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THE ASYMPTOTIC BEHAVIOUR OF OSCILLATORY SOLUTIONS OF THE EQUATION OF THE FOURTH ORDER

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Abstract. In the paper the structure and the behaviour of the oscillatory solutions of the differential equation of the fourth order are studied. The sufficient conditions are given under which the relation $\limsup_{t \to \infty} | y^{(i)}(t) | = \infty$, $i = 0, 1, 2, 3$ holds.

Key words. Ordinary differential equations, nonlinear oscillations, asymptotic properties.

Consider the differential equation

$$y^{(4)} = f(t, y, y', y'', y''') , \tag{1}$$

where f, defined on $D = \{(t, x_1, x_2, x_3, x_4) : t \in [0, \infty), \ | x_i | < \infty\}$ satisfies the local Carathéodory-conditions and

$$f(t, x_1, x_2, x_3, x_4)\, x_1 \leqq 0, \qquad f(t, 0, 0, 0, 0) \equiv 0 \qquad \text{on } D. \tag{2}$$

By a solution of (1) defined on $[0, b)$, $b \leqq \infty$ we shall mean a function y which, along with its derivatives to the third order is absolutely continuous on each segment of the interval $[0, b)$ and satisfies (1) for almost all t.

Definition 1. The solution y of (1) is called oscillatory on $[0, b)$, $b \leqq \infty$ if there exists a sequence $\{t_k\}_{k=1}^{\infty}$ of zeros of y such that $\lim_{k \to \infty} t_k = b$.

In the present paper the structure and the behaviour of the oscillatory solutions of (1), (2) will be studied. There are given conditions under the validity of which the relation $\limsup_{t \to \infty} | y^{(i)}(t) | = \infty$, $i = 0, 1, 2, 3$ holds. As the problem of the existence of oscillatory solutions of (1), (2) is concerned see e.g. [2].

Put $N = \{1, 2, \ldots\}$, $R_+ = [0, \infty)$ and $L[0, \infty)$ the set of all functions that are summable on each finite segment of R_+.

Definition 2. The oscillatory solution y, defined on $[0, b)$ is called of the 1-st type if the sequences $\{t_k^i\}$, $i = 0, 1, 2, 3, 4$ $k \in N$ exist such that

$$0 \leqq t_{k-1}^0 < t_k^3 \leqq t_k^4 < t_k^2 < t_k^1 < t_k^0, \qquad \lim_{k \to \infty} t_k^0 = b,$$

$$\tau y^{(i)}(t) > 0 \quad \text{for } t \in (t_{k-1}^0, t_k^i),\ y^{(i)}(t_k^i) = 0,\ i = 0, 1, 2, 3, \tag{3}$$

$$\tau y^{(j)}(t) < 0 \quad \text{for } t \in (t_k^j, t_k^0),\ j = 1, 2,$$

$$y'''(t) \equiv 0 \quad \text{on } [t_k^3, t_k^4],\ \tau y'''(t) < 0,\ t \in (t_k^4, t_k^0]$$

holds where $\tau = 1$ or $\tau = -1$ and $k \in N$.

Definition 3. The oscillatory solution y, defined $[0, b)$ is called of the 2-nd type if the sequences $\{t_k^i\}$, $i = 0, 1, 2, 3, 4$, $k \in N$ exist such that

$$0 \leqq t_k^0 < t_k^1 < t_k^2 < t_k^3 \leqq t_k^4 < t_{k+1}^0, \qquad \lim_{k \to \infty} t_k^0 = b,$$

$$(-1)^i \tau y^{(i)}(t) > 0 \quad \text{for } t \in [t_k^0, t_k^i)\ (t \in (t_{k+1}^0, t_{k+2}^0)) \text{ if } i = 1, 2, 3\ (i = 0), \tag{4}$$

$$y^{(j)}(t_k^j) = 0,\ (-1)^j \tau y^{(j)}(t) < 0 \quad \text{for } t \in (t_k^j, t_{k+1}^0),\ j = 1, 2,$$

$$y'''(t) = 0,\ t \in [t_k^3, t_k^4],\ \tau y'''(t) > 0 \quad \text{on } (t_k^4, t_{k+1}^0]$$

holds where $\tau = 1$ or $\tau = -1$ and $k \in N$.

