ldots, q^{\varrho}_{c})$ *such that* (6.9) *is satisfied,* $(\bar{U}, \varphi_{2c})$, *where* $\varphi_{2c} = (t, q^{\sigma}, \ldots, q^{\sigma}_{c-1}, p_{\sigma}, \ldots, p^{c-1}_{\sigma}, p^{c}_{\sigma})$, $1 \leqq \sigma \leqq m$ *is a coordinate chart on* $j^{2c}Y$,

(ii) $\mathscr{E}$ *is regular.*

Proof. Let us denote by Λ the Jacobi matrix of the transformation $\varphi_{s-1}\psi^{-1}_{s-1}$. Computing the Jacobian we obtain

$$\det \Lambda = \det\left(\frac{\partial p^{k}_{\nu}}{\partial q^{\sigma}_{i}}\right), \qquad 0 \leqq k \leqq s - c - 1, \qquad c \leqq i \leqq s - 1. \tag{6.10}$$

It is easy to prove by a direct computation, using (6.6), (6.8), (3.10), and

$$F^{ik}_{\sigma\nu} = 2\int_{0}^{1} (F^{ik}_{\sigma\nu} \circ \chi_{s-1})\, u\, du + \sum_{l=0}^{s-1} q^{\varrho}_{l} \int_{0}^{1} \left(\frac{\partial F^{ik}_{\sigma\nu}}{\partial q^{\varrho}_{l}} \circ \chi_{s-1}\right) u^{2}\, du, \tag{6.11}$$

that

$$\frac{\partial p^{k}_{\nu}}{\partial q^{\sigma}_{i}} - \frac{\partial p^{i}_{\sigma}}{\partial q^{\nu}_{k}} = 2F^{ik}_{\sigma\nu}, \qquad 0 \leqq k \leqq s - c - 1, \qquad 0 \leqq i \leqq s - c - 1, \tag{6.12}$$

$$\frac{\partial p^{k}_{\nu}}{\partial q^{\sigma}_{i}} = 2F^{ik}_{\sigma\nu}, \qquad 0 \leqq k \leqq s - c - 1, \qquad s - c \leqq i \leqq s - 1.$$

(1) Let $s = 2c$. Using (6.12) we obtain $\det \Lambda = (\det(2F^{ik}_{\sigma\nu}))^{c}$, $0 \leqq k \leqq c - 1$, $c \leqq i \leqq 2c - 1$. Hence the absolute value of $\det \Lambda$ satisfies

$$|\det \Lambda| = \left|\det\left(\frac{\partial E_{\sigma}}{\partial q^{\nu}_{2c}}\right)\right|^{c}. \tag{6.13}$$

Thus the matrix Λ is regular iff the form $\mathscr{E}$ is regular.

(2) Let $s = 2c + 1$. Then according to (6.12)

$$\det \Lambda = \begin{vmatrix} 2F^{2c,0}_{\sigma\nu} & \ldots & 2F^{c+1,0}_{\sigma\nu} & \dfrac{\partial p_{\nu}}{\partial q^{\sigma}_{c}} \\ & \ddots & \vdots & \\ & & 2F^{c+1,c-1}_{\sigma\nu} & \dfrac{\partial p^{c-1}_{\nu}}{\partial q^{\sigma}_{c}} \\ & 0 & & \dfrac{\partial p^{c}_{\nu}}{\partial q^{\sigma}_{c}} \end{vmatrix} \tag{6.14}$$

i.e. the absolute value of det A satisfies

$$|\det A| = \left|\det\left(\frac{\partial E_\sigma}{\partial q^\nu_{2c+1}}\right)\right|^c \cdot \left|\det\left(\frac{\partial p^c_\nu}{\partial q^\sigma_c}\right)\right|. \tag{6.15}$$

We shall show that a proper choice of a function $\varphi = \varphi(t, q^\varrho, \ldots, q^\varrho_c)$ gives rise to a set of momenta p^k_ν, $0 \leqq k \leqq c$, $1 \leqq \nu \leqq m$ of $\mathscr{E}$ satisfying (6.9). Let $\bar{p}^k_\nu$, $0 \leqq k \leqq c, 1 \leqq \nu \leqq m$ be momenta of $\mathscr{E}$ defined on U for which $\det(\partial \bar{p}^c_\nu / \partial q^\sigma_c)|_x = 0$ at a point $x \in U$. Put $A = (a_{\sigma\nu})$, where $a_{\sigma\nu} = \partial \bar{p}^c_\nu / \partial q^\sigma_c|_x$. Choose a $(m \times m)$-matrix $B = (b_{\sigma\nu})$ in such a way that $\det(A + B) \neq 0$. Let $\varphi = \varphi(t, q^\varrho, \ldots, q^\varrho_c)$ be a function satisfying $\partial^2\varphi / \partial q^\sigma_c\, \partial q^\nu_c|_x = b_{\sigma\nu}$. Then the continuity of the function det implies that

