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ON THE CONNECTIONS NATURALLY INDUCED ON THE SECOND ORDER FRAME BUNDLE

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Abstract. All natural operators transforming every linear connection into a connection on the semi-holonomic second order frame bundle form a 9-parameter family. Some geometric properties of these operators are deduced. There is a unique natural operator transforming every symmetric linear connection into a connection without torsion on the holonomic second order frame bundle.

Key words. Linear connection, connections on the second order frame bundle, natural operators.

Let H^1M be the first order frame bundle (i.e. the bundle of all linear frames) of a manifold M and H^2M or $\bar{H}^2M$ be the second order frame bundle or the semi-holonomic second order frame bundle, respectively. Further, let QH^1M, QH^2M and $Q\bar{H}^2M$ denote the corresponding bundles of connections. For every manifold M, consider an operator A_M transforming every linear connection $\Gamma : M \to QH^1M$ into a connection $A_M\Gamma : M \to Q\bar{H}^2M$ on the semi-holonomic second order frame bundle. Such a family of operators $A = \{A_M\}$ is said to be a natural operator, if every commutative diagram on the left is transformed into a commutative diagram on the right

$$
\begin{tikzcd}
QH^1M \arrow[r, "QH^1f"] & QH^1N \\
M \arrow[u, "\Gamma"] \arrow[r, "f"] & N \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Q\bar{H}^2M \arrow[r, "Q\bar{H}^2f"] & Q\bar{H}^2N \\
M \arrow[u, "A_M\Gamma"'] \arrow[r, "f"] & N \arrow[u, "A_N\Delta"']
\end{tikzcd}
$$

for every local diffeomorphism $f : M \to N$. In other words, if a connection Γ is transported into a connection Δ by a local diffeomorphism, then $A\Gamma$ is transported into $A\Delta$ by the same local diffeomorphism. The first author deduced analytically, under the assumption that one considers finite order operators only, [1], that all natural operators transforming every linear connection on M into a connection on $\bar{H}^2M$ form the following 9-parameter family

$$
\begin{aligned}
(1) \qquad & \bar{\Gamma}^i_{jk} = t\Gamma^i_{jk} + (1-t)\Gamma^i_{kj},\\
& \Gamma^i_{jkl} = a\Gamma^i_{jk,l} + b\Gamma^i_{jl,k} + c\Gamma^i_{kj,l} + d\Gamma^i_{kl,j} + e\Gamma^i_{lj,k} +\\
& + f\Gamma^i_{lk,j} + (t-c-e-d+\alpha)\Gamma^i_{jm}\Gamma^m_{kl} + (1-f-t-\alpha)\Gamma^i_{jm}\Gamma^m_{lk} +\\
& + (c+e-\alpha)\ {}^i_{mj}\Gamma^m_{kl} + \alpha\Gamma^i_{mj}\Gamma^m_{lk} + (c+d+e-1+t+\beta)\Gamma^i_{km}\Gamma^m_{jl} +\\
& + (1-e-t-\beta)\Gamma^i_{km}\Gamma^m_{lj} + (a+f-\beta)\Gamma^i_{mk}\Gamma^m_{jl} + \beta\Gamma^i_{mk}\Gamma^m_{lj} +\\
& + (t-a-b-d+\gamma)\Gamma^i_{lm}\Gamma^m_{jk} + (-c-\gamma)\Gamma^i_{lm}\Gamma^m_{kj} + (b+d-t-\gamma)\Gamma^i_{ml}\Gamma^m_{jk} +\\
& + \gamma\Gamma^i_{ml}\Gamma^m_{kj},
\end{aligned}
$$

with constant coeficients satisfying $a + b + c + d + e + f = 1$.

The basic purpose of the present paper is to study some geometric properties of the above operators. We first show that two basic operators of (1) can be determined by two original constructions related with arbitrary fibred manifolds. Our Proposition 4 describes in a direct geometric way a simple system of independent generators of (1). Then we discuss the most interesting classes of linear connections such that the values of some of those generators coincide or the induced family of connections on $\bar{H}^2M$ degenerates in a specific way. These problems lead frequently to some special properties of the curvatures of the connections in question. In the last section we deduce that there is a unique natural operator transforming every symetric linear connection into a connection without torsion on H^2M.

Our considerations are in the category C^∞.

