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A PURELY NUMBER THEORETIC ATTEMPT TO PROVE PICARD'S THEOREMS

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Abstract. In this paper a purely number-theoretic formulations for Picard's Little and Big theorems are given. Specifically, it is shown that Picard's Little theorem is equivalent to the vanishing of every term a_n of a sequence (a_n) of complex numbers where a_n is related to the terms b_n and c_n of two appropriate sequences (b_n) and (c_n) of complex numbers with vanishing n-th root limits of $|a_n|$, $|b_n|$ and $|c_n|$. Analogous formulation for Picard's Big theorem is also stated.

Key words. Picard's theorems. Zero's of analytic functions.

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In 1879 and 1880 Picard proved his celebrated Little and Big Theorems which initiated one of the most important stages in the history of the development of the theory of analytic functions of a complex variable. The initial proof of Picard (based on modular functions) is quite intricate and does not reveal in a straight forward way the reasons why the machinery of his proof works. Subsequently, there were massive attempts to give simpler and more tangible proofs of Picard's Theorems by Hadamard, Borel, Schottky, Caratheodory, Bloch, Landau and Nevanlinna [1]. These endeavors are still continuing. However, none of these attempts seem to reveal the true nature of the reasons behind the proofs. Moreover, all of these proofs require several preliminary lemmas which, in their turn, obscure the central line of reasoning.

In what follows we suggest a novel and purely number-theoretic approach to prove Picard's Theorems. We are not proving anything since we are unable to derive the desired conclusions from an infinite set of interrelated numbers. However, our suggestion strikes directly at the heart of the matter, and, in its purely number-theoretic formulation (see Theorems 1 and 2) requires no prior knowledge of any part of the theory of analytic functions. Indeed, our suggested approach reveals the true, tangible and easily comprehensible nature of Picard's Theorems. In fact, our Theorem 1 shows that Picard's Little Theorem is equivalent to the statement that if each of the limits of a certain three interrelated convergent sequences is equal to zero then each of these sequences is the zero sequence (i.e., every term of the sequence is equal to 0). Our Theorem 2 shows that analogous considerations apply to the Big Theorem of Picard.

We hope that a reader with a deep insight in number-theoretic combinatorics will come up with a proof of our Theorems 1 and 2, naturally without invoking Picard's Theorems.

In order to motivate our formulation of Picard's Little Theorem, we first recall its statement and then make some observations.

Picard's Little Theorem states that if an entire function f does not take on two values, say, 0 and 2 then f is a constant. Expressed in a more detailed form, we have:

Picard's little theorem. *Let f given by*

$$f(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n + \dots \tag{1}$$

be an entire function such that

$$f(z) \neq 0 \quad \text{and} \quad f(z) \neq 2 \quad \text{for every } z \tag{2}$$

Then

$$0 = a_1 = a_2 = a_3 = \dots = a_n = \dots \tag{3}$$

Now, let us observe that since f is entire, the radius of convergence of the power series appearing in (1) is ∞, Thus, by Cauchy—Hadamard formula:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = 0 = \lim_{n\to\infty} \sqrt[n]{|a_n|}, \tag{4}$$

since if the limit superior of a sequence of nonnegative terms is equal to 0 then the sequence converges to 0. Also, from (1) and (2) it follows that each of

$$1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \quad \text{and} \quad -1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

never vanishes. Consequently, each of

$$\frac{1}{1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots} \quad \text{and} \quad \frac{1}{-1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots} \tag{5}$$

is an entire function. Hence, each has a power series representation whose radius of convergence is ∞. Accordingly, let

$$\frac{1}{1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots \tag{6}$$

where the b_i's in terms of a_i's are given by:

$$\begin{aligned} b_1 + a_1 &= 0 \\ b_2 + b_1 a_1 + a_2 &= 0 \\ b_3 + b_2 a_1 + b_1 a_2 + a_3 &= 0 \\ &\dots \\ b_n + b_{n-1} a_1 + b_{n-2} a_2 + \dots + b_1 a_{n-1} + a_n &= 0 \\ &\dots \end{aligned} \tag{7}$$

The above equations are easily obtained by cross multiplying the two sides of the equation appearing in (6) and by setting the coefficients of the various powers of z equal to 0.

