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QUICKLY, MODERATELY AND SLOWLY OSCILLATORY SOLUTIONS OF A SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract. This paper considers the behaviour of quickly, moderately and slowly oscillatory solutions of the equation

$$(r(t)\,y'(t))' + f(t, y(\Delta(t, y(t)))) = Q(t), \qquad t \geqq t_0 \in R^1$$

in the cases when $\liminf_{t\to\infty} r(t) > 0$ and $\liminf_{t\to\infty} r(t) = 0$ and when the deviation $\Delta(t, u)$ depends on the unknown function and may be of a retarded, advanced or mixed type.

Key words. Oscillation, nonoscillation, quick oscillation, moderately oscillation, slow oscillation.

1. Introduction

We are concerned with the asymptotic behaviour of quickly, moderately and slowly oscillatory solutions of forced functional differential equations of second order

$$(r(t)\,y'(t))' + f(t, y(\Delta(t, y(t)))) = Q(t), \qquad t \geqq t_0 \in R^1. \tag{1}$$

As far as we know the first theorem about the asymptotic behaviour of quickly oscillatory solutions of the ordinary differential equations have been obtained by Lasota [4] and Luczynski [6]. Their results are generalized by Lasota, Yorke [5] and Bernfeld, Lasota [1]. Necessary conditions for existence of quickly oscillatory solution of the equation $y''(t) + q(t)\,y(t) = 0$ and sufficient conditions which causedall its solutions to oscillate quickly have been obtained by Singh [9].

For autonomous retarded systems with constant lag

$$y'(t) = f(y(t - h)) \qquad \text{when} \qquad f : R^n \to R^n$$

and

$$y'(t) = f(y(t), y(t - h)) \qquad \text{when} \qquad f : R^2 \to R^1$$

theorems classifying the behaviour of their quickly oscillatory solutions have been proved by Kaplan [2].

Sufficient conditions under which all bounded quickly oscillatory solutions of a more general equation with deviating arguments

$$[r_{n-1}(t)[\dots[r_1(t)[r_0(t)x(t)]']'\dots]']' + a(t)f(x(g_1(t)),\dots,x(g_m(t))) = b(t) \tag{2}$$

tend to zero as $t \to \infty$, have been given by Philos, Staikos [7].

Asymptotically vanishing of moderately oscillatory solutions of the equation

$$y''(t) + a(t)y^{\alpha}(t - g_0(t)) = Q(t) \tag{3}$$

where $\alpha \in (0, 1]$ and α is a ratio of odd integers,

$$y^{(n)}(t) + a(t)y(g(t)) = Q(t), \tag{4}$$

$$(r(t)y'(t))' + a(t)f(y(g(t))) = Q(t) \tag{5}$$

and (2) has been investigated in [10], [11], [3] and [8] respectively. Conditions under which all moderately oscillatory solutions of (5) for $r(t) = 1$ are bounded have been given in [13].

The slowly oscillatory solutions of (5) have been studied in [12], [13].

In most of the papers cited it is assumed that the function containing the deviation is multipled by some positive function on t and the deviation is a delay (in [3, 10–13] and $0 \leqq g_0(t) \leqq \text{const}$ in [10]). These hypotheses will not be required in the main results of this paper and furthermore the deviation $\Delta(t, v)$ depends on the unknown function and may be of a retarded, advanced or mixed type.

2. Assumptions, definitions and lemmas

Here $R^1 = (-\infty, \infty)$, $R^n = \underbrace{(-\infty, \infty) \times \dots \times (-\infty, \infty)}_{n \text{ times}}$ and $R_+ = [0, \infty)$. By $C(A; B)$ we denote the set of all continuous functions $f: A \to B$ where $A, B \subseteq R^1$, by $C^1(A; B)$ – the set of continuous differentiable functions $f: A \to B$ and by $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}$.