$$\det\left(\frac{\partial \bar{p}^c_\nu}{\partial q^\sigma_c} + \frac{\partial^2 \varphi}{\partial q^\sigma_c\, \partial q^\nu_c}\right) \neq 0 \tag{6.16}$$

on an open set $\bar{U} \subset U$, $\bar{U} \ni x$. Hence p^k_ν, $0 \leqq k \leqq c$, $1 \leqq \nu \leqq m$ defined by

$$p^k_\nu = \bar{p}^k_\nu + \frac{\partial \varphi}{\partial q^\nu_k} \tag{6.17}$$

form the desired set of momenta of $\mathscr{E}$. Obviously, (6.15) guarantees that these functions belong to a chart on $j^{2c}Y$ iff $\mathscr{E}$ is regular.

This completes the proof.

The local coordinates $\varphi_{s-1} = (t, q^\sigma_i, p^k_\nu)$, $0 \leqq i \leqq c - 1$, $0 \leqq k \leqq s - c - 1$, $1 \leqq \sigma, \nu \leqq m$ of Theorem 6, defined on an open subset U (resp. $\bar{U}$) of $V_{s-1} \subset j^{s-1}Y$ are called *Legendre coordinates related to the form* $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$, and the transformation $\varphi_{s-1}\psi^{-1}_{s-1}$ is called *Legendre transformation related to* $\mathscr{E}$.

We shall show that the above Legendre transformation can be, in the case of lagrangians (2.11) and (2.14), identified with the Legendre transformations (2.12) and (2.15) related to these lagrangians.

Example 3. Let $\lambda \in \Omega^1_X(j^rY)$ be a lagrangian, let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y, put $\lambda = L\,dt$ in this chart. Suppose that λ is regular in the standard sense, i.e. satisfies (2.11). Obviously, the Euler–Lagrange form $E_\lambda \in \Omega^{1,1}_Y(j^{2r}Y)$ of λ satisfies (5.1), i.e. λ is regular in the generalized sense. Computing $\alpha_{E_\lambda} = d\Theta_\lambda$ we obtain easily

$$F^{ik}_{\sigma\nu} = -\frac{1}{2}\left(\frac{\partial f^{i+1}_\sigma}{\partial q^\nu_k} - \frac{\partial f^{k+1}_\nu}{\partial q^\sigma_i}\right), \qquad 0 \leq i, k \leq 2r - 1, \tag{6.18}$$

where f^i_σ, $1 \leqq i \leqq r$ are given by (2.2), and $f^i_\sigma = 0$, $r + 1 \leqq i \leqq 2r - 1$. Let us compute the momenta p^k_ν, $0 \leqq k \leqq r - 1$, $1 \leqq \nu \leqq m$ of E_λ according to (6.6). We obtain

$$(6.19)\qquad f = -\sum_{k=r}^{2r-1}\sum_{i=0}^{2r-1-k} q_k^\nu q_i^\sigma \int_0^1\int_0^1 \left(\frac{\partial f_\sigma^{i+1}}{\partial q_k^\nu} \circ \chi_{2r-1} \circ \chi_{2r-1,r}\right) u\,du\,dv +$$
$$+ \varphi(t, q^\varrho, \dots, q_{r-1}^\varrho) = -\sum_{i=0}^{r-1} q_i^\sigma \int_0^1 (f_\sigma^{i-1} \circ \chi_{2r-1})\,du + \bar\varphi,$$

where $\bar\varphi(t, q^\varrho, \dots, q_{r-1}^\varrho)$ is an arbitrary function, and, putting $\bar\varphi = 0$,

$$(6.20)\quad p_\nu^k = \int_0^1 (f_\nu^{k+1} \circ \chi_{2r-1})\,du + \sum_{i=0}^{2r-1-k} q_i^\sigma \int_0^1 \left(\frac{\partial f_\nu^{k+1}}{\partial q_i^\sigma} \circ \chi_{2r-1}\right) u\,du = f_\nu^{k+1},$$
$$0 \leqq k \leqq r-1,$$

which is the formula (2.12).