Let y be an arbitrary solution of (1), (2). Define

$$F(t) = -y''(t)\, y(t) + y'^2(t), \qquad t \in [0, b). \tag{5}$$

Then

$$F'(t) = -y'''(t)\, y(t) + y'(t)\, y''(t),$$

$$F''(t) = -y^{(4)}(t)\, y(t) + y''^2(t) \geqq 0 \qquad \text{for almost all } t \in [0, b). \tag{6}$$

Thus F' is non-decreasing on $[0, b)$.

Lemma 1. *Let y be an oscillatory solution of the 1-st type on R_+. Then F and F' are positive non-decreasing on $[t_1^0, \infty)$ and $\lim_{t \to \infty} F(t) = \infty$ holds.*

Proof. The conclusion that F' is non-decreasing follows from (6) and according to (3) and (6)

$$F'(t_1^0) = y'(t_1^0)\, y''(t_1^0) > 0 \tag{7}$$

is valid. Thus $F(t)$ is non-decreasing on $[t_1^0, \infty)$,

$$F(t) = F(t_1^0) + F'(\xi)(t - t_1^0) \geqq F(t_1^0) + F'(t_1^0)(t - t_1^0) \geqq F(t_1^0) > 0$$

holds and it follows from (7) that F is positive and $\lim_{t \to \infty} F(t) = \infty$. The lemma is proved.

Lemma 2. *Let y be an oscillatory solution of the 2-nd type on $[0, b)$, $b \leqq \infty$. Then $F'(t) < 0$ on $[0, b)$.*

Proof. It follows from (6) and (4) that F' is non-decreasing, $F'(t_k^0) < 0, \lim_{k\to\infty} t_k^0 = b$ and thus the lemma is proved.

Theorem 1. *Let y be an oscillatory solution on $[0, b)$. Then one of following conclusions is valid:*

I. *y is an oscillatory solution of the 1-st type on $[0, b)$,*

II. *y is an oscillatory solution of the 2-nd type on $[0, b)$,*

III. *There exists a number $b_1 \in [0, b)$ such that $y(t) = 0$ for $t \in [b_1, b)$.*

Proof. Denote for $\tau = \pm 1$

1° $\tau y(t) \geqq 0, \tau y^{(i)}(t) > 0, i = 1, 2, 3,$

2° $\tau y^{(i)}(t) > 0, \tau y'''(t) \leqq 0, i = 0, 1, 2,$

3° $\tau y^{(i)}(t) > 0, \tau y''(t) \leqq 0, \tau y'''(t) < 0, i = 0, 1,$

4° $\tau y(t) > 0, \tau y'(t) \leqq 0, \tau y^{(i)}(t) < 0, i = 2, 3,$

5° $\tau y(t) \geqq 0, \tau y^{(i)}(t) > 0, \tau y''(t) < 0, i = 1, 3,$

6° $\tau y^{(i)}(t) > 0, \tau y'(t) \leqq 0, \tau y''(t) < 0, i = 0, 3,$

7° $\tau y^{(i)}(t) > 0, \tau y'(t) < 0, \tau y''(t) \geqq 0, i = 0, 3,$

8° $\tau y^{(i)}(t) > 0, \tau y'(t) < 0, \tau y'''(t) \leqq 0, i = 0, 2,$

9° $\tau y^{(i)}(t) \geqq 0, \tau y''(t) > 0, \tau y'''(t) \leqq 0, y(t)\, y'(t) = 0, i = 0, 1,$

10° $\tau y^{(i)}(t) \geqq 0, \tau y'''(t) > 0, i = 0, 1, 2,$

11° $y(t) = 0, \tau y'(t) > 0, \tau y''(t) \leqq 0, \tau y'''(t) < 0,$

12° $\tau y(t) > 0, \tau y'(t) \leqq 0, y''(t) = 0, \tau y'''(t) < 0,$

13° $\tau y(t) > 0, \tau y'(t) \leqq 0, \tau y''(t) < 0, y'''(t) = 0,$

14° $\tau y(t) \geqq 0, \tau y'(t) > 0, \tau y''(t) < 0, y'''(t) = 0,$

15° $\tau y(t) \geqq 0, \tau y'(t) > 0, y^{(i)}(t) = 0, i = 2, 3,$

16° $\tau y(t) > 0, \tau y'(t) < 0, y^{(i)}(t) = 0, i = 2, 3,$

17° $\tau y(t) > 0, y^{(i)}(t) = 0, i = 1, 2, 3,$

18° $y^{(i)}(t) = 0, i = 0, 1, 2, 3.$

These cases cover all the initial conditions at the point t. Let the relation j^0 be valid for y at $t = t_1$ and let the relation k^0 take place at $t = t_2$, $t_2 > t_1$. Then we shall write $j^0(t_1) \to k^0(t_2)$. Generally the notation $j^0(t_1) \to \{k_1^0(t_2), \ldots, k_s^0(t_2)\}$ denotes that $j^0(t_1) \to k_e^0(t_2)$ for suitable $e \in \{1, \ldots, s\}$ is valid.