1. The first author deduced, [1], that (1) can be constructed in a simple way from two basic operators. We now describe another constructions of both operators in a more general situation. We recall that a generalized connection on an arbitrary fibred manifold $Y \to M$ means any smooth section $\Gamma : Y \to J^1Y$ (= the first jet prolongation of Y). If Y is a principal bundle and Γ is right-invariant, we have a connection in the classical sense. If we want to distinguish the latter connection from an arbitrary generalized connection, we shall say that it is a principal connection. We first explain two geometric constructions of a generalized connection on the fibred manifold $J^1Y \to M$ from a generalized connection Γ on Y and a linear connection Λ on M.

If x^i are some local coordinates on M, y^p are some additional local fibre coordinates on Y and y^p_i are the induced local coordinates on J^1Y, then the equations of a generalized connection Γ on Y are

$$
(2) \qquad y^p_i = F^p_i(x^i, y^p) \qquad \begin{array}{l} i, j, \ldots = 1, \ldots, \dim M,\\ p, q, \ldots = 1, \ldots, \dim Y - \dim M. \end{array}
$$

The linear connection Λ on M can be interpreted as a map $\Lambda : TM \to J^1TM$ with the following coordinate expression

$$\xi^i_j = \Lambda^i_{kj}(x)\,\xi^k,\ x = (x^i), \tag{3}$$

where $\xi^i = \mathrm{d}x^i$ are the additional coordinates on TM. It is well-known that $J^1Y \to \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$, where VY means the vertical tangent bundle of Y. Section $\Gamma : Y \to J^1Y$ determines a point at each affine space, which identifies these spaces with the corresponding vector spaces. Hence Γ induces an identification $I_\Gamma : J^1Y \approx VY \otimes T^*M$. Construct the vertical prolongation $V\Gamma : VY \to VJ^1Y$ of the map Γ and use the canonical identification $VJ^1Y \approx J^1VY$. This gives a generalized connection $\mathscr{V}\Gamma : VY \to \to J^1VY$ on $VY \to M$, see also [3]. The coordinate expression of $\mathscr{V}\Gamma$ is (2) and

$$\eta^p_i = \frac{\partial F^p_i(x, y)}{\partial y^q}\,\eta^q, \tag{4}$$

where $\eta^p = \mathrm{d}y^p$ are the additional coordinates on VY. Further, Λ induces by duality a linear morphism $\Lambda^* : T^*M \to J^1T^*M$. Since (4) is linear in η^p, we can define the tensor product $\mathscr{V}\Gamma \otimes \Lambda^*$, [3], which is a generalized connection on $VY \otimes T^*M$. Then I_Γ transforms $\mathscr{V}\Gamma \otimes \Lambda^*$ into a connection $p(\Gamma, \Lambda)$ on $J^1Y \to M$ with the coordinate expression (2) and

$$y^p_{ij} = \frac{\partial F^p_j}{\partial y^q}(y^q_i - F^q_i) - \Lambda^k_{ij}(y^p_k - F^p_k) + \frac{\partial F^p_i}{\partial x^j} + \frac{\partial F^p_i}{\partial y^q}F^q_j. \tag{5}$$

On the other hand, consider the lifting map $\gamma : Y \oplus TM \to TY$ of Γ and construct its first jet prolongation $J^1\gamma : J^1Y \oplus J^1TM \to J^1TY$. If we add the linear connection $\Lambda : TM \to J^1TM$ and a natural map $\mu : J^1TY \to TJ^1Y$ introduced by Mangiarotti and Modugno, [6], we obtain $\mu \circ J^1\gamma \circ (\mathrm{id} \oplus \Lambda) : J^1Y \oplus TM \to \to TJ^1Y$. This map is linear in TM, so that it is a lifting map of a generalized connection $q(\Gamma, \Lambda)$ on J^1Y. (This connection was constructed in [3] in another way by the second author, who is grateful to Marco Modugno for suggesting this shorter procedure.) The coordinate expression of $q(P, \Lambda)$ is (2) and

$$y^p_{ij} = \frac{\partial F^p_j}{\partial x^i} + \frac{\partial F^p_j}{\partial y^q}\,y^q_i + \Lambda^k_{ji}(F^p_k - y^p_k). \tag{6}$$

Let $\tilde{\Lambda}$ denote the linear connection conjugate to Λ. If we compare (5) and (6) with the curvature form of generalized connection Γ, [3], we deduce.