Example 4. Let $\lambda \in \Omega_X^1(j^2 Y)$ be a lagrangian of type (2.13). Suppose that λ satisfies (2.14). Then E_λ is regular, i.e. λ is regular. Computing $\alpha_{E_\lambda} \in \Omega^2(j^1 Y)$ we obtain

$$(6.21)\qquad F_{\sigma\nu}^{00} = \frac{1}{2}\left(\frac{\partial f_\nu^1}{\partial q^\sigma} - \frac{\partial f_\sigma^1}{\partial q^\nu}\right), \qquad F_{\sigma\nu}^{01} = \frac{1}{2}\left(\frac{\partial f_\nu^2}{\partial q^\sigma} - \frac{\partial f_\sigma^1}{\partial q_1^\nu}\right),$$

where

$$f_\sigma^2 = \frac{\partial L}{\partial q_2^\sigma} = g_\sigma, \qquad f_\sigma^1 = \frac{\partial L}{\partial q_1^\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_2^\sigma}\right) = \frac{\partial L_0}{\partial q_1^\sigma} - \frac{\partial g_\sigma}{\partial t} - \frac{\partial g_\sigma}{\partial q^\nu} q_1^\nu.$$
(6.22)

Thus

$$(6.23)\qquad f = 2q_1^\nu q^\sigma \int_0^1\int_0^1 (F_{\sigma\nu}^{01} \circ \chi_1 \circ \chi_{1,1})\,u\,du\,dv + \varphi(t, q^\varrho) =$$
$$= q_1^\nu q^\sigma \int_0^1 \left(\frac{\partial f_\nu^2}{\partial q^\sigma} \circ \chi_1\right) u\,du - q^\sigma \int_0^1 (f_\sigma^1 \circ \chi_1)\,du + \bar\varphi(t, q^\varrho),$$

where $\bar\varphi$ is an arbitrary function. (We note that we have used the formulas

$$(6.24)\qquad \int_0^1 \left(\frac{\partial f_\nu^2}{\partial q^\sigma} \circ \chi_1\right) u\,du = \int_0^1 d\left[v\left(\int_0^1 \left(\frac{\partial f_\nu^2}{\partial q^\sigma} \circ \chi_1\right) u\,du\right) \circ \chi_{1,1}\right] =$$
$$= \int_0^1\int_0^1 \left(\frac{\partial f_\nu^2}{\partial q^\sigma} \circ \chi_1 \circ \chi_{1,1}\right) u\,du\,dv,$$

because f_ν^2 is independent of q_1^σ, and

$$(6.25)\qquad \int_0^1 (f_\sigma^1 \circ \chi_1)\,du = q_1^\nu \int_0^1 \left(\frac{\partial}{\partial q_1^\nu} \int_0^1 (f_\sigma^1 \circ \chi_1)\,du\right) \circ \chi_{1,1}\,dv + \varphi(t, q^\varrho) =$$
$$= q_1^\nu \int_0^1\int_0^1 \left(\frac{\partial f_\sigma^1}{\partial q_1^\nu} \circ \chi_1 \circ \chi_{1,1}\right) u\,du\,dv + \varphi(t, q^\varrho),$$

where φ is an arbitrary function). Putting $\bar\varphi = 0$ in (6.23) we obtain

$$(6.26)\qquad p_\nu = 2q^\sigma \int_0^1 (F^{00}_{\sigma\nu} \circ \chi_1) u\, du + 2q^\sigma_1 \int_0^1 (F^{10}_{\sigma\nu} \circ \chi_1) u\, du - \frac{\partial f}{\partial q^\nu} =$$

$$= \int_0^1 (f^1_\nu \circ \chi_1)\, du + q^\sigma \int_0^1 \left(\frac{\partial f^1_\nu}{\partial q^\sigma} \circ \chi_1\right) u\, du + q^\sigma_1 \int_0^1 \left(\frac{\partial f^1_\nu}{\partial q^\sigma_1} \circ \chi_1\right) u\, du -$$

$$- 2q^\sigma_1 \int_0^1 \left(\frac{\partial f^2_\sigma}{\partial q^\nu} \circ \chi_1\right) u\, du - q^\sigma_1 q^\varrho \int_0^1 \left(\frac{\partial^2 f^2_\sigma}{\partial q^\varrho\, \partial q^\nu} \circ \chi_1\right) u^2\, du =$$

$$f^1_\nu - q^\sigma_1 \frac{\partial f^2_\sigma}{\partial q^\nu} = \frac{\partial L_0}{\partial q^\nu_1} - \frac{\partial g_\nu}{\partial t} - \left(\frac{\partial g_\nu}{\partial q^\sigma} + \frac{\partial g_\sigma}{\partial q^\nu}\right) q^\sigma_1,$$

which is the formula (2.15).