We shall investigate the behaviour of y under the validity of all initial conditions $1^0 - 18^0$ at the point $t = 0$.

Let 1^0 be valid for $t = 0$ and put $\tau = 1$ for the simplicity. If $y(0) = 0$, then, with respect to $y'(0) > 0$ the inequalities $y^{(i)}(t) > 0$, $i = 0, 1, 2, 3$ are valid in some right neighbourhood of $t = 0$. According to (2) y''' is non-increasing in the interval at which $y(t) > 0$ holds. As y is oscillatory it follows from this that there exists a number $t^3 > 0$ with the property $y'''(t^3) = 0$, $y^{(i)}(t) > 0$ on $(0, t^3]$, $i = 0, 1, 2$. The case $y'''(t) = 0$ for $t^3 \leqq t < b$ is imposible with respect to the fact that y is oscillatory and thus the number t_4 exists such that $t^3 \leqq t^4 < b$, $y'''(t) = 0$ on

$[t^3, t^4]$, $y'''(t) < 0$ on $(t^4, t^4 + \varepsilon)$, $y^{(i)}(t) > 0$ on $(0, t^4 + \varepsilon)$, $i = 0, 1, 2$, $\varepsilon > 0$ being a suitable number. Thus y'' is decreasing in some right neighbourhood of the $t = t^4$. By the same procedure the existence of the points t^2, t^1, t^0 may be proved such that

$$t^4 < t^2 < t^1 < t^0 < b, \qquad y^{(i)}(t^i) = 0,$$

$y^{(i)}(t) > 0$ on $(0, t^i)$, $y^{(i)}(t) < 0$ on (t^i, t^0), $i = 2, 1, 0$ hold. Especially

$$y(t^0) = 0, \qquad y^{(i)}(t^0) < 0, \qquad i = 1, 2, 3$$

is valid and thus

$$1^0(0) \to 2^0(t^3) \to 3^0(t^2) \to 4^0(t^1) \to 1^0(t^0).$$

By repeating of the considerations we can conclude that y is an oscillatory solution of the 1-st type on some interval $[0, b_1)$, $b_1 \leqq b$ if the initial conditions $1^0, 2^0, 3^0$ or 4^0 are valid for $t = 0$.

When considering the sign of $y^{(i)}(0)$, $i = 0, 1, 2, 3$ and (2) it can be easily seen that for a suitable number $t_1 > 0$ the following relations hold:

$$9^0(0) \to 2^0(t_1), \qquad 10^0(0) \to 1^0(t_1), \qquad 11^0(0) \to 3^0(t_1),$$
$$12^0(0) \to 4^0(t_1), \qquad 13^0(0) \to \{4^0(t_1), \quad 9^0(t_1)\},$$
$$14^0(0) \to \{3^0(t_1), \quad 13^0(t_1)\}, \qquad 15^0(0) \to 3^0(t_1),$$
$$16^0(0) \to \{4^0(t_1), \quad 15^0(t_1)\}, \qquad 17^0(0) \to 4^0(t_1).$$

Thus in all cases with the exception of $5-8^0$ and 18^0 the solution y is the 1-st type on $[0, b_1)$, $b_1 \leqq b$.

Consider the case 5^0 for $t = 0$ and $\tau = 1$ (for the simplicity). If $y(0) = 0$, then in some right neighbourhood of $t = 0$

$$(8) \qquad y^{(i)}(t) > 0, \qquad i = 0, 1, 3, \qquad y''(t) < 0$$

holds. As y is oscillatory the number $t^1 > 0$ must exist such that $y'(t^1)\, y''(t^1)\, y'''(t^1) = 0$, $y^{(i)}(t) \neq 0$ for $t \in (0, t_1)$, $i \leqq 3$. First, let $y'''(t^1) = 0$ be valid. Then according to (8)

$$y(t^1) > 0, \qquad y'(t^1) \geqq 0, \qquad y''(t^1) \leqq 0$$

and it is clear, that one of the cases $13^0, 14^0, 15^0, 17^0$ is valid at $t = t^1$ and thus y is of the 1-st type. Similarly in case of $y''(t^1) = 0$ we have $y(t^1) > 0$, $y'(t^1) \geqq 0$, $y'''(t^1) \geqq 0$. Thus the cases 10^0, 15^0 or 17^0 take place at $t = t^1$ and y is of the first type, too. In the last case, when $y'(t^1) = 0$ is valid $y(t^1) > 0$, $y''(t^1) < 0$, $y'''(t^1) > 0$ holds and thus we have $5^0(0) \to 6^0(t^1)$.

