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ODD GRAPHS

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Abstract. Let k be an integer, $k \geqq 2$, $M_k = \{1, 2, \ldots, 2k - 1\}$, let $\mathscr{V}_k$ be the set of all $(k - 1)$-element subsets of M_k. The odd graph O_k is the graph whose vertex set is $\mathscr{V}_k$ and in which two vertices are adjacent if and only if they are disjoint as sets. Various properties of odd graph are studied.

Key words. Odd graph, chromatic number, distance, diameter, radius, geodetic graph, domination number, domatic number.

In [2] the concept of the odd graph is introduced. Here we shall show some of its properties.

Let k be an integer, $k \geqq 2$. Let $M_k = \{1, 2, \ldots, 2k - 1\}$, let $\mathscr{V}_k$ be the set of all subsets of M_k which have the cardinality $k - 1$. The odd graph O_k is the graph whose vertex set is $\mathscr{V}_k$ and in which two vertices are adjacent if and only if they are disjoint (as sets).

The graph O_2 is the complete graph K_3 with three vertices, the graph O_3 is the well-known Petersen graph.

First we determine the chromatic numbers of odd graphs.

Theorem 1. *The chromatic number of every odd graph is equal to 3.*

Proof. Consider an odd graph O_k. Let $\mathscr{U}_1$ be the set of all sets belonging to $\mathscr{V}_k$ and containing the number 1, let $\mathscr{U}_2$ be the set of all sets belonging to $\mathscr{V}_k - \mathscr{U}_1$ and containing the number 2, let $\mathscr{U}_3 = \mathscr{V}_k - (\mathscr{U}_1 \cup \mathscr{U}_2)$. Any two elements of $\mathscr{U}_1$ are non-adjacent (as vertices of O_k), because their intersection contains the number 1 and therefore it is non-empty. Hence $\mathscr{U}_1$ is an independent set in O_k and analogously so is $\mathscr{U}_2$. Now let $X \in \mathscr{U}_3$, $Y \in \mathscr{U}_3$. Then the sets X, Y are subsets of the set $M_k - \{1, 2\}$. This set has the cardinality $2k - 3$, while each of the sets X, Y has the cardinality $k - 1$. If X, Y were disjoint, their union $X \cup Y$ would have the cardinality $2(k - 1)$ which is greater than the cardinality of $M_k - \{1, 2\}$; this is impossible. Therefore $X \cap Y \neq \emptyset$ for any two elements X, Y of $\mathscr{U}_3$ and $\mathscr{U}_3$ is an independent set in O_k, too. The vertices of O_k can be coloured by three colours 1, 2, 3 in such a way that by the colour i ($i = 1, 2, 3$) the vertices belonging to $\mathscr{U}_i$ are coloured. This colouring is admissible; no two vertices of the same colour are adjacent. We have proved that $\chi(O_k) \leqq 3$, where $\chi(O_k)$ is the chromatic number of O_k.

Now we shall construct the sets $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ as follows. We put $X_1 = \{1, \ldots, k - 1\}$. If X_i is constructed for some i, then we put $Y_i = M_k - (X_i \cup \{2k - i\})$. If Y_i is constructed for some i, then we put $X_{i+1} = M_k - (Y_i \cup \{i\})$. The reader himself may verify that then $Y_k = X_1$. Further $X_i \cap Y_i = \emptyset$ for $i = 1, \ldots, k$ and $X_{i+1} \cap Y_i = \emptyset$ for $i = 1, \ldots, k - 1$. Therefore $X_1, Y_1, X_2, Y_2, \ldots, X_k, Y_k = X_1$ are vertices of a circuit in O_k having the length $2k - 1$ which is an odd number. Hence O_k is not bipartite and $\chi(O_k) \geqq 3$. Together with the previous inequality this yields $\chi(O_k) = 3$.

Now we shall study the distance in O_k.

Theorem 2. *Let U, V be two vertices of the graph O_k, let $| U \cap V | = m$. Then the distance of the vertices U, V in O_k is $\Delta(m) = \min(2m + 1, 2k - 2m - 2)$.*

Remark. The vertices of O_k are denoted by capital letters, because they are sets.

