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A NOTE ON HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM-LIOUVILLE FUNCTIONS III

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Abstract. The authors give sufficient conditions for a sequence

$$\{M_k\}_{k=1}^{\infty} = \{\int_{x'_k}^{x'_{k+1}} W(x)\,|\,y'(x)\,|^{\lambda}\,dx\}_{k=1}^{\infty}$$

to be n-times monotonic. Here $y(x)$ is a non-trivial solution of an oscillatory differential equation

$$(2.1) \qquad [g(x)\,y'(x)]' + f(x)\,y(x) = 0,$$

$f(x) > 0$, $g(x) > 0$, $f(x) \in C_2(a, \infty)$, $g(x) \in C_1(a, \infty)$, $x'_1, x'_2, \dots$ are consecutive zeros of $z'(x)$, where $z(x)$ is a non-trivial solution of (2.1) which may or may not be linearly independent of $y(x)$, $W(x)$ is a suitable function and $\lambda > -1$. A few intermediate results are also obtained.

Key words. n-times monotonic function and sequence; completely monotonic function and sequence.
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1. Definitions and notations

A function $\varphi(x)$ is said to be n-times monotonic (or monotonic of order n) on an interval I if

$$(1.1) \qquad (-1)^i \varphi^{(i)}(x) \geqq 0 \qquad i = 0, 1, \dots, n;\ x \in I.$$

For such a function we write $\varphi(x) \in M_n(I)$ or $\varphi(x) \in M_n(a, b)$ in case that I is an open interval (a, b). In case the strict inequality holds throughout (1.1) we write $\varphi(x) \in M_n^*(I)$ or $\varphi(x) \in M_n^*(a, b)$. We say that $\varphi(x)$ is completely monotonic on I if (1.1) holds for $n = \infty$.

A sequence $\{\mu_k\}_{k=0}^{\infty}$ denoted simple by $\{\mu_k\}$ is said to be n-times monotonic if

$$(1.2) \qquad (-1)^i \Delta^i \mu_k \geqq 0 \qquad i = 0, 1, \dots, n;\ k = 0, 1, 2, \dots$$

Here $\Delta\mu_k = \mu_{k+1} - \mu_k$; $\Delta^2\mu_k = \Delta(\Delta\mu_k)$, etc. For such a sequence we write $\{\mu_k\} \in$

$\in M_n$. In case the strict inequality holds throughout (1.2) we write $\{\mu_k\} \in M_n^*$. $\{\mu_k\}$ is called completely monotonic if (1.2) holds for $n = \infty$.

As usual we write $[a, b)$ to denote the interval $\{x \mid a \leqq x < b\}$. $\varphi(x) \in C_n(I)$ means that $\varphi(x)$ has (on I) continuous derivative of the n-th order. As usual $D_x[\varphi(x)]$ denotes the first derivative $\dfrac{d\varphi(x)}{dx}$.

2. New result

Consider a differential equation

$$[g(x)\,y'(x)]' + f(x)\,y(x) = 0 \qquad x \in (a, \infty), \tag{2.1}$$

where $f(x) \in C_2(a, \infty)$, $f(x) > 0$, $g(x) > 0$, $g(x) \in C_1(a, \infty)$.

Now if $y(x)$ is a solution of (2.1) then the function

$$u(x) = \frac{y'(x)\,g(x)}{\sqrt{f(x)}} \tag{2.2}$$

is a solution of

$$u'' + F(x)\,u = 0 \qquad x \in (a, \infty), \tag{2.3}$$

where

$$F(x) = \frac{f(x)}{g(x)} + \frac{1}{2}\frac{f''(x)}{f(x)} - \frac{3}{4}\left[\frac{f'(x)}{f(x)}\right]^2.$$

The change of variable

$$\xi = \int_a^x \frac{d\sigma}{\Psi^2(\sigma)}, \qquad \Psi(x) > 0, \qquad \Psi(x) \in C_2(a, \infty), \tag{2.4}$$

where the above integral is assumed to be convergent for $x \in (a, \infty)$, transforms (2.3) into

$$\frac{d^2\eta}{d\xi^2} + \Phi(\xi)\,\eta = 0, \qquad \xi \in (0, \infty), \tag{2.5}$$

where

$$\eta(\xi) = \frac{u(x)}{\Psi(x)} \qquad \text{and} \qquad \Phi(\xi) = \Psi''\Psi^3 + F(x)\,\Psi^4.$$

Theorem 2.1. *Let $y(x)$ and $z(x)$ be solutions of* (2.1) *on* (a, ∞), *where* $f(x) > 0$, $f(x) \in C_3(a, \infty)$, $g(x) > 0$, $g(x) \in C_1(a, \infty)$.

Suppose that for some $n \geqq 2$ there exists such a function $\Psi(x) > 0$, $\Psi(x) \in C_3(a, \infty)$ that $\Psi^2(x) \in M_n(a, \infty)$ and that functions $W(x)$ and

$$D_x\left\{\Psi''\Psi^3 + \left(\frac{f}{g} + \frac{1}{2}\frac{f''}{f} - \frac{3}{4}\left(\frac{f'}{f}\right)^2\right)\Psi^4\right\} \tag{2.6}$$

are positive and belong to the class $M_{n-2}(a, \infty)$. *Let*

$$0 < \lim_{x \to \infty} \Phi(\xi) \leqq \infty.$$

Let $z'(x)$ *have consecutive zeros at* $x'_1, x'_2, \dots$ *on* $[a, \infty)$. *Then, for fixed* $\lambda > -1$ *there holds*

$$\left\{ \int_{x'_k}^{x'_{k+1}} W(x) \left| \frac{y'(x)\, g(x)}{\Psi(x)\sqrt{f(x)}} \right|^{\lambda} \mathrm{d}x \right\} \in M^*_{n-2}, \qquad k = 1, 2, \dots \tag{2.7}$$

Remark 2.1. If $\Psi \equiv 1$ we obtain a slight modification of [3]. Theorem 3.3.

