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COMPARISON THEOREMS FOR STURM—LIOUVILLE EQUATIONS

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Abstract. Concerning the fferential equations $-(P(x)\,u'') + Q(x)\,u = 0$ and $-(p(x)\,u')' + q(x)\,u = 0$, $a \leqq x \leqq b$, Sturm-type comparison theorems are proved where the co. ditions on the coefficients in question are, for instance, $p \leqq P$ and mean value conditions for q and Q on certain subintervals of $[a, b]$. The results are closely related to well-known theorems of Levin and Fink.

Key words. Sturm-Liouville equation, comparison of solutions.
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Consider the differential equations

$$(1) \qquad L[u] \equiv -(P(x)\,u')' + Q(x)\,u = 0, \qquad P > 0, P \in C^1, Q \in C, \\ -\infty < a \leqq x \leqq b < \infty,$$

and

$$(2) \qquad l[u] \equiv -(p(x)\,u')' + q(x)\,u = 0, \qquad p > 0, p \in C^1, q \in C.$$

In the special case $P \equiv p \equiv 1$ a well-known comparison theorem of Levin [2] states the following (see [5]).

Theorem 1 (Levin): *Let $P \equiv p \equiv 1$ be fulfilled and suppose that there exists a nontrivial solution u of* (1) *with $u(a) = u(b) = u'(c) = 0$, $a < c < b$. If the inequality*

$$(3) \qquad \int_{x_1}^{x_2} q(x)\,dx \leqq -\left|\int_{x_1}^{x_2} Q(x)\,dx\right|$$

holds for all pairs of numbers x_1, x_2 with $a \leqq x_1 \leqq c \leqq x_2 \leqq b$, then every solution of (2) *has at least one zero on $[a, b]$.*

Condition (3) implies that all mean values of $q(x)$ on intervals $[x_1, x_2]$, $x_1 \leqq c \leqq x_2$, are non-positive. In the following we shall prove a corresponding comparison theorem where mean values of $q(x)$ can also be positive. We give the following preparation.

Let $u(x)$ be a nontrivial solution of the boundary problem

$$L[u] = 0, \qquad u(a) = 0 = u(b),$$

with fixed sign on (a, b); assume that u is positive on (a, b). Choose a positive function f belonging to $C^2[a, b]$. Then because of

$$\lim_{x \downarrow a} \frac{u'}{u} = \infty, \qquad \lim_{x \uparrow b} \frac{u'}{u} = -\infty,$$

it is easily seen that there exist points t_1, t_2 with $a < t_1 \leqq t_2 < b$ such that

$$\text{(4)} \qquad \frac{f'(t_i)}{f(t_i)} = \frac{u'(t_i)}{u(t_i)}, \qquad i = 1, 2, \quad \frac{f'(x)}{f(x)} \leqq \frac{u'(x)}{u(x)}, \qquad a < x \leqq t_1, \qquad \frac{f'(x)}{f(x)} \geqq \frac{u'(x)}{u(x)}, \qquad t_2 \leqq x < b.$$

Note that there can exist several points t_1 or t_2 with the properties (4), respectively. Set

$$\text{(5)} \qquad c_i = f(t_i)\, u^{-1}(t_i), \qquad i = 1, 2,$$

and define the function

$$\text{(6)} \qquad v(x) = \begin{cases} c_1 u(x), & a \leqq x < t_1, \\ f(x), & t_1 \leqq x \leqq t_2, \\ c_2 u(x), & t_2 < x \leqq b. \end{cases}$$