Proof. If for two pairs U_1, V_1 and U_2, V_2 of vertices of Q_k we have $| U_1 \cap V_1 | = | U_2 \cap V_2 |$, then evidently there exists a permutation of the set M_k which maps U_1 onto U_2 and V_1 onto V_2; this permutation induces an automorphism of O_k which again maps U_1 onto U_2 and V_1 onto V_2. This implies that the distance of two vertices of O_k is a function of the cardinality of their intersection and we may denote it by $\Delta(m)$, where m is this cardinality. Now let us have two vertices U, V of O_k, let $m = | U \cap V |$. If $m = 0$, then $U \cap V = \emptyset$ and the vertices U, V are adjacent; their distance is 1, therefore $\Delta(0) = 1$, which fulfills the assertion. If $m = k - 1$, then $U = V$, because $| U | = | V | = k - 1$. The distance of U and V is 0, therefore $\Delta(k - 1) = 0$, which again fulfils the assertion. Now let m be an arbitrary integer such that $2 \leqq m \leqq k - 2$. We have $| U - V | = | V - U | = k - 1 - m$, $| M_k - (U \cup V) | = m + 1$. Let P be the shortest path in O_k connecting U and V. Let U_0 (or V_0) be the vertex of P adjacent to U (or V respectively). Evidently $d(U, V) = d(U_0, V_0) + 2$, where d denotes the distance of two vertices. We have $U \cap U_0 = V \cap V_0 = \emptyset$, therefore the intersection $U_0 \cap V_0 \subseteq M_k - (U \cup V)$ and $| U_0 \cap V_0 | \leqq m + 1$. On the other hand, the set U_0 can have at most $k - 1 - m$ elements in common with V and the other vertices of U_0 belong to $M_k - (U \cup V)$, hence $| U_0 \cap (M_k - (U \cup V)) | \geqq m$ and analogously $| V_0 \cap (M_k - (U \cup V)) | \geqq m$. This implies $| U_0 \cap V_0 | \geqq m - 1$. Thus there are three possibilities for the cardinality of $U_0 \cap V_0$, namely $m - 1$ or m or $m + 1$. As P is the shortest path connecting U and V, the sets U_0, V_0 must be chosen so that their distance might be the least possible, i.e. $d(U_0, V_0) = \min(\Delta(m - 1), \Delta(m), \Delta(m + 1))$. As $\Delta(m) = d(U, V) = d(U_0, V_0) + 2$, the equalities $d(U_0, V_0) = \Delta(m)$ and $| U_0 \cap V_0 | = m$ are impossible. There can be only either $d(U_0, V_0) = m - 1$ and $\Delta(m) = \Delta(m - 1) + 2$, or $d(U_0, V_0) = m + 1$ and $\Delta(m) = \Delta(m + 1) + 2$. Suppose that $\Delta(m) = \Delta(m - 1) + 2$ holds, hence $d(U_0, V_0) = \Delta(m - 1)$ and $| U_0 \cap V_0 | = m - 1$. If $m = 1$, then U_0, V_0 are adjacent and $d(U, V) = \Delta(1) = 3$ (evidently it cannot be less) which fulfills the assertion. If $m \geqq 2$, consider the interrelation between $\Delta(m - 1)$ and $\Delta(m - 2)$. Analogously there is $\Delta(m - 1) = \Delta(m - 2) + 2$ or $\Delta(m - 1) = \Delta(m) + 2$. But, as we have supposed $\Delta(m) = \Delta(m - 1) + 2$, we must have $\Delta(m - 1) = \Delta(m - 2) + 2$. Inductively we can prove that if $\Delta(m) = \Delta(m - 1) + 2$ for some m, then $\Delta(p) = \Delta(p - 1) + 2$ for each integer p such that $2 \leqq p \leqq m$. Analogously if $\Delta(m) = \Delta(m + 1) + 2$ for some m, then $\Delta(q) = \Delta(q + 1) + 2$ for each integer q such that $m \leqq q \leqq$ $\leqq k - 2$. As we have proved $\Delta(0) = 1$, $\Delta(k - 1) = 0$, the function $\Delta(m)$ is uniquely determined as $\Delta(m) = \min(2m + 1, 2k - 2m - 2)$.