Theorem 7. *Let the form $\mathscr{E} \in \Omega^{1,1}_Y(j^s Y)$ be regular, consider a Legendre chart (U, ψ_{s-1}) on $j^{s-1}Y$ related to $\mathscr{E}$. Let $\delta \in \Gamma(\pi_{s-1})$ be a section defined on an open set $I \subset X$ such that $\delta(I) \subset U$. δ is a Hamilton extremal associated with $\mathscr{E}$ iff it satisfies the system of equations*

$$(6.27)\qquad -\frac{\partial H}{\partial q^\sigma_i} - \frac{d}{dt}(p^i_\sigma \circ \delta) = 0, \qquad -\frac{\partial H}{\partial p^i_\sigma} + \frac{d}{dt}(q^\sigma_i \circ \delta) = 0,$$

$$1 \leqslant \sigma \leqslant m, \qquad 0 \leqslant i \leqslant c-1,$$

if s is even ($s = 2c$), resp.

$$(6.28)\quad -\frac{\partial H}{\partial q^\sigma_i} - \frac{\partial q^\nu_c}{\partial q^\sigma_i}\frac{d}{dt}(p^c_\nu \circ \delta) - \frac{d}{dt}(p^i_\sigma \circ \delta) = 0, \qquad -\frac{\partial H}{\partial p^i_\sigma} + \frac{d}{dt}(q^\sigma_i \circ \delta) = 0,$$

$$1 \leqslant \sigma \leqslant m, \qquad 0 \leqslant i \leqslant c-1,$$

$$-\frac{\partial H}{\partial p^c_\sigma} + \sum_{k=0}^{c-1} \frac{\partial q^\sigma_c}{\partial q^\nu_k}\frac{d}{dt}(q^\nu_k \circ \delta) + \left(\frac{\partial q^\sigma_c}{\partial p^c_\nu} - \frac{\partial q^\nu_c}{\partial p^c_\sigma}\right)\frac{d}{dt}(p^c_\nu \circ \delta) = 0, \qquad 1 \leqq \sigma \leqq m,$$

if s is odd ($s = 2c + 1$).

Proof. Let us denote by $((U)_1, (\psi_{s-1})_1)$, $(\psi_{s-1})_1 = (t, q^\sigma_i, p^k_\nu, q^\sigma_{i,1}, p^k_{\nu,1})$, where $1 \leqq \sigma, \nu \leqq m, 0 \leqq i \leqq c-1, 0 \leqq k \leqq s-c-1$, the chart on $j^1(j^{s-1}Y)$ associated with the Legendre chart (U, ψ_{s-1}). Writing $\alpha_\mathscr{E}$ in the canonical form (6.1) we compute the Hamilton form $\mathscr{H}$ associated with $\mathscr{E}$ according to (4.2). Considering q^σ_i, $c \leqq i \leqq s-1$ as functions of the Legendre coordinates we obtain, taking into account (6.12), the relation

$$(6.29)\qquad \frac{\partial q^\sigma_i}{\partial p^k_\nu} = 0, \qquad c \leqq i \leqq s-1,\ i + k < s - 1.$$

Using (6.29) for $i = c$ we get for the Hamilton form of $\mathscr{E}$ the following chart expression

$$(6.30)\ \mathscr{H} = \left\{\sum_{i=0}^{c-1}\left[\left(-\frac{\partial H}{\partial q^\sigma_i} - \frac{\partial q^\nu_c}{\partial q^\sigma_i} p^c_{\nu,1} - p^i_{\sigma,1}\right) dq^\sigma_i + \left(-\frac{\partial H}{\partial p^i_\sigma} + q^\sigma_{i,1}\right) dp^i_\sigma\right] + \right.$$

$$+\sum_{i=c}^{s-c-1}\left[-\frac{\partial H}{\partial p^i_\sigma}+\sum_{k=0}^{c-1}\frac{\partial q^\sigma_i}{\partial q^\nu_k}q^\nu_{k,1}+\sum_{k=c}^{s-c-1}\left(\frac{\partial q^\sigma_i}{\partial p^k_\nu}-\frac{\partial q^\nu_k}{\partial p^i_\sigma}\right)p^k_{\nu,1}\right]dp^i_\sigma\Bigg\}\wedge dt$$

in the chart $((U)_1, (\varphi_{s-1})_1)$. Consequently, according to (4.14), resp. (4.15), δ is a Hamilton extremal associated with $\mathscr{E}$ iff it satisfies the system (6.27) and (6.28) if $s = 2c$ and $c = 2c + 1$, respectively.

The equations (6.27), resp. (6.28), i.e. Hamilton equations associated with the (regular) form $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$ expressed in Legendre coordinates of $\mathscr{E}$, will be called *Hamilton canonical equations.*

7. LAGRANGIANS OF MINIMAL ORDER

Similarly as in the previous sections let $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$ denote a locally variational form satisfying (4.1), and c an integer such that $(s/2) - 1 < c \leqq (s/2)$.

We shall show that the Hamilton function and momenta of $\mathscr{E}$ can be equivalently expressed by means of certain local lagrangians of $\mathscr{E}$.