Proof: Mappings (2.2) and (2.4) transform (2.1) into (2.5) where $\xi \in (0, \xi(\infty))$ and $\eta(\xi) = \dfrac{y'(x)\, g(x)}{\Psi(x)\sqrt{f(x)}}$.

Now $D_\xi(\Phi(\xi)) = D_x(\Phi(\xi)) \dfrac{\mathrm{d}x}{\mathrm{d}\xi} = D_x(\Phi(\xi))\, \Psi^2(x)$. But under hypotheses of the theorem $D_x[\Phi(\xi)] \in M_{n-2}$ and $\Psi^2(x) \in M_n$ on (a, ∞). So we have $D_\xi(\Phi(\xi)) \in M_{n-2}(0, \xi(\infty))$. Now we can apply [4] Theorem (or [1] Theorem 2.1 with $g(x) \equiv 1$) from which (2.7) follows immediately.

As example we introduce the differential equation (2.8) where we can apply Theorem 2.1, but [3] Theorem 3.3 does not lead to any result.

Consider the differential equation

$$(x^2 y')' + x^m y = 0 \qquad x \in (0, \infty),\ m \geqq 2 \tag{2.8}$$

is a real value.

If we choose $\Psi(x) = x^{\frac{2-m}{4}}$, then for $m \geqq 2$ we have that $\Psi^2(x) \in M_\infty(0, \infty)$. Further there holds

$$\Phi(\xi) = 1 - \frac{(11m^2 - 8m + 12)}{x^m}.$$

It is obvious that

$$\lim_{x \to \infty} \Phi(\xi) = 1$$

and

$$D_x[\Phi(\xi)] = \frac{(11m^2 - 8m + 12)\, m}{x^{m+1}} \in M_\infty(0, \infty).$$

Therefore when $W(x) \in M_\infty(0, \infty)$ then for $\lambda > -1$ we get

$$\left\{ \int_{x'_k}^{x'_{k+1}} W(x) \left| \frac{y'(x)\, x^2}{x^{\frac{2-m}{4}}\, x^{\frac{m}{2}}} \right|^{\lambda} \mathrm{d}x \right\} \in M^*_\infty. \tag{2.9}$$

Now if we choose $W(x) = [x^{\frac{2-m}{4}} x^{\frac{m}{2}} x^{-2}]^\lambda = x^{\frac{(m-6)\lambda}{4}}$, then in case $\dfrac{m-6}{4}\lambda \leqq 0$ we obtain $W(x) \in M_\infty(0, \infty)$ and there holds

$$(2.10) \qquad \{ \int_{x_k'}^{x_{k+1}'} | y'(x) |^\lambda \, dx \} \in M_\infty^*.$$

We can see that in case $\lambda \geqq 0$ there must be fulfilled $m \in [2; 6]$.

Theorem 2.2. *Let the hypotheses of Theorem 2.1 be fulfilled. Moreover let*

$$(2.11) \qquad -f(x) D_x \left[\left(\left(\frac{\Psi'}{f} \right)' + \frac{\Psi}{g} \right) \frac{\Psi^3}{f} \right]$$

be positive and belong to the class $M_{n-2}(a, \infty)$. Let $x_1' > a$. Then

$$(2.12) \qquad [z(x_k') \Psi(x_k')]^2 \in M_{n-1}^*, \qquad k = 1, 2, \ldots$$

Proof: Using ([2] Lemma 3.2 Theorem 3.2) we have for $x_1' > a$ that

$$-([z(x_{k+1}') \Psi(x_{k+1}')]^2 - [z(x_k') \Psi(x_k')]^2 =$$

$$= - \int_{x_k'}^{x_{k+1}'} \left[\frac{g(x) z'(x)}{\Psi(x)} \right]^2 D_x \left[\left(\left(\frac{\Psi'}{f} \right)' + \frac{\Psi}{g} \right) \frac{\Psi^3}{f} \right] dx = \int_{x_k'}^{x_{k+1}'} W(x) \left| \frac{z'(x) g(x)}{\Psi(x) \sqrt{f(x)}} \right|^2 dx,$$

where $W(x) = -f(x) D_x \left[\left(\left(\frac{\Psi'}{f} \right)' + \frac{\Psi}{g} \right) \frac{\Psi^3}{f} \right]$. The result now follows from Theorem 2.1 with $\lambda = 2$ once it is shown that $W(x) > 0$, and $W(x) \in M_{n-2}(a, \infty)$. But it is guaranteed by (2.11). This completes the proof.

As example again consider the equation (2.8). By easy calculation we obtain that

$$(2.11) = \frac{(2m-2)(m-2)(5m+2)}{16} x^{-m-3} + (m+2) x^{-3},$$

which belongs to $M_\infty(0, \infty)$ when $m \geqq 2$. Now all hypotheses of Theorem 2.2 are fulfilled, therefore (2.12) holds.

Remark 2.2. If $\Psi \equiv 1$, we obtain a slight modification of [3] Theorem 4.3.

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