Corollary. *The diameter and the radius of the graph O_k are both equal to $k - 1$.*

The number $k - 1$ is evidently the maximum of $\Delta(m)$; it is attained in $m = \frac{1}{2}(k - 1)$ for k odd and in $m = \frac{1}{2}k - 1$ for k even. As O_k is vertex-transitive, its radius is equal to its diameter.

Theorem 3. *The graph O_k for every integer $k \geqq 2$ is geodetic.*

Proof. In the proof of Theorem 2 we have shown that for given vertices U, V the vertices U_0, V_0 (the vertices adjacent to U and V respectively in the shortest path connecting U and V) are determined uniquely. Thus by induction we can prove that whole the shortest path between U and V is uniquely determined.

The graph O_k is an example of a geodetic graph of the diameter $k - 1$ which is simultaneously regular of the degree k.

In the sequel we shall use a certain labelling of edges of O_k.

Let e be an edge of O_k, let U, V be its end vertices. Then by $\lambda(e)$ we denote the element of the one-element set $M_k - (U \cup V)$.

An edge-dominating set in a graph G is a subset D of the edge set $E(G)$ of G with the property that to each edge $e \in E(G) - D$ there exists an edge $f \in D$ such that the edges e, f have a common end vertex. The minimal number of vertices of an edge-dominating set in G is called the edge-domination number of G.

Analogously to the domatic number of a graph [1] we may define the edge-domatic number of a graph G.

An edge-domatic partition of a graph G is a partition of the edge set $E(G)$ of G, all of whose classes are edge-dominating sets in G. The maximal number of classes of an edge-domatic partition of G is called the edge-domatic number of G.

Theorem 4. *The edge-domination number of the graph O_k is equal to $\frac{1}{2}\binom{2k - 2}{k - 1}$ and its edge-domatic number is equal to $2k - 1$.*

Proof. Let $j \in M_k$ and let E_j be the set of all edges e of O_k such that $\lambda(e) = j$. Let f be an edge of O_k not belonging to E_j, let $\lambda(f) = k$. Then $k \neq j$. Let U, V be the end vertices of f. Exactly one of the sets U, V contains the element j; without loss of generality let it be U. Let $W = M_k - (V \cup \{j\})$; then V and W are joined by an edge belonging to E_j. As f was chosen arbitrarily, we have proved that E_j is an edge-dominating set (for an arbitrary j).

Now let us look for the cardinality of E_j. If X is an arbitrary subset of $M_k - \{j\}$ of the cardinality $k - 1$ and $Y = M_k - (X \cup \{j\})$, then the vertices X, Y are joined by an edge belonging to E_j and vice versa. The number of subsets of $M_k - \{j\}$ of the cardinality $k - 1$ is equal to $\binom{2k-2}{k-1}$. Having in mind that for a subset X of $M_k - \{j\}$ of the cardinality $k - 1$ the set $Y = M_k - (X \cup \{j\})$ is also a subset of $M_k - \{j\}$ of the cardinality $k - 1$, we find that the number of unordered pairs $\{X, Y\}$ of described sets is $\frac{1}{2}\binom{2k-2}{k-1}$ and this is also the cardinality of E_j. This number does not depend on j, thus all the sets E_j for $j = 1, \ldots, 2k - 1$ have equal cardinalities. The edge-domination number of O_k is thus at most $\frac{1}{2}\binom{2k-2}{k-1}$ and its edge-domatic number is at least $2k - 1$.

The edge-domatic number of a graph is evidently equal to the domatic number [1] of its line-graph. The degree of each vertex of the line-graph of O_k is $2k - 2$ and this implies [1] that its domatic number (and thus the edge-domatic number of O_k) is at most $2k - 1$. We have proved that the edge-domatic number of O_k is $2k - 1$.