Since, as mentioned above, the radius of convergence of the power series appearing in (6) is ∞, we (as in the case of (4)) have:

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = 0. \tag{8}$$

Applying the same considerations to the second fraction appearing in (5), we obtain:

$$\frac{1}{-1 + a_1 z + a_2 z^2 + a_3 z^3 + \dots} = -1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \tag{9}$$

where the c_i's in terms of a_i's are given gy:

$$\begin{aligned} c_1 + a_1 &= 0 \\ c_2 - c_1 a_1 + a_2 &= 0 \\ c_3 - c_2 a_1 - c_1 a_2 + a_3 &= 0 \\ &\dots \\ c_n - c_{n-1} a_1 - c_{n-2} a_2 - \dots - c_1 a_{n-1} + a_n &= 0 \qquad (10) \\ &\dots \end{aligned}$$

Here again, we have:

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = 0. \tag{11}$$

Denoting the set of all positive integers by N, from the above it immediately follows that Picard's Little Theorem can be equivalently stated as:

Theorem 1. (Picard's Little Theorem). *Let $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ and $(c_n)_{n\in N}$ be sequences of complex numbers such that for every $n \in N$*

$$b_n + b_{n-1} a_1 + b_{n-2} a_2 + \dots + b_1 a_{n-1} + a_n = 0 \tag{12}$$

and

$$c_n - c_{n-1} a_1 - c_{n-2} a_2 - \dots - c_1 a_{n-1} + a_n = 0 \tag{13}$$

and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \sqrt[n]{|c_n|} = 0. \tag{14}$$

Then

$$0 = a_1 = a_2 = \dots = a_n = \dots \tag{15}$$

Remark 1. We observe that Theorem 1, although equivalent to Picard's Little Theorem is a statement detached from any function-analytic considerations. It requires and is based only on the notion of the limit of a sequence. Indeed, Theorem 1 says that if two sequences (b_n) and (c_n) are related to a third sequence (a_n) via (12) and (13) and if the limit of each of the sequences $(\sqrt[n]{|a_n|})$ and $(\sqrt[n]{|b_n|})$ and $(\sqrt[n]{|c_n|})$ is equal to 0 then every term a_n of the sequence (a_n) is equal to 0. The latter, clearly, in view of (12) and (13) also implies that every b_n and c_n is also equal to 0.

It seems incredible that in order to prove that (12), (13), (14) imply (15) one has to develop all the sophisticated machinery which is usually used in the standard proofs of Picard's Little Theorem. It seems almost certain that there should be a direct and straightforward way to prove that (12), (13), (14) imply (15).

Next, in order to motivate our formulation of Picard's Big Theorem, we first recall its statement and then make some observations.

Picard's Big Theorem states that if f is an analytic function on a disk punctured at the center and if f does not take on two values, say, 0 and -1 then the center of the disk is either a pole or a removable singularity of f. Expressed in a more detailed form, we have:

Picard's big theorem. *Let f given by*

$$f(z) = \ldots + \frac{a_{-n}}{z^n} + \ldots + \frac{a_{-1}}{z} + a_0 + a_1 z + \ldots + a_n z^n + \ldots \tag{16}$$

be a function analytic in the punctured disk $0 < |z| < r$ where r is a positive real number. Let

$$f(z) \neq 0 \quad \text{and} \quad f(z) \neq -1 \quad \text{for every } z \text{ with } 0 < |z| < r \tag{17}$$

Then

$$0 = a_{-P} = a_{-P-1} = a_{-P-2} = \ldots = a_{-P-n} = \ldots \text{ for some positive interger } P. \tag{18}$$

Now, let us observe that since f is analytic in the punctured disk given by $0 < |z| < r$, the radius of convergence of the series of negative powers of z appearing in (16) is ∞. Thus, by Cauchy—Hadamard formula we have:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_{-n}|} = 0 = \lim_{n\to\infty} \sqrt[n]{|a_{-n}|} \tag{19}$$

as in the case of (4). Moreover, since r appearing in (17) is positive, the radius of convergence of the series of nonnegative powers of z appearing in (16) is positive. Thus, by Cauchy—Hadamard formula we have:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} < \infty. \tag{20}$$

From (17) and (16) it follows that each of

$$\frac{1}{\sum_{-\infty}^{\infty} a_n z^n} \quad \text{and} \quad \frac{1}{1 + \sum_{-\infty}^{\infty} a_n z^n}, \tag{21}$$

is a function analytic on the punctured disk $0 < | z | < r$. Hence, each has a power (Laurent) series representation. Accordingly, let