Lemma 1. *Consider the form $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$. $j^{s-c}Y$ can be covered by open sets Z such that on each Z there exists a lagrangian $\lambda_{\min}$ satisfying (up to a projection) $\mathscr{E}\,|_Z = E_{\lambda\,\min}$.*

Proof. The local variationality of $\mathscr{E}$ means that j^sY can be covered by open sets W such that on each W there exists a lagrangian $\lambda \in \Omega^1_X(W)$ satisfying $\mathscr{E}\,|_W = E_\lambda$ (up to a projection). Let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y such that $V_s \cap W = U \neq \emptyset$. Then λ can be constructed by setting

$$\lambda = A\mathscr{E} = (q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\,du)\,dt \tag{7.1}$$

on U (see (2.8)). Put

$$\lambda_{\min} = \lambda - h(df) \tag{7.2}$$

where the function f is given by (6.5). Obviously, $\lambda_{\min}$ is a local lagrangian for $\mathscr{E}$; we shall show that it is $\pi_{s,s-c}$-projectable. Put $\lambda_{\min} = L_{\min}\,dt$. Then

$$L_{\min} = q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\,du - \tag{7.3}$$

$$-\frac{d}{dt}\Big(\sum_{k=s-c}^{s-1}\sum_{i=0}^{s-1-k} 2q^\nu_k q^\sigma_i \int_0^1\int_0^1 (F^{ik}_{\sigma\nu}\circ \chi_{s-1}\circ \chi_{s-1,s-c})\,u\,du\,dv - \frac{d\varphi}{dt},$$

where $\varphi = \varphi(t, q^\varrho, \ldots, q^\varrho_{s-c-1})$ is an arbitrary function. Computing $\partial L_{\min}/\partial q^\nu_k$ for $s - c + 1 \leqq k \leqq s$ we obtain using (6.8)

$$\frac{\partial L_{\min}}{\partial q_k^\nu} = \frac{\partial}{\partial q_k^\nu}\Big(q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du\Big) - \frac{d}{dt}\Big(\frac{\partial f}{\partial q_k^\nu}\Big) - \frac{\partial f}{\partial q_{k-1}^\nu} =$$
$$= q^\sigma \int_0^1 \Big(\frac{\partial E_\sigma}{\partial q_k^\nu} - 2F_{\sigma\nu}^{0,k-1} - 2\frac{d}{dt}(F_{\sigma\nu}^{0k})\Big)\circ \chi_s u\, du -$$
$$- \sum_{i=1}^{s-k-1} 2q_i^\sigma \int_0^1 \Big(F_{\sigma\nu}^{i-1,k} + F_{\sigma\nu}^{i,k-1} + \frac{d}{dt}(F_{\sigma\nu}^{ik})\Big)\circ \chi_s u\, du -$$
$$- 2q_{s-k}^\sigma \int_0^1 (F_{\sigma\nu}^{s-k-1,k} + F_{\sigma\nu}^{s-k,k-1})\circ \chi_{s-1} u\, du = 0, \tag{7.4}$$

for $s - c + 1 \leqq k \leqq s - 1$, because of (3.6), (3.8) and (5.3); similarly

$$\frac{\partial L_{\min}}{\partial q_s^\nu} = q^\sigma \int_0^1 \Big(\frac{\partial E_\sigma}{\partial q_s^\nu} - 2F_{\sigma\nu}^{0,s-1}\Big)\circ \chi_s u\, du = 0 \tag{7.5}$$

because of (3.7). Hence $\lambda_{\min}$, defined on an open set $Z \subset j^{s-c}Y$, $Z = \pi_{s,s-c}U$, is the desired lagrangian.

Notice that for each function $\varphi = \varphi(t, q^\varrho, \dots, q_{s-c-1}^\varrho)$ $\lambda_{\min}$ is a (local) lagrangian of minimal possible order for $\mathscr{E}$; explicitly, $\lambda_{\min}$ is of order c (resp. $c + 1$) if $s = 2c$ (resp. $s = 2c + 1$). We note that such a lagrangian was firstly constructed in [12].

Lemma 2. *Let (V, ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y, let (6.6) (resp. (6.7)) (for an arbitrary but fixed function φ) be the momenta (resp. the Hamilton function) of the form $\mathscr{E} \in \Omega_Y^{1,1}(j^sY)$, defined on an open set $U \subset V_{s-1}$. Then it holds*

$$p_\nu^k = (f_{\min})_\nu^{k+1}, \qquad 1 \leqq \nu \leqq m,\ 0 \leqq k \leqq s - c - 1, \tag{7.6}$$

$$H = -L_{\min} + \sum_{i=0}^{s-c-1} (f_{\min})_\sigma^{i+1} q_{i+1}^\sigma \tag{7.7}$$

for a lagrangian $\lambda_{\min} \in \Omega_X^1(\pi_{s-1,s-c}U)$ of $\mathscr{E}$, $\lambda_{\min} = L_{\min}\,\mathrm{d}t$, where $L_{\min}$ and $(f_{\min})_\sigma^i$, $1 \leqq i \leqq s - c$ are defined by (7.3) *and*

$$(f_{\min})_\sigma^i = \sum_{k=0}^{s-c-1-i} (-1)^k \frac{d^k}{dt^k}\Big(\frac{\partial L_{\min}}{\partial q_{k+1}^\sigma}\Big), \qquad 1 \leqq i \leqq s - c,\ 1 \leqq \sigma \leqq m \tag{7.8}$$

respectively, in the chart (V, ψ).