The functions $r(t)$, $f(t, u)$, $\Delta(t, v)$ and $Q(t)$ in equation (1) will be assumed to satisfy hypotheses (A):

A1. $r(t) \in C([t_0, \infty); (0, \infty))$;

A2. $f(t, u) \in C([t_0, \infty) \times R^1; R^1)$, $uf(t, u) > 0$ for $u \neq 0$ and $f(t, .)$ is nondecreasing for $t \geqq t_0$;

A3. $\Delta(t, v) \in C([t_0, \infty) \times R^1; R^1)$, $\lim_{t \to \infty} \Delta(t, v) = \infty$ for any fixed $v \in R^1$;

A4. $Q(t) \in C([t_0, \infty); R^1)$;

A5. One of the following conditions holds:

(a) $\Delta(t, .)$ is nondecreasing,

(b) $\Delta(t, .)$ is nonincreasing,

(c) $\Delta(t, v_1) \leqq \Delta(t, v_2)$ for $|v_1| \leqq |v_2|$, $v_1 v_2 \geqq 0$,
(d) $\Delta(t, v_1) \leqq \Delta(t, v_2)$ for $|v_1| \geqq |v_2|$, $v_1 v_2 \geqq 0$
for $t \geqq t_0$.

In what follows the term „solution" applies only to nontrivial solutions of equation (1) which can be continuously extended for $t \geqq t_0$. Denote by S the set of these solutions of (1).

The dependence of the deviation on the unknown solution $y(t)$ of (1) yields the stickness to the zero solution. So it is necessary to precise the term „consecutive" zeros of $y(t)$. We say that the point $\xi_v \in (t_0, \infty)$ is a pick point for $y(t)$, if $y'(\xi_v) = 0$ and $|y(\xi_v)| = \sup_{[\tau_v, \tau_{v+1}]} |y(t)|$ where $\tau_v \in [t_0, \xi_v)$ is the largest zero of $y(t)$ on the left from ξ_v and $\tau_{v+1} \in (\xi_v, \infty)$ is the smallest zero of $y(t)$ on the right from ξ_v (if the zeros τ_v and τ_{v+1} exist on $[t_0, \infty)$). If $y(t)$ has only one zero $T \in [t_0, \infty)$ then we propose that $\tau_v = T$ and $\tau_{v+1} = \infty$ when T is on the left from ξ_v and $\tau_v = t_0$, $\tau_{v+1} = T$ when T is on the right from ξ_v. If $y(t)$ has no zeros on $[t_0, \infty)$ then we assume that $\tau_v = t_0$ and $\tau_{v+1} = \infty$. The zeros τ_v and τ_{v+1} are said to be consecutive. The left and the right end of any interval of stickness to the zero solution are said to be consecutive zeros, too. Then the solution $y(t)$ is said to be oscillatory if there exists an infinite set $\{\tau_v\}_{v=1}^{\infty}$ of its consecutive zeros such that $\lim_{v \to \infty} \tau_v = \infty$; otherwise it is said to be nonoscillatory. The oscillatory solution $y(t)$ is said to be quickly, moderately and slowly oscillating if the distance $|\tau_{v+1} - \tau_v|$ between every pair of its consecutive zeros tends to zero as $v \to \infty$, it is bounded and unbounded, respectively.

Denote by $\tilde{S}$, $\tilde{S}_q$, $\tilde{S}_m$ and $\tilde{S}_s$ the sets of oscillatory, quickly, moderately and slowly oscillatory solutions of (1), respectively. It is clear that $\tilde{S}_q \subset S_m$ and $\tilde{S} = \tilde{S}_m \cup \tilde{S}_s$.

To obtain the main results we need the following two lemmas:

Lemma 2.1. [8] *Let* $h \in C'[t_0, \infty)$ *be a moderately oscillatory function and* $\lim_{t \to \infty} h'(t) = 0$. *Then* $\lim_{t \to \infty} h(t) = 0$.

Lemma 2.2. [7] *Let* $h(t) \in C'[t_0, \infty)$ *be a quickly oscillatory function and* $h'(t)$ *be bounded. Then* $\lim_{t \to \infty} h(t) = 0$.

3. Moderate and quick oscillations

First of all we study the behaviour of moderately oscillatory solutions of equation (1). Researches in this direction on equations (2)–(5) are made by Singh [10, 11, 13] and Kusano, Onose [3] in the cases when $\limsup_{t \to \infty} \frac{1}{r(t)} < \infty$, $0 <$

$< \liminf_{t\to\infty} r(t) \leqq \limsup_{t\to\infty} r(t) < \infty$, $r(t) = 1$ and $\limsup_{t\to\infty} \int_t^{t+c} \frac{ds}{r(s)} < \infty$ for any $c > 0$, respectively. We distinguish the two cases $\liminf_{t\to\infty} r(t) > 0$ and $\liminf_{t\to\infty} r(t) = 0$ and so we include the cases which are not discussed in cited papers.