(22) $$\frac{1}{\sum\limits_{-\infty}^{\infty} a_n z^n} = \sum_{-\infty}^{\infty} b_n z^n \qquad \textit{for every } z \textit{ with } 0 < | z | < r,$$

where the b_i's in terms of a_i's are given by:

(23) $$1 = \sum_{-\infty}^{\infty} a_i b_{-i} \qquad \textit{and} \qquad 0 = \sum_{-\infty}^{\infty} a_i b_{n-i} \qquad \textit{for every } n = \pm 1, \pm 2, \pm 3, \ldots$$

The above equations are easily obtained by cross multiplying the two sides of the equation appearing in (22) and by setting the coefficients of the various powers of z equal to 0.

From (22), as in the case of (16), in view of (19) and (20) we have:

(24) $$\lim_{n\to\infty} \sqrt[n]{| b_{-n} |} = 0 \qquad \textit{and} \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{| b_n |} < \infty.$$

Applying the same considerations to the second fraction appearing in (21), we obtain:

(25) $$\frac{1}{1 + \sum\limits_{-\infty}^{\infty} a_n z^n} = \sum_{-\infty}^{\infty} c_n z^n \qquad \textit{for every } z \textit{ with } 0 < | z | < r,$$

where the c_i's in terms of a_i's are given by:

(26) $$1 = c_0 + \sum_{-\infty}^{\infty} a_i c_{-i} \quad \textit{and} \quad 0 = c_n + \sum_{-\infty}^{\infty} a_i c_{n-i} \quad \textit{for every } n = \pm 1, \pm 2, \pm 3, \ldots$$

Here again, we have:

(27) $$\lim_{n\to\infty} \sqrt[n]{| c_{-n} |} = 0 \qquad \textit{and} \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{| c_n |} < \infty.$$

From the above it immediately follows that Picard's Big Theorem can be equivalently stated as:

Theorem 2. (Picard's Big Theorem). *Let* $\{a_i \mid i = 0, \pm 1, \pm 2, \ldots\}$ *and* $\{b_i \mid i = 0, \pm 1, \pm 2, \ldots\}$ *and* $\{c_i \mid i = 0, \pm 1, \pm 2, \ldots\}$ *be sets of complex numbers such that*

(28) $$1 = \sum_{-\infty}^{\infty} a_i b_{-i} \qquad \textit{and} \qquad 0 = \sum_{-\infty}^{\infty} a_i b_{n-i} \qquad \textit{for every } n = \pm 1, \pm 2, \pm 3, \ldots$$

and

(29) $$1 = c_0 + \sum_{-\infty}^{\infty} a_i c_{-i} \quad \textit{and} \quad 0 = c_n + \sum_{-\infty}^{\infty} a_i c_{n-i} \quad \textit{for every } n = \pm 1, \pm 2, \pm 3, \ldots$$

and

(30) $$\lim_{n\to\infty} \sqrt[n]{|a_{-n}|} = \lim_{n\to\infty} \sqrt[n]{|b_{-n}|} = \lim_{n\to\infty} \sqrt[n]{|c_{-n}|} = 0$$

and

(31) $$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} < \infty \quad \text{and} \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} < \infty \quad \text{and} \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < \infty.$$

Then

(32) $0 = a_{-P} = a_{-P-1} = a_{-P-2} = \ldots = a_{-P-n} = \ldots$ *for some positive integer P.*

Remark 2. Let us observe again that Theorem 2 states Picard's Big Theorem in a form detached from any function-analytic considerations. Indeed, it says that if two sets $\{b_i \mid i = 0, \pm 1, \pm 2, \ldots\}$ and $\{c_i / i = 0, \pm 1, \pm 2, \ldots\}$ of complex numbers are related to a third set $\{a_i \mid i = 0, \pm 1, \pm 2, \ldots\}$ via (28) and (29) and if the related limits and limit superiors satisfy the relations given by (30) and (31) then all but a finite (possibly none) number of a_{-i}'s are equal to 0.

Remark 3. Comparing the statements of Theorems 1 and 2, one realizes, in a rather concrete way, how enormously more complicated is the Big Theorem of Picard compared to the Litttle Theorem. The discrepancy is so overwhelming that one is almost reassured that Theorem 1 should have a rather easy proof utilizing our number-theoretic formulation and approach. We would like to point out also that the strengthwise discrepancy between the two Theorems of Picard are revealed in a much more lucid way by means of our number-theoretic formulation than by their usual formulation.

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