Proof. From (6.3) we obtain (up to $\pi_{s,s-1}$)

$$A\alpha_{\mathscr{E}} - df = \Big(-H + \sum_{k=0}^{s-c-1} p_\nu^k q_{k+1}^\nu\Big) dt + \sum_{k=0}^{s-c-1} p_\nu^k \omega_k^\nu, \tag{7.9}$$

where f is given by (6.5). Let $\lambda_{\min} \in \Omega_X^1(\pi_{s-1,s-c}U)$ be defined by (7.2). Then

$$\Theta_{\lambda\min} = L_{\min}\, dt + \sum_{i=0}^{s-c-1} (f_{\min})_\sigma^{i+1} \omega_i^\sigma \tag{7.10}$$

with $L_{\min}$ (resp. $(f_{\min})^i_\sigma$) given by (7.3) (resp. (7.8)). Notice that $\partial(f_{\min})^{i+1}_\sigma/\partial q^\nu_s = -2F^{is}_{\sigma\nu} = 0$ for $0 \leqq i \leqq s - c - 1$, i.e. $\Theta_{\lambda\min}$ is projectable onto $j^{s-1}Y$. Thus using (7.2) and the fact that $A\alpha_{\mathscr{E}}$ is a Lepagean 1-form we obtain (up to a projection) $A\alpha_{\mathscr{E}} - df = \Theta_{A\mathscr{E}} - \Theta_{h(df)} = \Theta_{\lambda\min}$. This completes the proof.

Theorem 8. *Let $\lambda_{\min}$ be an arbitrary lagrangian of minimal order for $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$, defined on an open set $Z \subset j^{s-c}Y$, denote by $\lambda_{\min} = L_{\min}\,dt$ its chart expression in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y such that $V_{s-c} \cap Z = U \neq \emptyset$.*

(1) *Let $s = 2c$. The following conditions are equivalent:*

(i) *$\lambda_{\min}$ is regular on U,*

(ii) *at each point of U*

$$\det\left(\frac{\partial^2 L_{\min}}{\partial q^\sigma_c\,\partial q^\nu_c}\right) \neq 0, \tag{7.11}$$

(iii) *$(\pi^{-1}_{2c-1,c}U, \bar{\psi}_{2c-1})$, where $\bar{\psi}_{2c-1} = (t, q^\sigma, \ldots, q^\sigma_{c-1}, (f_{\min})^1_\sigma, \ldots, (f_{\min})^c_\sigma)$, $1 \leqq \sigma \leqq m$ is a Legendre chart of $\mathscr{E}$.*

(2) *Let $s = 2c + 1$. The following conditions are equivalent:*

(i) *$\lambda_{\min}$ is regular on U,*

(ii) *at each point of U*

$$\det\left(\frac{\partial^2 L_{\min}}{\partial q^\sigma_{c+1}\,\partial q^\nu_c} - \frac{\partial^2 L_{\min}}{\partial q^\sigma_c\,\partial q^\nu_{c+1}}\right) \neq 0, \tag{7.12}$$

(iii) *there exists a lagrangian $\lambda^0_{\min}$ of $\mathscr{E}$ on an open set $\bar{U} \subset U$ such that $(\pi^{-1}_{2c,c+1}\bar{U}, \bar{\psi}_{2c})$, where $\bar{\psi}_{2c} = (t, q^\sigma, \ldots, q^\sigma_{c-1}, (f^0_{\min})^1_\sigma, \ldots, (f^0_{\min})^{c+1}_\sigma)$, $1 \leqq \sigma \leqq m$ is a Legendre chart of $\mathscr{E}$.*

Proof. Theorem 8 is a consequence of the regularity condition (5.1), and of Lemma 2 and Theorem 6.