Now suppose that there exists an edge-dominating set D of a cardinality $d < \frac{1}{2}\binom{2k-2}{k-1}$. For each edge $e \in D$ the set consisting of e and all edges having a common end vertex with e has the cardinality $2k - 1$. As each edge of O_k either is in D, or has an end vertex in common with an edge of D, the number of edges of O_k is at most $d(2k-1) < \frac{1}{2}(2k-1)\binom{2k-2}{k-1} = \frac{1}{2}k\binom{2k-1}{k-1}$. But the number at the right-hand side of this inequality is the number of edges of O_k. (The number of vertices is $\binom{2k-1}{k-1}$ and the graph is regular of the degree k.) As $d(2k - 1)$ is less, we have a contradiction. Thus each E_j is an edge-dominating set of the least cardinality and the edge-domination number of O_k is $\frac{1}{2}\binom{2k-2}{k-1}$.

Theorem 5. *Let T_k be a tree with the vertex set $\{a, b, c_1, \ldots, c_{k-1}, d_1, \ldots, d_{k-1}\}$ and with the edges ab, ac_i, bd_i for $i = 1, \ldots, k - 1$. Then the graph O_k can be decomposed into $\frac{1}{2}\binom{2k-2}{k-1}$ pairwise edge-disjoint subgraphs which are all isomorphic to T_k. Moreover, each of these subgraphs contains exactly one edge from each set E_j for $j = 1, \ldots, 2k - 1$.*

Proof. Let $j \in \{1, \ldots, 2k - 1\}$, let E_j have the same meaning as in the proof of Theorem 4. Let e_1, e_2 be two elements of E_j. Suppose that these edges have a common end vertex U. Let V_1 (or V_2) be the end vertex of e_1 (or e_2 respectively) distinct from U. Then $M_k - (U \cup V_1) = M_k - (U \cup V_2) = \{j\}$ and $U \cap V_1 = U \cap V_2 = \emptyset$. This implies $V_1 = V_2$ and also $e_1 = e_2$, because O_k is a graph without multiple edges. We have proved that there exist no two distinct edges of E_j which would have an end vertex in common. Now suppose that to the edges e_1, e_2 of E_j there exists an edge f which has common end vertices with both e_1, e_2. Let U_1 (or U_2) be the common end vertex of e_1 (or e_2 respectively) and f. Let V_1 (or V_2) be the end vertex of e_1 (or e_2 respectively) distinct from U_1 and U_2. Then $M_k - (U_1 \cup V_1) = M_k - (U_2 \cup V_2) = \{j\}$, $U_1 \cap V_1 = U_2 \cap V_2 = U_1 \cap U_2 = \emptyset$. This implies that none of the sets U_1, U_2, V_1, V_2 contains j. As $U_1 \cap U_2 = \emptyset$, we have $M_k - (U_1 \cup U_2) = \{j\}$ and $f \in E_j$. According to the above proved this is possible only if $e_1 = e_2 = f$. Therefore if $\lambda(e_1) = \lambda(e_2)$ and $e_1 \neq e_2$, then the distance between an arbitrary end vertex of e_1 and an arbitrary vertex of e_2 is at least 2.

Now let e be an edge of O_k. Let $G(e)$ be the subgraph of O_k consisting of the edge e, all edges having a common end vertex with e and of end vertices of all of these edges. This is a tree isomorphic to T_k. If e_1, e_2 are two distinct edges of $G(e)$, then either they have a common end vertex, or there exists an edge of $G(e)$ which has common end vertices with both of them. According to the above proved the labellings of edges of $G(e)$ are pairwise different.

Let $\mathcal{T}(j)$ be the set of subtrees $G(e)$ for all edges $e \in E_j$. Any two distinct trees from $\mathcal{T}(j)$ are edge-disjoint; otherwise there would exist two distinct edges of E_j with a common end vertex or with the property that there exists an edge having common vertices with both of them. The cardinality of $\mathcal{T}(j)$ is equal to that of E_j, namely $\frac{1}{2}\binom{2k-2}{k-1}$. Each tree from $\mathcal{T}(j)$ has $2k - 1$ edges. Hence the union of all trees from $\mathcal{T}(j)$ has $\frac{1}{2}\binom{2k-2}{k-1} . (2k-1) = \frac{1}{2} k \binom{2k-1}{k-1}$ edges and this is the number of edges of O_k. We have proved that $\mathcal{T}(j)$ is the required decomposition.

To contract an edge of a graph means to delete this edge and to identify its end vertices.