Proposition 1. *It holds $p(\Gamma, \Lambda) = q(\Gamma, \tilde{\Lambda})$ if and only if Γ is integrable.*

Every local coordinates x^i on M are canonically extended into local coordinates x^i, x^i_j on H^1M and into local coordinates x^i, x^i_j, x^i_{jk} on $\bar{H}^2M$, $\det x^i_j \neq 0$, the subbundle $H^2M \subset \bar{H}^2M$ being characterized by $x^i_{jk} = x^i_{kj}$. The equations of a connection Γ on H^1M are

$$dx^i_j = \Gamma^i_{lk}(x)\, x^l_j dx^k, \tag{7}$$

where Γ^i_{jk} differ by sign and by the order of subscripts from the classical Christoffel's of Γ (i.e. the Christoffel symbols of Γ as defined e.g. in [4] are $-\Gamma^i_{kj}$). The equations of a connection on $\bar{H}^2M$ are

$$\begin{aligned} dx^i_j &= \Gamma^i_{lk}(x)\, x^l_j dx^k, \\ dx^i_{jk} &= (\Gamma^i_{lmn}(x)\, x^l_j x^m_k + \Gamma^i_{ln}(x)\, x^l_{jk})\, dx^n, \end{aligned} \tag{8}$$

and such a connection is reducible to $H^2M \subset \bar{H}^2M$ iff $\Gamma^i_{jkl} = \Gamma^i_{kjl}$, [1].

There is a canonical identification $J^1H^1M \approx \bar{H}^2M$ over the identity of H^1M, [5], the coordinate expression of which is $x^i_{jk} = x^i_{j,l}x^l_k$, where $x^i_{j,k}$ are the induced jet coordinates on J^1H^1M. If we take $Y = H^1M$, then a direct evaluation shows that $p(\Gamma, \Lambda)$ is a principal connection on $\bar{H}^2M \approx J^1H^1M$ if and only if $\Gamma = \Lambda$. This gives a natural operator p of (1), $p(\Gamma) = p(\Gamma, \Gamma)$, with the following coordinate expression

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}, \qquad \Gamma^i_{jkl} = \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{ml}\Gamma^m_{jk}, \tag{9}$$

where $\Gamma^i_{jk,l}$ are the induced coordinates on J^1QH^1M. (This operator was constructed in another way in [2].) In the same situation, one finds easily that $q(\Gamma, \Lambda)$ is a principal connection on $\bar{H}^2M$ if and only if $\Gamma = \tilde{\Lambda}$. This gives another natural operator q of (1), $q(\Gamma) = q(\Gamma, \tilde{\Gamma})$, with the following coordinate expression

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}, \qquad \Gamma^i_{jkl} = \Gamma^i_{jl,k} + \Gamma^i_{jm}\Gamma^m_{kl}. \tag{10}$$

This operator was constructed in another way by Oproiu, [7].

By Proposition 1, we obtain immediately.

Proposition 2. *It holds $p\Gamma = q\Gamma$ iff Γ is integrable.*

The second author deduced in the case of an arbitrary fibered manifold, [3], that $q(\Gamma, \tilde{\Lambda})$ is an integrable connection iff both Γ and Λ are integrable. This implies

Proposition 3. *$q\Gamma$ is integrable iff Γ is integrable.*

2. The canonical involution of semi-holonomic 2-jets, [8], determines an involutive automorphism $\bar{H}^2M \to \bar{H}^2M$, $(x^i, x^i_j, x^i_{jk}) \to (x^i, x^i_j, x^i_{kj})$. This map is extended into an involutive automorphism $i : Q\bar{H}^2M \to Q\bar{H}^2M$, whose coordinate effect consists in the exchange of the first two subscripts in Γ^i_{jkl}. Denote by $\bar{p}$ or $\bar{q}$ the operators $\Gamma \to p\tilde{\Gamma}$ or $\Gamma \to q\tilde{\Gamma}$, respectively. Further, write $\Sigma = \frac{1}{2}\Gamma + \frac{1}{2}\tilde{\Gamma}$ for the classical symmetrization of linear connection Γ and define $s\Gamma = p\Sigma$ and $\bar{s}\Gamma = q\Sigma$. Hence we have 12 natural operators $p, q, \bar{p}, \bar{q}, s, \bar{s}, ip, iq, i\bar{p}, i\bar{q}, is, i\bar{s}$ transforming any linear connection into a connection on the semi-holonomic second order frame bundle.