Corollary. *Let $\mathscr{E} \in \Omega^{1,1}_Y(j^sY)$ be regular, consider a covering of $j^{s-1}Y$ by open sets U such that for each U $(U, \bar{\psi}_{s-1})$ is a Legendre chart of $\mathscr{E}$. If $s = 2c$ (resp. $s = 2c + 1$) then each system of Legendre coordinates $\bar{\psi}_{s-1}$ of $\mathscr{E}$ defined on U arises from a lagrangian of minimal order $\lambda_{\min} \in \Omega^1_X(\pi_{2c-1,c}U)$ of $\mathscr{E}$ (resp. from a lagrangian of minimal order $\lambda^0_{\min} \in \Omega^1_X(\pi_{2c,c+1}U)$ of $\mathscr{E}$ which satisfies the condition*

$$\det\left(\frac{\partial^2 L^0_{\min}}{\partial q^\sigma_{c+1}\,\partial q^\nu_c}\right) \neq 0 \tag{7.13}$$

at each point $x \in \pi_{2c,c+1}U$).

Example 5. Let $\lambda \in \Omega^1_X(j^2Y)$ be a lagrangian of type (2.13), $\mathscr{E} = E_\lambda \in \Omega^{1,1}_Y(j^2Y)$ its Euler-Lagrange form. Let us construct a (local) lagrangian of minimal order of $\mathscr{E}$. According to (7.3) and (6.23) we obtain $\lambda_{\min} = L_{\min}\,dt$, where

(7.14)
$$L_{\min} = q^\sigma \int_0^1 \left(\frac{\partial L_0}{\partial q^\sigma} \circ \chi_1\right) du + q_1^\sigma \int_0^1 \left(\frac{\partial L_0}{\partial q_1^\sigma} \circ \chi_1\right) du -$$
$$- q_1^\sigma \left[\int_0^1 \left(\frac{dg_\sigma}{dt} \circ \chi_1\right) du + q^\nu \int_0^1 \left(\frac{\partial}{\partial q^\nu}\left(\frac{dg_\sigma}{dt}\right) \circ \chi_1\right) u\, du + \right.$$
$$\left. + q_1^\nu \int_0^1 \left(\frac{\partial}{\partial q_1^\nu}\left(\frac{dg_\sigma}{dt}\right) \circ \chi_1\right) u\, du \right] = L_0 - q_1^\sigma \frac{dg_\sigma}{dt} = L - \frac{d}{dt}(g_\sigma q_1^\sigma)$$

up to $d\varphi/dt$ where $\varphi = \varphi(t, q^\varrho)$ is an arbitrary function and L is defined by (2.13); hence $\lambda_{\min} \sim \lambda$. Moreover

(7.15)
$$(f_{\min})_\sigma^1 = \frac{\partial L_{\min}}{\partial q_1^\sigma} = \frac{\partial L_0}{\partial q_1^\sigma} - \frac{\partial g_\sigma}{\partial t} - \left(\frac{\partial g_\sigma}{\partial q^\nu} + \frac{\partial g_\nu}{\partial q^\sigma}\right) q_1^\nu = p_\sigma,$$

for $1 \leqq \sigma \leqq m$ (compare with (6.26)). According to Theorem 8 (t, q^σ, p_σ) are Legendre coordinates of $\mathscr{E}$ iff at each point of the domain of definition of the functions p_σ $\lambda_{\min}$ satisfies $\det(\partial^2 L_{\min}/\partial q_1^\sigma \partial q_1^\nu) \neq 0$.

Our last example is meant to illustrate in a simple situation the notions and techniques introduced in this paper.

Example 6. Consider the fibered manifold $\pi : R \times R^2 \to R$, denote by t (resp. (t, x, y), resp. $(t, x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dddot{x}, \dddot{y})$) the canonical coordinates on R (resp. $R \times R^2$, resp. the associated coordinates on $j^3(R \times R^2)$). Let $E \in \Omega_Y^{1,1}(j^3(R \times R^2))$ be of the form

(7.16)
$$E = (-m_1 \ddot{x} - \dddot{y})\, dx \wedge dt + (-m_2 \ddot{y} + \dddot{x})\, dy \wedge dt,$$

where m_1, m_2 are positive constants. The form (7.16) is obviously variational and regular satisfying (3.2) and (5.1), respectively. Thus one may search for the Hamilton canonical equations of E. From (3.3) we obtain

(7.17)
$$F_{xx}^{10} = \frac{1}{2} m_1, \quad F_{yy}^{10} = \frac{1}{2} m_2, \quad F_{xy}^{11} = F_{yx}^{02} = F_{yx}^{20} = \frac{1}{2},$$

the remaining ones being equal to zero (up to the relation $F_{\sigma\nu}^{ik} = -F_{\nu\sigma}^{kl}$). Let us compute the momenta and the Hamilton function of E. Substituting (7.17) into (6.5) we obtain