Our first theorem gives sufficient conditions for all moderately oscillatory solutions of (1) to be bounded.

Theorem 3.1. *Let conditions (A) be fulfilled, $\Delta(t, v) \leqq t$ for any fixed $v \in R^1$ and there exist $F(t)$ $\varphi(t) \in C([t_0, \infty); R_+)$ such that $\lim_{|u|\to\infty} \frac{f(t,u)}{u} = \varphi(t)$ $\frac{f(t,u)}{u} \leqq F(t)$ for $u \neq 0$ uniformly for $t \geqq t_0$ and $\int_{t_0}^{\infty} F(t)\,dt < \infty$. Let either*

1. $\liminf_{t\to\infty} r(t) = \alpha > 0$, $\int_{t_0}^{\infty} \varphi(t)\,dt < \infty$ and $\int_{t_0}^{\infty} |Q(t)|\,dt < \infty$

or

2. $\liminf_{t\to\infty} r(t) = 0$, $\int_{t_0}^{\infty} R(t)\varphi(t)\,dt < \infty$ and $\int_{t_0}^{\infty} R(t)|Q(t)|\,dt < \infty$.

Then all solutions from $\tilde{S}_m$ are bounded.

Proof. Suppose that there exists $y(t) \in \tilde{S}_m$ such that

$$\limsup_{t\to\infty} |y(t)| = \infty \tag{6}$$

and $\sup_{\nu} |\tau_{\nu+1} - \tau_\nu| \leqq M$ for every pair of its consecutive zeros, where $M = \text{const} > 0$.

Using conditions 1 and 2 of theorem 3.1 we may find a number $t_1 \geqq t_0$ such that

$$\int_{t_1}^{\infty} \varphi(t)\,dt < \frac{\alpha}{2M} \tag{7}$$

and

$$\int_{t_1}^{\infty} R(t)\varphi(t)\,dt < \frac{1}{2}, \tag{8}$$

respectively.

In view of A3 we may find a zero $t_2 \geqq t_1$ of $y(t)$ such that $\Delta(t, v) \geqq t_1$ for $t \geqq t_2$ and any fixed $v \in R^1$, and in view of (6) we may choose the sequences $\{\tau_\nu\}_{\nu=1}^{\infty}$ (of zeros of $y(t)$) and $\{\xi_\nu\}_{\nu=1}^{\infty}$ (of pick points of $y(t)$) in such way that $t_2 \leqq \tau_1 < \tau_2 < \ldots$ and $M_1 \geqq \sup_{[t_1,t_2]} |y(t)|$, $M_{\nu+1} \geqq M_\nu$ $(\nu = 1, 2, \ldots)$, $\lim_{\nu\to\infty} M_\nu = \infty$, where $M_\nu = |y(\xi_\nu)|$. Let ξ_ν be any fixed pick point of $y(t)$ and let for instance $y(\xi_\nu) > 0$ (The proof is similar when $y(\xi_\nu) < 0$). In view of A5 we get

$$t_1 \leqq \Delta(t, 0) \leqq \Delta(t, y(t)) \leqq \Delta(t, M_\nu) \leqq t \leqq \xi_\nu$$

in the cases (a) and (c) and

$$t_1 \leqq \Delta(t, M_\nu) \leqq \Delta(t, y(t)) \leqq \Delta(t, 0) \leqq t \leqq \xi_\nu$$

in the cases (b) and (d) for $t \in [\tau_\mu, \xi_\nu)$. (Here τ_μ is the largest zero of $y(t)$ which is on the left from ξ_ν. The index μ may be greater than ν since it is possible to exist sticknesses to the zero solution). Then $| y(\Delta(t, y(t))) | \leqq M_\nu$ for $t \in [\tau_\mu, \xi_\nu]$ and using A2 we obtain

$$f(t, -M_\nu) \leqq f(t, y(\Delta(t, y(t)))) \leqq f(t, M_\nu) \qquad \text{for } t \in [\tau_\mu, \xi_\nu]. \tag{9}$$