Theorem 6. *The graph $O_k'(j)$ obtained from O_k by contracting every edge e with $\lambda(e) = j$, where j is an integer between* 1 *and* $2k - 1$, *is a bipartite graph.*

Proof. By the described contractions each tree from $\mathcal{T}(j)$ is transformed into a star. Hence $O_k'(j)$ is a graph which is the union of edge-disjoint stars with the property that each of them contains all edges incident with its centre in $O_k'(j)$. Every graph with this property is bipartite.

Let $\mathcal{P}(n)$ be the set of all linear orderings of the set $\{1, ..., n\}$. Let π_1, π_2 be elements of $\mathcal{P}(n)$. We say that π_1, π_2 are dihedrally equivalent, if either $\pi_1 = \pi_2$, or π_2 can be obtained from π_1 by a cyclic permutation, by reversing or by a superposition of a cyclic permutation and a reversing. The relation thus defined is evidently an equivalence on the set $\mathscr{P}(n)$.

Let C be a circuit of the length n whose edges are labelled by pairwise different numbers from the set $\{1, \ldots, n\}$. If we run around C and write the labels of the traversed edges, we may obtain different linear orderings of the set $\{1, \ldots, n\}$ according to in which vertex we have started and in which sense we have gone. These orderings form one class of the dihedral equivalence. We may say that to C a class of the dihedral equivalence on $\mathscr{P}(n)$ corresponds.

The number of classes of the dihedral equivalence on $\mathscr{P}(n)$ is evidently equal to $\frac{1}{2}(n-1)!$.

Theorem 7. *The graph O_k with the labelling λ is the union of $\frac{1}{2}(2k-2)!$ circuits of the length $2k-1$ which correspond to pairwise different classes of the dihedral equivalence on $\mathscr{P}(2k-1)$. Each edge of O_k belongs to $(k-1)!^2$ and each vertex to $\frac{1}{2}k!(k-1)!$ such circuits.*

Proof. Let $\mathscr{C}$ be a class of the dihedral equivalence on $\mathscr{P}(2k-1)$. Let $\pi \in \mathscr{C}$ and $[a_1, \ldots, a_{2k-1}] = \pi$. Let $U_1 = \{a_i \mid i \text{ even}, 2 \leqq i \leqq 2k-2\}$. We construct the sets $U_2, \ldots, U_{2k-1}$ recursively. If U_i is constructed for some i, then $U_{i+1} = M_k - (U_i \cup \{i\})$. Any two vertices U_i, U_{i+1} are adjacent in O_k. Further it may be easily proved that $M_k - (U_{2k-1} \cup \{2k-1\}) = U_1$ and the vertices U_{2k-1}, U_1 are adjacent, too. We have obtained a circuit in O_k; evidently this circuit corresponds to $\mathscr{C}$. We may construct such a circuit for each class of the dihedral equivalence on $\mathscr{P}(2k-1)$. From the construction it is evident that circuits corresponding to the same class are identical and that each edge of O_k is contained in some of these circuits. The family of the mentioned circuits will be denoted by $\mathfrak{C}$.

The graph O_k is evidently vertex-transitive and edge-transitive. (A graph is vertex-transitive, if to any two of its vertices there exists its automorphism which maps one vertex onto the other. Analogously the edge-transitivity is defined.) This implies that for any two vertices V_1, V_2 of O_k the number of circuits of $\mathfrak{C}$ containing V_1 is equal to the number of those containing V_2 and an analogous assertion holds for edges, too. Thus the number of circuits from $\mathfrak{C}$ containing any vertex is obtained by dividing the sum of lengths of all circuits of $\mathfrak{C}$, namely $\frac{1}{2}(2k-2)!(2k-1)$, by the number of vertices of O_k, namely $\binom{2k-1}{k-1}$; the result is $\frac{1}{2}k!(k-1)!$. If we divide the number $\frac{1}{2}(2k-2)!(2k-1)$ by the number of edges of O_k, namely $\frac{1}{2}k\binom{2k-1}{k-1}$, we obtain the number of circuits of $\mathfrak{C}$ containing any edge, namely $(k-1)!^2$.

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