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A REMARK CONCERNING x-SYSTEMS

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Abstract. The paper deals with certain closure operators in commutative semigroups introduced by K. E. Aubert. It is shown that they form a complete lattice with respect to a natural ordering.

Key words. X-system in a commutative semigroup, x-system of finite character.

K. E. Aubert studied so-called x-systems in commutative semigroups in [1]. In Chapter 1, there is a proposition (Proposition 8) claiming that the family $\mathscr{F}_S$ of all x-systems of finite character in a commutative semigroup S forms a complete sublattice of the complete lattice $\mathscr{L}_S$ of all x-systems in S with respect to a certain ordering $\succ$. In this remark, an example is given showing that the mentioned proposition is incorrect. A weaker theorem is proved below.

An *x-system* in a commutative semigroup S is defined to be a mapping x of the set of all subsets of S into itself satisfying the conditions:

1.1 $A \subseteq A_x$ for any $A \subseteq S$,

1.2 $A \subseteq B_x \Rightarrow A_x \subseteq B_x$ for any $A \subseteq S$, $B \subseteq S$,

1.3′ $AB_x \subseteq B_x$ for any $A \subseteq S$, $B \subseteq S$,

1.3″ $AB_x \subseteq (AB)_x$ for any $A \subseteq S$, $B \subseteq S$.

An x-system in S is said to be *of finite character* if, for any $A \subseteq S$, $A_x = \bigcup N_x$ ($N \subseteq A$, N is finite). The family of all x-systems in S (of all x-systems of finite character in S) is denoted by $\mathscr{L}_S(\mathscr{F}_S)$. An ordering $\succ$ is defined in $\mathscr{L}_S(\mathscr{F}_S)$ as follows:

$$x_1 \succ x_2 \quad \text{if} \quad A_{x_1} \subseteq A_{x_2} \quad \text{for any} \quad A \subseteq S.$$

Theorem. *The family $\mathscr{F}_S$ of all x-systems of finite character in S forms a sublattice of the complete lattice $\mathscr{L}_S$ of all x-systems in S with respect to the ordering $\succ$, i.e. when $\{x_i\}_{i \in I}$ is a finite family of x-systems of finite character then $\bigwedge_{i \in I} x_i$ and $\bigvee_{i \in I} x_i$ are both x-systems of finite character. Moreover, $\mathscr{F}_S$ is a meet subsemilattice of $\mathscr{L}_S$.*

Proof. In [1], the proof of Proposition 8, it is shown that $\mathscr{F}_S$ is a meet subsemilattice of $\mathscr{L}_S$. Assume that $x = \bigvee_{i \in I} x_i$ where $\{x_i\}_{i \in I}$ is a finite family of x-systems of finite character. By [1], Proposition 7, $A_x = \bigcap_{i \in I} A_{x_i}$ for all $A \subseteq S$. Clearly $A_x \supseteq \bigcup_{N \subseteq A} N_x$ where N denotes a finite set. Conversely, assume that $a \in A_x = \bigcap_{i \in I} A_{x_i}$. Then $a \in A_{x_i}$ for all $i \in I$. It follows that for all $i \in I$ there is a finite set $N^i \subseteq A$ such that $a \in N^i_{x_i}$. Consider $N^0 = \bigcup_{i \in I} N^i$. N^0 is finite, $N^0 \subseteq A$ and $a \in N^0_{x_i}$ for all $i \in I$. Hence $a \in \bigcap_{i \in I} N^0_{x_i} = N^0_x \subseteq \bigcup_{N \subseteq A} N_x$, where N denotes a finite set. Thus x is of finite character.

Example. The following example shows that if $\{x_i\}_{i \in I}$ is an infinite family of x-systems of finite character, then $x = \bigvee_{i \in I} x_i$ need not be an x-system of finite character.