Integrating (1) from $t \in (\tau_\mu, \xi_\nu)$ to ξ_ν and using (9) we get

$$r(t)\, y'(t) = \int_t^{\xi_\nu} f(s, y(\Delta(s, y(s))))\, ds - \int_t^{\xi_\nu} Q(s)\, ds \leqq \int_t^{\xi_\nu} [f(s, M_\nu) + | Q(s) |]\, ds. \tag{10}$$

Dividing (10) by $r(t)$ and integrating from τ_μ to ξ_ν we have

$$M_\nu \leqq \int_{\tau_\mu}^{\xi_\nu} \frac{1}{r(t)} \int_t^{\xi_\nu} [f(s, M_\nu) + | Q(s) |]\, ds\, dt =$$

$$= \int_{\tau_\mu}^{\xi_\nu} \left(\int_t^{\xi_\nu} [f(s, M_\nu) + | Q(s) |]\, ds \right) d\left(\int_{\tau_\mu}^{t} \frac{ds}{r(s)} \right) = \int_{\tau_\mu}^{\xi_\nu} \left(\int_{\tau_\mu}^{t} \frac{ds}{r(s)} \right) [f(t, M_\nu) + | Q(t) |]\, dt.$$

This inequality yields

$$1 \leqq \int_{\tau_\mu}^{\xi_\nu} \left(\int_{\tau_\mu}^{t} \frac{ds}{r(s)} \right) \frac{f(t, M_\nu)}{M_\nu}\, dt + \frac{1}{M_\nu} \int_{\tau_\mu}^{\xi_\nu} \left(\int_{\tau_\mu}^{t} \frac{ds}{r(s)} \right) | Q(t) |\, dt.$$

Then using (7) and (8) we obtain the contradictions

$$1 \leqq \frac{M}{\alpha} \left[\int_{t_1}^{\infty} \frac{f(t, M_\nu)}{M_\nu}\, dt + \frac{1}{M_\nu} \int_{t_0}^{\infty} | Q(t) |\, dt \right] \underset{\nu \to \infty}{\to} \frac{M}{\alpha} \int_{t_1}^{\infty} \varphi(t)\, dt < \frac{1}{2}$$

and

$$1 \leqq \int_{t_1}^{\infty} \left(\int_{t_0}^{t} \frac{ds}{r(s)} \right) \frac{f(t, M_\nu)}{M_\nu}\, dt + \frac{1}{M_\nu} \int_{t_0}^{\infty} \left(\int_{t_0}^{t} \frac{ds}{r(s)} \right) | Q(t) |\, dt \underset{\nu \to \infty}{\to} \int_{t_1}^{\infty} R(t)\, \varphi(t)\, dt < \frac{1}{2},$$

respectively.

Therefore all solutions from $\tilde{S}_m$ are bounded and theorem 3.1 is proved.

Now we will find conditions for vanishing at infinity of bounded solutions from S_m.

Theorem 3.2. *In addition to conditions* (A) *suppose that either*

1. $\liminf_{t \to \infty} r(t) = \alpha > 0$, $\int_{t_0}^{\infty} | Q(t) |\, dt < \infty$ *and* $\int_{t_0}^{\infty} | f(t, u) |\, dt < \infty$ *for any fixed* $u \in R^1$

or

2. $\liminf_{t \to \infty} r(t) = 0$, $\limsup_{t \to \infty} \frac{1}{r(t)} \int_t^{\infty} | Q(s) |\, ds = 0$

and $\limsup_{t\to\infty} \frac{1}{r(t)} \int_t^\infty |f(s,u)|\,ds = 0$ *for any fixed* $u \in R^1$.

Then all bounded solutions from $\tilde{S}_m$ *tend to zero as* $t \to \infty$.

Proof. Let $y(t) \in \tilde{S}_m$ be bounded. Then we may find $l = \text{const} > 0$ and $t_1 \geqq t_0$ such that $|y(t)| \leqq l$ for $t \geqq t_1$. In view of A3, A5 and A2 we get (9) for $M_\nu = l$ and $t \geqq t_2$ where $t_2 \geqq t_1$ is such that $\Delta(t, v) \geqq t_1$ for $t \geqq t_2$ and any fixed $v \in R^1$.