It follows from (4) and (5) that $v(x)$ is a continuously differentiable function on $[a, b]$. Seting

$$v(x) = \mu(x) f(x), \qquad a \leqq x \leqq b,$$

we have

$$\mu' \mu^{-1} = v' v^{-1} - f' f^{-1}$$

and (4) implies that

$$\text{(7)} \qquad \begin{aligned} &\mu \in C^1[a, b]; \qquad \mu'(x) \geqq 0, a \leqq x \leqq t_1; \\ &\mu(x) = 1, \ t_1 \leqq x \leqq t_2; \qquad \mu'(x) \leqq 0, \ t_2 \leqq x \leqq b. \end{aligned}$$

v will be used as a test function to estimate the quadratic form of equation (2). Supposing

$$\text{(8)} \qquad p(x) \leqq P(x), \qquad a \leqq x \leqq b,$$

we have

$$\begin{aligned} \int_a^b [p(v')^2 + qv^2]\, dx &= \int_a^b [(p - P)(v')^2 + (q - Q) v^2]\, dx + \int_a^b [P(v')^2 + Qv^2]\, dx \leqq \\ \text{(9)} \qquad &\leqq \int_a^b (q - Q) v^2\, dx + \int_a^b [P(v')^2 + Qv^2]\, dx. \end{aligned}$$

(7) shows that the function $\mu^2(x)$ is monotone increasing on $[a, t_1]$ from $\mu^2(a) = 0$ to $\mu^2(t_1) = 1$ and monotone decreasing on $[t_2, b]$ from $\mu^2(t_2) = 1$ to $\mu^2(b) = 0$.

Therefore, be a mean value theorem of integral calculus, there exist points τ_1, $a \leqq \tau_1 \leqq t_1$, and τ_2, $t_2 \leqq \tau_2 \leqq b$, such that

$$\int_a^b (q - Q)\, v^2\, dx = \int_{\tau_1}^{\tau_2} (q - Q)\, f^2\, dx, \qquad a \leqq \tau_1 \leqq t_1, \qquad t_2 \leqq \tau_2 \leqq b. \tag{10}$$

The second integral on the right-hand side of (9) is handled by integration by parts as follows.

$$\begin{aligned} \int_a^b [P(v')^2 + Qv^2]\, dx &= c_1^2 \int_a^{t_1} [P(u')^2 + Qu^2]\, dx + \int_{t_1}^{t_2} [P(f')^2 + Qf^2]\, dx + \\ &+ c_2^2 \int_{t_2}^{b} [P(u')^2 + Qu^2]\, dx = c_1^2 P(t_1)\, u'(t_1)\, u(t_1) + P(t_2)\, f'(t_2)\, f(t_2) - \\ &- P(t_1)\, f'(t_1)\, f(t_1) + \int_{t_1}^{t_2} L[f]\, f\, dx - c_2^2 P(t_2)\, u'(t_2)\, u(t_2) = \int_{t_1}^{t_2} L[f]\, f\, dx. \end{aligned} \tag{11}$$

Thus, we obtain

$$\int_a^b [p(v')^2 + qv^2]\, dx \leqq \int_{\tau_1}^{\tau_2} (q - Q)\, f^2\, dx + \int_{t_1}^{t_2} L[f]\, f\, dx, \tag{12}$$
$$a \leqq \tau_1 \leqq t_1 \leqq t_2 \leqq \tau_2 \leqq b,$$

where the numbers t_1 and t_2 are defined by (4).

Theorem 2: *Let u be a nontrivial solution of equation* (1) *with fixed sign on* (a, b) *and $u(a) = 0 = u(b)$ and let f be a positive function belonging to $C^2[a, b]$. If* (8) *is fulfilled and the inequality*

$$\int_{x_1}^{x_2} (q - Q)\, f^2\, dx + \int_{t_1}^{t_2} L[f]\, f\, dx \leqq 0 \tag{13}$$

holds for all pairs of numbers x_1, x_2 with $a \leqq x_1 \leqq t_1$ and $t_2 \leqq x_2 \leqq b$ where t_1 and t_2 are defined by

$$\frac{f'(t_i)}{f(t_i)} = \frac{u'(t_i)}{u(t_i)}, \qquad i = 1, 2,\ t_1 \leqq t_2, \tag{14}$$
$$\frac{f'(x)}{f(x)} \leqq \frac{u'(x)}{u(x)}, \qquad a < x \leqq t_1, \qquad \frac{f'(x)}{f(x)} \geqq \frac{u'(x)}{u(x)}, \qquad t_2 \leqq x < b,$$

then every solution v of equation (2) *has a zero in* (a, b) *or v has the properties*

i) *v is a constant multiple of u on $[a, t_1]$,*
ii) *v is a constant multiple of f on $[t_1, t_2]$,*
iii) *v is a constant multiple of u on $[t_2, b]$.*