Let S be the set of all ordinals less than or equal to the least infinite ordinal ω with their usual ordering. Let $M = S - \{\omega\}$. Define $ab = \min(a, b)$ for $a \in S$, $b \in S$. S is then clearly a commutative semigroup. Further, define a family $\{x_i\}_{i \in M}$ of mappings of the set of all subsets of S into itself as follows:

$$A_{x_i} = \begin{cases} S & \text{if there is } a \in A \text{ such that } i \leqq a, \\ \{y \in S : \exists\, z \in A : y \leqq z\} & \text{if there is no } a \in A \text{ such that } i \leqq a. \end{cases}$$

First, we shall show that $\{x_i\}_{i \in M}$ is a family of x-systems in S. Evidently, $A \subseteq A_{x_i}$ for all $A \subseteq S$ and all $i \in M$, so that the condition 1.1 is satisfied. Assume that $A \subseteq B_{x_i}$ for some $A, B \subseteq S$ and some $i \in M$. Let $r \in A_{x_i}$. If there is $a \in A \subseteq B_{x_i}$ such that $i \leqq a$, then we have two possibilities:

(a) There is $b \in B$ such that $i \leqq b$. Then $B_{x_i} = S$.

(b) There is $z \in B$ such that $a \leqq z$. Then $i \leqq z$ and $B_{x_i} = S$ again.

Consequently, $r \in B_{x_i}$ in both cases. If there is $z \in A \subseteq B_{x_i}$ such that $r \leqq z$, then we have two possibilities:

(a) There is $a \in B$ such that $i \leqq a$. Then $B_{x_i} = S$.

(b) There is $u \in B$ such that $z \leqq u$. Then $r \leqq u$ and $r \in B_{x_i}$.

Again, $r \in B_{x_i}$ in both cases. Hence, the condition 1.2 is satisfied. Now, assume that $r \in AB_{x_i}$ for some $A, B \subseteq S$ and some $i \in M$. There are $s \in A$, $t \in B_{x_i}$ such that $r = st$. If there is $a \in B$ such that $i \leqq a$, then $B_{x_i} = S$ and $r \in B_{x_i}$. If there is no $a \in B$ such that $i \leqq a$, then there is $z \in B$ such that $t \leqq z$. Then, however, $r \leqq z$ and $r \in B_{x_i}$. The condition 1.3′ is satisfied. Again, assume that $r \in AB_{x_i}$. Then there are $s \in A$, $t \in B_{x_i}$ such that $r = st$. If there is $a \in AB$ such that $i \leqq a$, then $(AB)_{x_i} = S$ and $r \in (AB)_{x_i}$. If there is no $a \in AB$ such that $i \leqq a$ and there is $b \in B$ such that $i \leqq b$, then there is no $c \in A$ such that $i \leqq c$. From this it follows that $s < i$. Two possibilities can occur:

(a) $t > y$ for all $y \in B$. Then $t \geqq b \geqq i > s$, so that $r = st = s = sb \in AB \subseteq (AB)_{x_i}$.

(b) There is $d \in B$ such that $t \leqq d$. Then $r = st \leqq sd \in AB$. Thus $r \in (AB)_{x_i}$.

If there is no $b \in B$ such that $i \leqq b$, then there is $z \in B$ such that $t \leqq z$. From this it follows that $r = st \leqq sz \in AB$. Thus $r \in (AB)_{x_t}$. The condition 1.3″ is satisfied.

Now, we shall show that $\{x_i\}_{i \in M}$ is a family of x-systems of finite character. Assume that $A \subseteq S$ and $i \in M$. Clearly $A_{x_i} \supseteq \bigcup_{N \subseteq A} N_{x_i}$ where N denotes a finite set. Let $r \in A_{x_i}$. If there is $a \in A$ such that $i \leqq a$, then $r \in \{a\}_{x_i} = S$. If there is no $a \in A$ such that $i \leqq a$, then there is $z \in A$ such that $r \leqq z$. Thus $r \in \{z\}_{x_i}$. Hence $r \in \bigcup_{N \subseteq A} N_{x_i}$ where N is a finite set in both cases.

$x = \bigvee_{i \in M} x_i$ is, however, not an x-system of finite character. In fact, $M_x = \bigcap_{i \in M} M_{x_i} = S$, while $N_x = \bigcap_{i \in M} N_{x_i} = \{y \in S: \exists\, z \in N: y \leqq z\}$ for any $N \subseteq M$ such that N is finite, so that $\bigcup_{N \subseteq M} N_x = M \neq S = M_x$.

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