Since the oscillation of $y(t)$ yields the oscillation of $y'(t)$ we may choose a zero $t_3 > t_2$ of $y'(t)$. Integrating (1) from $t \in (t_2, t_3)$ to t_3 and using (9) we obtain (10) for $M_\nu = l$ and $\xi_\nu = t_3$. Dividing (10) by $r(t)$ we have

$$y'(t) \leqq \frac{1}{r(t)} \int_t^\infty [f(s,l) + |Q(s)|]\,ds.$$

This yields in view of conditions 1 and 2 of theorem 3.2 $y'(t) \to 0$ as $t \to \infty$. From lemma 2.1 we conclude that $\lim_{t\to\infty} y(t) = 0$.

Theorem 3.2 is proved.

From theorems 3.1 and 3.2 it follows that all solutions from $\tilde{S}_m$ tend to zero as $t \to \infty$.

Since $S_q \subset \tilde{S}_m$ we observe that the above results are valid for quickly oscillatory solutions of (1).

The following theorem contains conditions which guarantee the vanishing at infinity of all bounded solutions from $\tilde{S}_q$.

Theorem 3.3. *In addition to conditions* (*A*) *suppose that either*

1. $\liminf_{t\to\infty} r(t) = \alpha > 0$, $\sup_{t\geqq t_0} |Q(t)| < \infty$ *and* $\sup_{t\geqq t_0} |f(t,u)| < \infty$ *for any fixed* $u \in R^1$ *or*

2. $\liminf_{t\to\infty} r(t) = 0$, $\limsup_{t\to\infty} \frac{1}{r(t)} \int_t^\infty |Q(s)|\,ds < \infty$

and $\limsup_{t\to\infty} \frac{1}{r(t)} \int_t^\infty |f(s,u)|\,ds < \infty$ *for any fixed* $u \in R^1$.

Then all bounded solutions from S_q *tend to zero as* $t \to \infty$.

Proof. Let $y(t) \in \tilde{S}_q$ be bounded and $|y(t)| \leqq l$ for $t \geqq t_1 \geqq t_0$ and some $l = \text{const} > 0$. As in the proof of theorem 3.2 we obtain (9) for $M_\nu = l$ and $t \geqq t_2 \geqq t_1$. Let $\tau_\nu \geqq t_2$ and $\tau_{\nu+1}$ be consecutive zeros of $y(t)$ and $\xi_\nu \in (\tau_\nu, \tau_{\nu+1})$ be a pick point of $y(t)$. Integrating (1) from $t \in (\tau_\nu, \xi_\nu)$ to ξ_ν and using (9) we obtain (10) for $M_\nu = l$. Dividing (10) by $r(t)$ and having in mind conditions 1 and 2 of theorem 3.3. we conclude that $y'(t)$ is bounded for $t \geqq t_2$. According to lemma 2.2 $\lim_{t\to\infty} y(t) = 0$ and theorem 3.3 is proved.

The following examples illustrate the above theorems.

Example 3.1. Consider the equation

$$(11)\qquad \left(\frac{y'(t)}{t^2}\right)' + \frac{y\left(t - \frac{1}{t^8}\right)}{t^5} =$$

$$= \frac{28\sin t - 10t\cos t - t^2\sin t}{t^8} + \frac{t^{27}\sin\left(t - \frac{1}{t^8}\right)}{(t^9 - 1)^4}, \qquad t \geqq 2.$$

The theorems proved in [3, 8, 10, 11, 13] do not apply to (11) since $r(t) = \frac{1}{t^2}$

Here $r(t) = \frac{1}{t^2}$, $f(t, u) = \frac{u}{t^5}$, $\Delta(t, v) = t - \frac{1}{t^8}$ and

$$Q(t) = \frac{28\sin t - 10t\cos t - t^2\sin t}{t^8} + \frac{t^{27}\sin\left(t - \frac{1}{t^8}\right)}{(t^9 - 1)^4}$$

satisfy conditions (A) and since $R(t) = \frac{t^3 - 8}{3}$ and $\varphi(t) = F(t) = \frac{1}{t^5}$,it is easy to see that condition 2 of theorem 3.1 is fulfilled. According to this theorem all moderately oscillatory solutions of (11) are bounded. Indeed, $y(t) = \frac{\sin t}{t^4}$ is such a solution of (11).