Proof: In view of (13) it follows from (12) that

$$\int_a^b [p(v')^2 + qv^2]\, dx \leqq 0, \tag{15}$$

where v is the test function (6). v belongs to the domain of the closure of the form

$$l(\varphi, \psi) = \int_a^b (p\varphi'\bar{\psi}' + q\varphi\bar{\psi})\,\mathrm{d}x, \qquad \varphi, \psi \in C_0^\infty(a, b),$$

of equation (2). Because of (15) two cases are possible,

$$\inf_{\varphi \in C_0^\infty,\ \|\varphi\| = 1} l(\varphi, \varphi) < 0 \qquad \text{or} \qquad \inf_{\varphi \in C_0^\infty,\ \|\varphi\| = 1} l(\varphi, \varphi) = 0,$$

where $\| \varphi \|$ denotes the norm of φ in the Hilbert space $L_2(a, b)$. In the first case equation (2) has a nontrivial solution with at least two zeros in (a, b) (cp. [3]). Then by Sturm's comparison theorem every solution of (2) has a zero in (a, b). In the second case the infimum of the form is realized by the (normalized) function v. Consequently, this function v is an eigenfunction of the Friedrichs extension A of the operator A_0,

$$A_0\varphi = l[\varphi], \qquad \varphi \in C_0^\infty(a, b),$$

in the Hilbert space $L_2(a, b)$. The corresponding eigenvalue is zero. Now it is easily seen that v belongs to $C^2[a, b]$. v is a classical solution of (2). This proves Theorem 2.

Corollary 1: *Let u be a nontrivial solution of* (1) *with fixed sign on (a, b) and $u(a) = 0 = u(b)$ and let f be a positive function belonging to $C^2[a, b]$. Assume that there exists a point c, $a < c < b$, such that*

$$\frac{f'(c)}{f(c)} = \frac{u'(c)}{u(c)}, \quad \frac{f'(x)}{f(x)} \leqq \frac{u'(x)}{u(x)}, \quad a < x \leqq c, \quad \frac{f'(x)}{f(x)} \geqq \frac{u'(x)}{u(x)}, \quad c \leqq x < b.$$

If (8) *is fulfilled and the inequality*

$$\int_{x_1}^{x_2} qf^2\,\mathrm{d}x \leqq \int_{x_1}^{x_2} Qf^2\,\mathrm{d}x$$

holds for all pairs x_1, x_2 with $a \leqq x_1 \leqq c \leqq x_2 \leqq b$, then every solution v of equation (2) *has a zero in (a, b), or v is a constant multiple of u.*

Proof: Set $t_1 = t_2 = c$ in Theorem 2.

Corollary 2: *Let $P \equiv p \equiv 1$ and assume that u is a nontrivial solution of equation* (1) *with fixed sign on (a, b) and $u(a) = 0 = u(b)$. If the inequality*

$$(16) \qquad \int_{[x_1, x_2]} q(x - x_0)^2\,\mathrm{d}x \leqq \int_{[x_1, x_2] \setminus [t_1, t_2]} Q(x - x_0)^2\,\mathrm{d}x$$

holds for a point $x_0 \notin [a, b]$ and all pairs x_1, x_2 with $a \leqq x_1 \leqq t_1 \leqq t_2 \leqq x_2 \leqq b$ where t_1 and t_2 are defined by

$$(17) \qquad \frac{1}{t_i - x_0} = \frac{u'(t_i)}{u(t_i)}, \qquad i = 1, 2,$$

$$\frac{1}{x - x_0} \leqq \frac{u'(x)}{u(x)}, \qquad a < x \leqq t_1, \qquad \frac{1}{x - x_0} \geqq \frac{u'(x)}{u(x)}, \qquad t_2 \leqq x < b,$$

then every solution v of equation (2) *has a zero in* (a, b), *or v has the following properties*:

i) *v is a constant multiple of u on* $[a, t_1]$,

ii) *v is a constant multiple of* $x - x_0$ *on* $[t_1, t_2]$,

iii) *v is a constant multiple of u on* $[t_2, b]$.