Lemma 1. *It holds* $s = \frac{1}{4}(p + ip + \bar{p} + i\bar{p})$.

Proof. This follows easily from (9).

Applying i to the relation of Lemma 1, we obtain $s = is$.

Proposition 4. *The operators* $p, q, \bar{p}, \bar{q}, \bar{s}, ip, iq, i\bar{p}, i\bar{q}, i\bar{s}$ *generate affinely the 9-parameter family* (1).

Proof. The family affinely generated by our operators is $a_1 p + \dots + a_{10} i\bar{s}$ with $\sum_{i=1}^{10} a_i = 1$. Using direct evaluation one verifies that the latter family is equivalent to (1), QED.

Let S denote the torsion tensor of a linear connection Γ, i.e. $S^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$.

Proposition 5. *It holds* $p\Gamma = ip\Gamma$ iff $\nabla_\Gamma S = 0$.

Proof. This follows directly from (9).

We recall that the coordinate expression of the curvature tensor $\Omega(\Gamma)$ of Γ is

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}. \tag{11}$$

Lemma 2. *It holds* $iq\Gamma = t(p\Gamma) + (1 - t)\, q\Gamma$ *with* $t \in R$ *iff* $tR^i_{jkl} + \tilde{R}^i_{ljk} = 0$, *where* $\tilde{R}^i_{jkl}$ *is the curvature tensor of the conjugate connection* $\tilde{\Gamma}$.

Proof. By (9) and (10) we have

$$\frac{\partial \Gamma^i_{kl}}{\partial x^j} + \Gamma^i_{km}\Gamma^m_{jl} = t\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{mk}\Gamma^m_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{ml}\Gamma^m_{jk}\right) + (1 - t)\left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{jm}\Gamma^m_{kl}\right) \tag{12}$$

Comparing with (11) we prove our assertion.

For $t = 0$ Lemma 2 implies

Proposition 6. *It holds* $iq\Gamma = q\Gamma$ *iff* $\tilde{\Gamma}$ *is integrable.*

3. If Γ is a symmetric linear connection, then $\Gamma = \tilde{\Gamma} = \Sigma$ and Proposition 5 implies $p\Gamma = ip\Gamma$, so that all connections naturally induced on $\bar{H}^2 M$ by Γ are affinely generated by $p\Gamma$, $q\Gamma$ and $iq\Gamma$, i.e. they form the following two-parameter family

$$Ap\Gamma + Bq\Gamma + (1 - A - B)\, iq\Gamma, \qquad A, B \in R. \tag{13}$$

Moreover, if Γ is also integrable, then $p\Gamma = q\Gamma$ by Proposition 2 and $q\Gamma = iq\Gamma$ by Proposition 6. Taking into account Proposition 3, we prove.

Proposition 7. *An integrable symmetric linear connection determines naturally a unique connection on* $\bar{H}^2 M$, *which is also integrable.*

Conversely, if all connections of (13) coincide, we have $p\Gamma = q\Gamma$, which implies by Proposition 2 that Γ is integrable.

It remains to discuss under what conditions (13) is a one-parameter family. In this case $p\Gamma \neq q\Gamma$ and $iq\Gamma = tp\Gamma + (1 - t)\, q\Gamma$, $t \in R$. By Lemma 2 the latter condition is equivalent to $tR^i_{jkl} + R^i_{ljk} = 0$. This gives.

Proposition 8. *Connections* (13) *form a one-parameter family iff* $tR^i_{jkl} + R^i_{ljk} = 0$.

4. Consider an arbitrary linear connection Γ. In (1) there is an 8-parameter family of connections over every linear connection $t\Gamma + (1 - t)\,\tilde{\Gamma}$, $t \in R$. If all connections of each of these families coincide, we shall say that the prolongation of Γ is unique in the pencil sense.

Lemma 3. (*Generalized Bianchi identity.*) *It holds*

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} + \tilde{R}^i_{jkl} + \tilde{R}^i_{klj} + \tilde{R}^i_{ljk} = -S^i_{mj}S^m_{kl} - S^i_{mk}S^m_{lj} - S^i_{ml}S^m_{jk}. \tag{14}$$

Proof consists in direct evaluation.