Since condition 2 of theorem 3.2 is also fulfilled we conclude by this theorem that all bounded moderately oscillatory solutions of (11) tend to zero as $t \to \infty$. In fact, $y(t) = \frac{\sin t}{t^4}$ is such solution of equation (11).

So, all moderately oscillatory solutions of (11) tend to zero as $t \to \infty$.

Example 3.2. The equation

$$(12)\qquad y''(t) + y(t - y^2(t)) = \sin(t - \sin^2 t) - \sin t$$

has $y(t) = \sin t$ as a solution. This solution is bounded and moderately oscillatory. Theorem 3.2 does not apply since condition 1 of it is violated. Condition 1 of theorem 3.3 holds, but this theorem guarantees vanishing at infinity only for bounded quickly oscillatory solutions of (12).

Example 3.3. Consider the equation

$$(13)\qquad \left(\frac{y'(t)}{t}\right)' + \frac{y\left(t + \frac{1}{t^8} - y^2(t)\right)}{t^4} = \frac{24\sin t^2 - 16t^2\cos t^2 - 4t^4\sin t^2}{t^7} +$$

$$+\frac{\sin\left[\left(t+\frac{\cos^2 t^2}{t^8}\right)^2\right]}{t^4\left(t+\frac{\cos^2 t^2}{t^8}\right)^4}, \qquad t \geqq 2.$$

Here $R(t) = \int_2^t \frac{ds}{r(s)} = \frac{t^2-4}{2}$, $\varphi(t) = F(t) = \frac{1}{t^4}$, $\Delta(t, v) = t + \frac{1}{t^8} - v^2$ and $\Delta(t, v) \geqq t$ for $t \in \left[2, \sqrt[4]{\frac{1}{|v|}}\right]$, $\Delta(t, v) \leqq t$ for $t \geqq \sqrt[4]{\frac{1}{|v|}}$, $v \neq 0$. Conditions (A) are fulfilled. Since condition 2 of theorem 3.1 is not satisfied, this theorem does not apply. But condition 1 of theorem 3.2 holds and hence all bounded moderately (and hence, quickly) oscillatory solutions of (13) tend to zero as $t \to \infty$. Indeed, $y(t) = \frac{\sin t^2}{t^4}$ is such bounded quickly oscillatory solution of (13).

4. Slow oscillation

Sufficient conditions under which all oscillatory solutions of (5) belong to $\tilde{S}_s$ are given in [12] in the case when $\int^\infty \frac{dt}{r(t)} = \infty$ and $a(t) \geqq 0$ and in [13] in the case when $a(t) = a_1(t) + a_2(t)$ where $a_1(t) > 0$ and $\frac{a_2(t)}{a_1(t)}$ is bounded for large t. Theorem 2 of [13] guarantees the unboundedness of slowly oscillatory solutions of (5) on the above assumptions for $a(t)$.

We extend the Singh's investigations for equation (1) in the cases when $\liminf_{t\to\infty} r(t) > 0$ and $\liminf_{t\to\infty} r(t) = 0$.

Theorem 4.1. *Let conditions (A) be fulfilled,* $\int_{t_0}^\infty |f(t, u)|\, dt < \infty$ *for any fixed* $u \in R^1$, $\Delta(t, v) \leqq t$ *for any fixed* $v \in R^1$, $|Q(t)| > 0$ *on* $[t_0, \infty)$ *and* $\int_{t_0}^\infty |Q(t)|\, dt = \infty$, *and there exist* $\varphi(t), F(t) \in C([t_0, \infty); R_+)$ *such that* $\lim_{|u|\to\infty} \frac{f(t, u)}{u} = \varphi(t)$, $\frac{f(t,u)}{u} \leqq F(t)$ *for* $u \neq 0$ *uniformly for* $t \geqq t_0$ *and* $\int_{t_0}^\infty F(t)\, dt < \infty$. *Let either*

1. $\liminf_{t\to\infty} r(t) = \alpha > 0$ *and* $\int_{t_0}^\infty \varphi(t)\, dt < \infty$

or

2. $\liminf_{t\to\infty} r(t) = 0$ *and* $\int_{t_0}^\infty R(t)\varphi(t)\, dt < \infty$.