Proof: By choosing $f(x) = x - x_0$ in Theorem 2 it follows that

$$\int_{t_1}^{t_2} L[f]\, f \,dx = \int_{t_1}^{t_2} Q(x - x_0)^2 \,dx.$$

Thus, (16) implies (13), and Corollary 2 follows from Theorem 2. The geometrical meaning of (17) is that there exist tangents $y_i(x) = \lambda_i(x - x_0)$, $i = 1, 2$, touching the curve of u at t_i, respectively.

The special case $f \equiv 1$ leads to the following corollaries.

Corollary 3: *Let u be a nontrivial solution of* (1) *with fixed sign on* (a, b) *and* $u(a) = 0 = u(b)$ *and assume that*

$$u'(t_1) = 0 = u'(t_2), \quad a < t_1 \leqq t_2 < b; \quad u'(x) \geqq 0, \quad a \leqq x \leqq t_1; \quad u'(x) \leqq 0, \quad t_2 \leqq x \leqq b.$$

If (8) is fulfilled and the inequality

$$\int_{[x_1, x_2]} q \,dx \leqq \int_{[x_1, x_2] \setminus [t_1, t_2]} Q \,dx \tag{18}$$

holds for all pairs of numbers x_1, x_2 with $a \leqq x_1 \leqq t_1 \leqq t_2 \leqq x_2 \leqq b$, then v has a zero in (a, b) or v has the following properties:

i) *v is a constant multiple of u on* $[a, t_1]$,

ii) $v =$ const *on* $[t_1, t_2]$,

iii) *v is a constant multiple of u on* $[t_2, b]$.

Proof: Set $f \equiv 1$ in Theorem 2.

A special case of Corollary 3 is the case $t_1 = t_2 = c$, $a < c < b$. Then inequality (18) has the form

$$\int_{x_1}^{x_2} q \,dx \leqq \int_{x_1}^{x_2} Q \,dx, \qquad a \leqq x_1 \leqq c \leqq x_2 \leqq b.$$

In this special case Corollary 3 is closely related to a result of Fink [1] concerning the smallest positive eigenvalues λ_1 and λ_2 of the problems

$$(p(x)\, u')' + \lambda_1 q_1(x)\, u = 0, \qquad u(a) = 0 = u(b),$$

and

$$(p(x)\, u')' + \lambda_2 q_2(x)\, u = 0, \qquad u(a) = 0 = u(b).$$

Concerning the importance of the quantity of these eigenvalues for oscillation or disconjugacy of the corresponding equations compare [4, p. 53].

In the following the restriction $x_0 \notin [a, b]$ supposed in Corollary 2 is to be omitted. Assume that there exist points $x_0 \in (a, b)$ and t with $x_0 < t < b$ such that

$$\frac{1}{t - x_0} = \frac{u'(t)}{u(t)} \quad \text{and} \quad \frac{1}{x - x_0} \geqq \frac{u'(x)}{u(x)}, \qquad t \leqq x < b,$$

where u is the solution of equation (1) from above. Then, the function

$$v(x) = \begin{cases} 0, \; a \leqq x < x_0, \\ x - x_0, \; x_0 \leqq x \leqq t, \\ (t - x_0)\, u^{-1}(t)\, u(x), \; t \leqq x \leqq b, \end{cases} \tag{19}$$

belongs to the Sobolev space $\mathring{W}_2^1(a, b)$[1]) which is identical with the domain of the closure of the form of equation (2). By using this function v the estimate (12) gets the form

$$\int_a^b [p(v')^2 + q(x)\, v^2]\, dx \leqq \int_{x_0}^{\tau} (q - Q)(x - x_0)^2\, dx + \int_{x_0}^{t} Q(x - x_0)^2\, dx, \tag{20}$$
$$a < x_0 < t \leqq \tau \leqq b.$$

Of course, an analogous estimate holds when the point t is situated to the left of x_0. Finally, the point x_0 can be identical with one of the endpoints of the interval (a, b). The following corollary corresponds to the case $a < x_0 < t < b$.