Proposition 9. *The prolongation of* Γ *is unique in the pencil sense iff all connections* $t\Gamma + (1 - t)\,\tilde{\Gamma}$, $t \in R$, *are integrable.*

Proof. In general, all connections of a pencil are integrable if three connections of the pencil are integrable, [3]. If the prolongation fo Γ is unique in the pencil sense, it holds $p\Gamma = q\Gamma$, $\bar{p}\Gamma = \bar{q}\Gamma$ and $s\Gamma = \bar{s}\Gamma$, which implies $\Omega(\Gamma) = 0$, $\Omega(\tilde{\Gamma}) = 0$ and $\Omega(\Sigma) = 0$. Thus, all connections $t\Gamma + (1 - t)\,\tilde{\Gamma}$ are integrable. Conversely, let $\Omega(\Gamma) = \Omega(\tilde{\Gamma}) = \Omega(\Sigma) = 0$. Then Proposition 2 gives $p\Gamma = q\Gamma$, $\bar{p}\Gamma = \bar{q}\Gamma$, so that $ip\Gamma = iq\Gamma$, $i\bar{p}\Gamma = i\bar{q}\Gamma$. Further, Proposition 6 implies $q\Gamma = iq\Gamma$, $\bar{q}\Gamma = i\bar{q}\Gamma$, $\bar{s}\Gamma = i\bar{s}\Gamma$. Hence the family of connections naturally induced by Γ is reduced to $\lambda q\Gamma + \mu\bar{q}\Gamma + \varrho\bar{s}\Gamma$, $\lambda + \mu + \varrho = 1$. Setting $t = \lambda + \dfrac{\varrho}{2}$, the latter family can be rewritten as

$$tq\Gamma + (1 - t)\,\bar{q}\Gamma + \frac{\varrho}{4}(S^i_{mj}S^m_{kl}), \tag{15}$$

where $(S^i_{mj}S^m_{kl})$ is a tensor field on M with the indicated coordinate expression. In general, denoting by $\bar{R}^i_{jkl}$ the curvature tensor of Σ, we deduce by direct evaluation

$$\frac{1}{2}R^i_{jkl} + \frac{1}{2}\tilde{R}^i_{jkl} - \bar{R}^i_{jkl} = \frac{1}{4}(S^i_{mk}S^m_{jl} + S^i_{ml}S^m_{kj}). \tag{16}$$

Since all Γ, $\tilde{\Gamma}$ and Σ are integrable, (16) implies $S^i_{mk}S^m_{lj} + S^i_{ml}S^m_{jk} = 0$ and Lemma 3 gives $S^i_{mj}S^m_{kl} = 0$. Then our assertion follows from (15), QED.

By (15), if the assumptions of Proposition 9 are satisfied, then the induced pencil of connections is

$$tq\Gamma + (1 - t)\,\bar{q}\Gamma. \tag{17}$$

5. If Γ is integrable, we have $p\Gamma = q\Gamma$, $ip\Gamma = iq\Gamma$, $\bar{q}\Gamma = i\bar{q}\Gamma$ by Propositions 2 and 6, so that the induced connections form a 6-parameter family. If $\tilde{\Gamma}$ is also integrable, we obtain similarly $\bar{p}\Gamma = \bar{q}\Gamma$, $i\bar{p}\Gamma = i\bar{q}\Gamma$, $q\Gamma = iq\Gamma$. In this case the induced connections form a 3-parameter family

(18) $$a_1 q\Gamma + a_2 \bar{q}\Gamma + a_3 \bar{s}\Gamma + a_4 i\bar{s}\Gamma, \qquad a_1 + a_2 + a_3 + a_4 = 1.$$

We shall discuss under what conditions this family degenerates. If $\bar{s}\Gamma$ is an affine combination of $q\Gamma$ and $\bar{q}\Gamma$, then $q\Gamma = iq\Gamma$ and $\bar{q}\Gamma = i\bar{q}\Gamma$ implies $i\bar{s}\Gamma = \bar{s}\Gamma$, which is the case of Proposition 9. *If $\bar{s}\Gamma \neq i\bar{s}\Gamma$ and* (18) *degenerates, the fact that the underlying connections Γ, $\tilde{\Gamma}$ and Σ of $q\Gamma$, $\bar{q}\Gamma$ and $\bar{s}\Gamma$ are different implies*