Then all oscillatory solutions of (1) *belong to* $\tilde{S}_s$.

Proof. Let $Q(t) > 0$ on $[t_0, \infty)$ (The proof is similar when $Q(t) < 0$ on $[t_0, \infty)$). Suppose that there exists $y(t) \in \tilde{S}_m$ and let $M = \sup_\nu |\tau_{\nu+1} - \tau_\nu|$ for every pair of consecutive zeros of $y(t)$ where $M = \text{const} > 0$.

First we will prove that $y(t)$ is bounded. Suppose it is not.

In view of conditions 1 and 2 of theorem 4.1 we may find $t_1 \geqq t_0$ such that (7) and (8) hold, respectively.

From (1) using the positiveness of $Q(t)$ we obtain the inequality

$$(r(t)\,y'(t))' \geqq -f(t, y(\Delta(t, y(t)))) \qquad \text{for } t \geqq t_0. \tag{14}$$

As in the proof of theorem 3.1 we may choose τ_μ, ξ_ν and M_ν such that (9) holds. Integrating (14) from $t \in (\tau_\mu, \xi_\nu)$ to ξ_ν and using (9) we have

$$r(t)\,y'(t) \leqq \int_t^{\xi_\nu} f(s, M_\nu)\,\mathrm{d}s. \tag{15}$$

Dividing (15) by $r(t)$ and integrating from τ_μ to ξ_ν we get

$$M_\nu \leqq \int_{\tau_\mu}^{\xi_\nu} \left(\int_{\tau_\mu}^{t} \frac{\mathrm{d}s}{r(s)} \right) f(t, M_\nu)\,\mathrm{d}t$$

which yields

$$1 \leqq \int_{\tau_\mu}^{\xi_\nu} \left(\int_{\tau_\mu}^{t} \frac{\mathrm{d}s}{r(s)} \right) \frac{f(t, M_\nu)}{M_\nu}\,\mathrm{d}t.$$

Then using (7) and (8) we obtain the contradictions

$$1 \leqq \frac{M}{\alpha} \int_{t_1}^{\infty} \frac{f(t, M_\nu)}{M_\nu}\,\mathrm{d}t \underset{\nu \to \infty}{\to} \frac{M}{\alpha} \int_{t_1}^{\infty} \varphi(t)\,\mathrm{d}t < \frac{1}{2}$$

and

$$1 \leqq \int_{t_1}^{\infty} \left(\int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} \right) \frac{f(t, M_\nu)}{M_\nu}\,\mathrm{d}t \underset{\nu \to \infty}{\to} \int_{t_1}^{\infty} R(t)\,\varphi(t)\,\mathrm{d}t < \frac{1}{2},$$

respectively.

Hence, $y(t)$ is bounded and we may find $l = \text{const} > 0$ and $t_1 \geqq t_0$ such that $|y(t)| \leqq l$ for $t \geqq t_1$. As in the proof of theorem 3.2 we get (9) for $M_\nu = l$ and $t \geqq t_2 \geqq t_1$. Let $t_3 > t_2$ be a zero of $y'(t)$.

Integrating (1) from t_3 to $t > t_3$ and using (9) we have

$$r(t)\,y'(t) = \int_{t_3}^{t} [Q(s) - f(s, y(\Delta(s, y(s))))]\,\mathrm{d}s \geqq \int_{t_3}^{t} [Q(s) - f(s, l)]\,\mathrm{d}s \underset{t \to \infty}{\to} \infty$$

from which it follows that $y(t)$ is nonoscillatory.

This contradiction proves theorem 4.1.

Theorem 4.2. *In addition to conditions of theorem 3.1 suppose that* $|Q(t)| > 0$ *on* $[t_0, \infty)$ *and* $\lim_{t\to\infty} \frac{|Q(t)|}{|f(t,u)|} = \infty$ *for any fixed* $u \neq 0$.

Then all solutions from $\tilde{S}$ *belong to* $\tilde{S}_s$.