Corollary 4: *Let $P \equiv p \equiv 1$ and assume that u is a nontrivial solution of equation* (1) *with fixed sign on (a, b) and $u(a) = 0 = u(b)$. Let further x_0, $a < x_0 < b$, and t, $x_0 < t < b$, be points with the properties*

$$\frac{1}{t - x_0} = \frac{u'(t)}{u(t)} \quad \text{and} \quad \frac{1}{x - x_0} \geqq \frac{u'(x)}{u(x)}, \qquad t \leqq x < b.$$

If the inequality

$$\int_{x_0}^{\xi} q(x - x_0)^2\, dx \leqq \int_{t}^{\xi} Q(x - x_0)^2\, dx \tag{21}$$

holds for all points ξ with $t \leqq \xi \leqq b$, then every solution v of equation (2) *has a zero on $[x_0, b)$.*

Proof: It follows from (20) and (21) that

$$\int_{x_0}^{b} [p(v')^2 + qv^2]\, dx \leqq 0,$$

where v is defined by (19). In the case

[1]) $\mathring{W}_2^1(a, b)$ is the completion of $C_0^\infty(a, b)$ by using the norm

$$\| \varphi \|_1 = \left(\int_a^b (| \varphi' |^2 + | \varphi |^2)\, dx\right)^{1/2}.$$

$$\inf_{\varphi \in C_0^\infty(x_0, b),\ \|\varphi\|=1} \int_{x_0}^{b} (p\,|\varphi'|^2 + q|\varphi|^2)\,dx < 0,$$

there exists a nontrivial solution of (2) on $[x_0, b]$ with at least two zeros in (x_0, b) and, consequently, every solution of (2) has a zero in (x_0, b) (compare the proof of Theorem 2). Assuming the case

$$\inf_{\varphi \in C_0^\infty(x_0, b),\ \|\varphi\|=1} \int_{x_0}^{b} (p\,|\varphi'|^2 + q\,|\varphi|^2)\,dx = 0$$

the (normalized) function $v(x)$, $x_0 \leqq x \leqq b$, of (19) realizes the infimum. Hence v is a nontrivial solution of (2) on $[x_0, b]$ which has the zero x_0. This proves Corollary 4.

In the case $x_0 = a$ we obtain the following result.

Corollary 5: *Let the suppositions of Corollary 4 be fulfilled for* $x_0 = a$. *Then every solution* v *of equation* (2) *has a zero in* (a, b), *or* v *has the following properties*:

i) v *is a constant multiple of* $x - a$ *on* $[a, t]$,

ii) v *is a constant multiple of* u *on* $[t, b]$.

$t = a$: *If*

$$\frac{1}{x-a} \geqq \frac{u'(x)}{u(x)}, \qquad a < x < b, \tag{22}$$

and the inequality

$$\int_a^\xi q(x-a)^2\,dx \leqq \int_a^\xi Q(x-a)^2\,dx \tag{23}$$

holds for all ξ, $a < \xi < b$, *then every solution* v *of* (2) *has a zero in* (a, b) *or* v *is a constant multiple of* u.

The proof of Corollary 5 is analogous to the proof of Corollary 4.