(19) $$i\bar{s}\Gamma = tq\Gamma + u\bar{q}\Gamma + (1 - t - u)\,\bar{s}\Gamma, \qquad t, u \in R.$$

Then the underlying linear connections satisfy $\Sigma = t\Gamma + u\tilde{\Gamma} + (1 - t - u)\,\Sigma$, which gives $u = t$. Thus,

(20) $$i\bar{s}\Gamma = tq\Gamma + t\bar{q}\Gamma + (1 - 2t)\,\bar{s}\Gamma.$$

Applying i to (20), we obtain

(21) $$\bar{s}\Gamma = tq\Gamma + t\bar{q}\Gamma + (1 - 2t)\,i\bar{s}\Gamma.$$

Since $q\Gamma, \bar{q}\Gamma$ and $\bar{s}\Gamma$ are affinely independent, (20) and (21) imply $t^2 - t = 0$. For $t = 0$ we obtain the case $\bar{s}\Gamma = i\bar{s}\Gamma$. For $t = 1$ we have $i\bar{s}\Gamma = q\Gamma + \bar{q}\Gamma - \bar{s}\Gamma$. Reformulating this expression, we prove

Proposition 10. *If both Γ and $\tilde{\Gamma}$ are integrable and $\Omega(\Sigma) \neq 0$, then the induced connections form a* 3*-parameter family* (18). *This family degenerates into a two-parameter family iff*

(22) $$\frac{1}{2}(q\Gamma + \bar{q}\Gamma) = \frac{1}{2}(\bar{s}\Gamma + i\bar{s}\Gamma)$$

6. Consider the canonical $R^n \oplus \mathfrak{l}^1_n$ — valued form Θ on $\bar{H}^2 M$, $n = \dim M$, [11], where $\mathfrak{l}^1_n$ is the Lie algebra of $L^1_n = Gl(n, R)$. The coordinate expression of Θ is, [10],

(23) $$\Theta^i = \tilde{x}^i_j dx^j$$
$$\Theta^i_j = \tilde{x}^i_k dx^k_j - \tilde{x}^i_k x^k_{jl} \tilde{x}^l_m dx^m,$$

where $\tilde{x}^i_j$ means the inverse matrix to x^i_j. The torsion form of a connection on $\bar{H}^2 M$ is defined as the covariant differential of Θ with respect to the connection, [11]. One finds easily that the torsion form of a connection (8) vanishes iff it holds

(24) $$\Gamma^i_{jk} = \Gamma^i_{kj}, \qquad \Gamma^i_{jkl} - \Gamma^i_{mk}\Gamma^m_{jl} = \Gamma^i_{jlk} - \Gamma^i_{ml}\Gamma^m_{jk}.$$

We determine all natural operators transforming every linear connection into a connection without torsion on $\bar{H}^2 M$. We deduce by (1) and (24) $t = \dfrac{1}{2}$, $a = b$, $c = e$, $d = f$, $\alpha = c$ and $\beta = \dfrac{1}{4} + \gamma$. This is a 3-parameter family. Assume further that Γ is symmetric. One finds easily that connections (13) are without torsion iff $A = B$ or Γ is integrable. The second case was discussed in Proposition 7. Now we can add to Proposition 7 that the unique connection naturally induced on

$\bar{H}^2M$ is without torsion and is reducible to H^2M. On the other hand, a non-integrable symmetric linear connection determines naturally a one-parameter family

$$Ap\Gamma + Aq\Gamma + (1 - 2A)\, iq\Gamma \tag{25}$$

of connections without torsion on $\bar{H}^2M$. One verifies directly that (25) is reducible to H^2M iff $A = \frac{1}{3}$. This proves

Proposition 11. *There is a unique natural operator transforming every symmetric linear connection into a connection without torsion on H^2M. Its coordinate expression is*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}, \qquad \Gamma^i_{jkl} = \frac{1}{6}\Gamma^i_{(jk,l)} + \frac{2}{3}\Gamma^i_{m(j}\Gamma^m_{k)l} - \frac{1}{3}\Gamma^i_{lm}\Gamma^m_{jk}, \tag{26}$$

where the round bracket means symmetrization.

We remark that this operator was constructed in another way by Rybnikov, [9].

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