Proof. Let $Q(t) > 0$ on $[t_0, \infty)$. First we will prove that all solutions of $\tilde{S}$ are upper unbounded. Suppose the contrary and let $y(t) \in \tilde{S}$ and $y(t) \leqq l$ for $t \geqq t_1 \geqq t_0$ and some $l = \text{const} > 0$. As in the proof of theorem 3.2 we obtain that

$$f(t, y(\Delta(t, y(t)))) \leqq f(t, l) \qquad \text{for } t \geqq t_2 \geqq t_1. \tag{16}$$

Dividing (1) by $f(t, l)$ and using (16) we get

$$\frac{(r(t)\,y'(t))'}{f(t,l)} = \frac{Q(t)}{f(t,l)} - \frac{f(t, y(\Delta(t, y(t))))}{f(t,l)} \geqq \frac{Q(t)}{f(t,l)} - 1 \underset{t\to\infty}{\to} \infty$$

from which it follows that $y(t)$ is nonoscillatory. This contradiction implies the upper unboundedness of all oscillatory solutions of (1).

Similarly we establish that all solutions from $\tilde{S}$ are lower unbounded for $Q(t) < 0$ on $[t_0, \infty)$.

If we suppose that there exists $y(t) \in \tilde{S}_m$ then according to theorem 3.1 we conclude that $y(t)$ is bounded which is a contradiction.

Theorem 4.2 is proved.

Finally we discuss the above theorems via several examples.

Example 4.1. Consider the equation

$$y''(t) + \frac{y(t - y^2(t)) + t y^{1/3}(t - y^2(t))}{t^3} = 4 + \sin(\ln t) + 3\cos(\ln t) + \\ + \frac{\sin(t - t^4(1 + \sin(\ln t))^2) + t \sin^{1/3}(t - t^4(1 + \sin(\ln t))^2)}{t^3} \tag{17}$$

for $t \geqq 2$. Here $r(t) = 1$ and $\varphi(t) = F(t) = \frac{1}{t^3}$; Conditions of theorem 4.1 in the case 1 are fulfilled.

Verify that $Q(t) > 0$. We have

$$Q(t) = 2 + \sin(\ln t) + 3\cos(\ln t) + \frac{\sin(t - t^4(1 + \sin(\ln t))^2)}{t^3} + \\ + \frac{\sin^{1/3}(t - t^4(1 + \sin(\ln t))^2)}{t^2} > 4 + \sin(\ln t) + \\ + 3\cos(\ln t) - \frac{1}{t^3} - \frac{1}{t^2} > \frac{29}{8} + \sin(\ln t) + 3\cos(\ln t).$$

Denote by $\Psi(t)$ the function $\Psi(t) = \frac{29}{8} + \sin(\ln t) + 3\cos(\ln t)$. Since the relative extrems of $\Psi(t)$ occur for $\operatorname{cotg}(\ln t) = 3$ we have that $\Psi(t) > \frac{3}{8}$ for $t \geqq 2$. Then $Q(t) > 0$ and $\int_2^\infty Q(t)\,dt = \infty$.

By theorem 4.1 all oscillatory solutions of (17) are slowly oscillatory.

We note that theorems from [12, 13] do not apply to (17) since $f(t, u) = \frac{u + tu^{1/3}}{t^3}$.

Example 4.2. Consider the equation

$$\left(\frac{1}{t} y'(t)\right)' + \frac{y(t - y^2(t))}{t^5} = \frac{1}{t^4} \qquad \text{for } t \geqq 1. \tag{18}$$

Here $r(t) = \frac{1}{t}$, $R(t) = \frac{t^2 - 1}{2}$, $\varphi(t) = F(t) = \frac{1}{t^5}$ and $Q(t) > 0$. Conditions of theorem 3.1 in the case 2 hold and

$$\lim_{t \to \infty} \frac{Q(t)}{|f(t, u)|} = \lim_{t \to \infty} \frac{\frac{1}{t^4}}{\frac{|u|}{t^5}} = \lim_{t \to \infty} \frac{t}{|u|} = \infty \qquad \text{for any fixed } u \neq 0.$$

Then by theorem 4.2 all oscillatory solutions of (18) are upper unbounded, and slowly oscillatory.

This example does not treat by theorem 4.1. since $\int_1^\infty |Q(t)|\,dt < \infty$.

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