(7.18)
$$f = 2\ddot{x} y \int_0^1 \int_0^1 (F_{yx}^{02} \circ \chi_2 \circ \chi_{2,2})\, u\, du\, dv +$$
$$+ 2\ddot{y} x \int_0^1 \int_0^1 (F_{xy}^{02} \circ \chi_2 \circ \chi_{2,2})\, u\, du\, dv + \varphi(t, x, y, \dot{x}, \dot{y}) =$$
$$= \frac{1}{2}(\ddot{x} y - \ddot{y} x) + \varphi(t, x, y, \dot{x}, \dot{y}),$$

where φ is an arbitrary function. Thus (according to (6.6) and (6.7))

$$\bar{p}_x^1 = -\frac{1}{2}\dot{y} - \frac{1}{2}m_1 x - \frac{\partial\varphi}{\partial\dot{x}}, \qquad \bar{p}_x^0 = \ddot{y} + \frac{1}{2}m_1\dot{x} - \frac{\partial\varphi}{\partial x}, \tag{7.19}$$

$$\bar{p}_y^1 = \frac{1}{2}\dot{x} - \frac{1}{2}m_2 y - \frac{\partial\varphi}{\partial\dot{y}}, \qquad \bar{p}_y^0 = -\ddot{x} + \frac{1}{2}m_2\dot{y} - \frac{\partial\varphi}{\partial y},$$

and

$$H = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \dot{x}\ddot{y} - \dot{y}\ddot{x} + \frac{\partial f}{\partial t}. \tag{7.20}$$

Let us choose φ in the form

$$\varphi = -\frac{1}{2}(m_1\dot{x}x + m_2\dot{y}y). \tag{7.21}$$

Then we obtain from (7.19)

$$p_x^1 = -\frac{1}{2}\dot{y}, \qquad p_x^0 = \ddot{y} + m_1\dot{x}, \tag{7.22}$$

$$p_y^1 = \frac{1}{2}\dot{x}, \qquad p_y^0 = -\ddot{x} + m_2\dot{y}.$$

Theorem 6 guarantees that $(t, x, y, p_x^0, p_y^0, p_x^1, p_y^1)$ are Legendre coordinates of E on $j^2(R\times R^2) \approx R\times R^2\times R^2\times R^2$. Obviously, the inverse Legendre transformation formulas are given by

$$\dot{x} = 2p_y^1, \qquad \ddot{x} = -p_y^0 - 2m_2 p_x^1, \tag{7.23}$$

$$\dot{y} = -2p_x^1, \qquad \ddot{y} = p_x^0 - 2m_1 p_y^1.$$

Expressing the Hamilton function (7.20) in the Legendre coordinates we obtain

$$H = 2p_x^0 p_y^1 - 2p_y^0 p_x^1 - 2m_1(p_y^1)^2 - 2m_2(p_x^1)^2. \tag{7.24}$$

Consider a section $\delta : R \ni t \to (t, \delta_0(t)) \in R\times(R^2\times R^2\times R^2)$, where $\delta_0(t) = (x(t), y(t), p_x^0(t), p_y^0(t), p_x^1(t), p_y^1(t))$ in the Legendre coordinates. Using Theorem 7 we get that δ is a Hamilton extremal of E iff the functions $x(t)$, $y(t)$, $p_x^0(t)$, $p_y^0(t)$, $p_x^1(t), p_y^1(t)$ satisfy the system of Hamilton canonical equations

$$\frac{dp_x^0}{dt} = 0, \qquad \frac{dp_y^0}{dt} = 0, \qquad \frac{dx}{dt} = 2p_y^1, \qquad \frac{dy}{dt} = -2p_x^1, \tag{7.25}$$

$$\frac{dp_x^1}{dt} = -\frac{1}{2}p_x^0 + m_1 p_y^1, \qquad \frac{dp_y^1}{dt} = -\frac{1}{2}p_y^0 - m_2 p_x^1.$$

Notice that $\delta = j^2\gamma$ where $\gamma : R \ni t \to (t, \gamma_0(t)) \in R\times R^2$, $\gamma_0(t) = (x(t), y(t))$ is a section satisfying the Euler–Lagrange equations $-m_1\ddot{x} - \dddot{y} = 0$, $-m_2\ddot{y} + \dddot{x} = 0$. Solving the system (7.25) one can find all Hamilton extremals of E. We note that another choice of the function $\varphi = \varphi(t, x, y, \dot{x}, \dot{y})$ such that (6.9) is satisfied would lead to the Hamilton extremals of E expressed in another coordinates on $i^2(R \times R^2)$.

The minimal lagrangian of E corresponding to the choice (7.21) is of the form $\lambda_{\min} = L_{\min}\, dt$ where

$$L_{\min} = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2}(\ddot{y}\dot{x} - \ddot{x}\dot{y}); \tag{7.26}$$

evidently $(f_{\min})_\nu^{k+1} = p_\nu^k$, $\nu = x, y$, $k = 0, 1$.

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