Example: Every solution of the equation

$$-u'' + q(x)\,u = 0, \qquad q \not\equiv -1, \qquad -\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}, \tag{24}$$

has a zero in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if there exists a point c, $-\frac{\pi}{2} \leqq c \leqq \frac{\pi}{2}$, $c \neq 0$, such that

$$\max_{-\frac{\pi}{2} \leqq x_1 \leqq c \leqq x_2 \leqq \frac{\pi}{2}} \int_{x_1}^{x_2} (q+1)(x - c - \cot c)^2\,dx \leqq 0 \tag{25}$$

or if

$$\sup_{-\frac{\pi}{2} \leqq x_1 < 0 < x_2 \leqq \frac{\pi}{2}} \left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} q\,dx\right) \leqq -1. \tag{26}$$

Proof: Compare equation (24) with the equation

$$-u'' - u = 0, \qquad u\left(-\frac{\pi}{2}\right) = 0 = u\left(\frac{\pi}{2}\right),$$

and take $u = \cos x$. In the case where $|c| < \frac{\pi}{2}$, $c \neq 0$, apply Corollary 2. Condition (17) is fulfilled for $t_1 = t_2 = c$ and $x_0 = c + \cot c$. Then (25) corresponds to (16) with $Q \equiv -1$. In the case where $c = -\frac{\pi}{2}$ apply Corollary 5 under the supposition $t = a$. In this case the condition (25) has the form

$$\max_{-\frac{\pi}{2} \leqq \xi \leqq \frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\xi} (q+1)\left(x + \frac{\pi}{2}\right)^2 dx \leqq 0.$$

An analogous condition is valid in the case $c = \frac{\pi}{2}$. Inequality (26) corresponds to (18) of Corollary 3.

Corollary 6: *Let $P \equiv p \equiv 1$ and consider the solution u of equation* (1) *determined by the initial values $u(c) = \alpha > 0$, $u'(c) = \beta > 0$, $a < c < b$. If the inequalities*

$$(27) \qquad -\frac{\pi^2}{(b-a)^2} \leqq \frac{\int_{x_1}^{x_2} Q(x - c + \alpha\beta^{-1})^2\, dx}{\int_{x_1}^{x_2} (x - c + \alpha\beta^{-1})^2\, dx} \leqq 0$$

hold for all numbers x_1, x_2 with

$$\max(a, c - \alpha\beta^{-1}) \leqq x_1 < c < x_2 \leqq b,$$

then the solution u does not vanish in at least one of the intervals (a, c) or (c, b). In the case where $u(c) = \alpha > 0$, $u'(c) = 0$, the same conclusion is true when

$$(28) \qquad -\frac{\pi^2}{(b-a)^2} \leqq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} Q\, dx \leqq 0$$

for all x_1, x_2 with $a \leqq x_1 < c < x_2 \leqq b$.

Proof: Assume that u has a zero a' in (a, c) and a zero b' in (c, b). We may assume that u is positive on (a', b'). Now apply the Corollaries 2–5. First let $u'(c) > 0$. It follows from

$$\frac{1}{c - x_0} = \frac{u'(c)}{u(c)} = \frac{\beta}{\alpha}$$

that $x_0 = c - \alpha\beta^{-1}$. Thus, replacing a by a' and b by b', Corollary 2 can be applied when $c - \alpha\beta^{-1} < a'$. The points t_1 and t_2 can be determined such that (17) is fulfilled with $a = a'$ and $b = b'$. Now it follows from (27) that (16) is fulfilled by setting

$$q(x) = -\frac{\pi^2}{(b-a)^2}.$$

The solution $v = \sin\left(\pi \frac{x-a}{b-a}\right)$ of equation (2), however, does not vanish on $[a', b']$ contradictory to the conclusion of Corollary 2. Assume now $a' < c - \alpha\beta^{-1}$ and apply Corollary 4 with $a = a'$ and $b = b'$. The point t, $c \leqq t < b'$, can be determined and (27) implies (21) with

$$q(x) = -\frac{\pi^2}{(b-a)^2}.$$

Thus, considering the solution $v = \sin\left(\pi \frac{x-a}{b-a}\right)$ of equation (2) we again obtain a contradiction. Finally, in the case $a' = c - \alpha\beta^{-1}$ apply Corollary 5 with $a = a'$ and $b = b'$. Analogously, the assertion of Corollary 6 under the supposition $\beta = 0$ follows from Corollary 3. This completes the proof of Corollary 6.

The case $u(c) = \alpha > 0$, $u'(c) = \beta < 0$ can be handled analogously.

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