It can be proved in the same way, that either y is the oscillatory solution of the 1-st type on some interval $[0, b_1)$, $b_1 \leqq b$ or the following relations

$$6^0(t^1) \to 7^0(t^2) \to 8^0(t^3) \to 5^0(t_5)$$

and (4) for $t_k^i = t^i$, $i = 0, 1, 2, 3$, $t_{k+1}^0 = t^5$ hold. We can conclude that in cases $5-8^0$ y is of the 1-st or 2-nd type on some interval $[0, b_1)$, $b_1 \leqq b$. The last case is 18^0, i.e.

(9) $$y^{(i)}(0) = 0, \qquad i = 0, 1, 2, 3.$$

Let us exclude the trivial solution $y \equiv 0$ on $[0, b)$ from our considerations—the theorem is valid in this case. Then there exists τ, $0 \leqq \tau < b$ such that $y(t) \equiv 0$ on $[0, \tau]$, $\sup_{[\tau, \tau+\varepsilon]} |y(t)| > 0$ for an arbitrary ε, $0 < \varepsilon < b - \tau$ holds. Suppose, that there exists a number $\varepsilon > 0$ such that $y(t) \neq 0$ for $t \in J = (0, \varepsilon)$. Put for the simplicity

(10) $$y(t) > 0 \qquad \text{on } J.$$

According to (2) y''' is non-increasing on J, $y'''(t) \leqq 0$ on J. Then successively $y^{(i)}(t) \leqq 0$ on J, $i = 2, 1, 0$, that contradicts to (9), (10). Thus there exists a sequence $\{t_k^0\}_{k=-\infty}^0$ such that $\lim_{k \to -\infty} t_k^0 = 0$, $t_k^0 > 0$, $y(t_k^0) = 0$ and the point τ such that $y(\tau) \neq 0$. Thus we have for $t = \tau$ one of the investigated cases $1-17^0$ and y is oscillatory solution of the 1-st or 2-nd type. Moreover, according to (5), (6) and (9) $F'(t) \geqq 0$ on $[0, b)$ and thus according to Lemma 2 we can conclude that y must be of the 1-st type on some interval $[0, b_1)$, $b_1 \leqq b$.

Now, let y be of the 1-st type on $[0, b_1)$, $b_1 \leqq b$. Then according to (3), (5), (6)

$$F'(t) \geqq F'(t_0^0) = y'(t_0^0)\, y''(t_0^0) = K > 0,$$
$$F(t) \geqq F(t_0^0) + K(t - t_0^0) \geqq F(t_0^0) = y'^2(t_0^0) > 0,$$
$$y'^2(t_k^0) = F(t_k^0) \geqq y'^2(t_0^0) > 0, \qquad \lim_{k \to \infty} t_k^0 = b_1$$

holds and thus $b_1 = b$ must be valid.

Let y be oscillatory solution of the 2-nd type on $[0, b)$. Then it follows from the continuity of y at $t = b_1$ that

$$y^{(i)}(b_1) = 0, \qquad i = 0, 1, 2, 3.$$

But this solution was met in the case 18^0. The theorem is proved.

Remark 1. Let y be an oscillatory solution on $[0, b)$ and let τ exist such that $y^{(j)}(\tau) = 0$, $j = 0, 1, 2, 3$, $\tau \in [0, b)$. Then numbers τ_1, τ_2, $0 \leqq \tau_1 \leqq \tau_2 \leqq b$ exist with the properties: $y(t) \equiv 0$ on $[\tau_1, \tau_2]$, y is non-trivial in every right (left) neighbourhood of the point $\tau_2(\tau_1)$, y is oscillatory of the 1-st (2-nd) type in the interval (τ_2, b) ($[0, \tau_1)$) and the sequence $\{t_k\}_{-\infty}^0$ of zeros of y exists such that $t_k > \tau_2$, $\lim_{k \to -\infty} t_k = \tau_2$.

This statement was proved in the course of the proof of Theorem 1. We must only prove that y is oscillatory on $[0, \tau_1)$. Suppose on the contrary that $y > 0$ on $J = [\tau_1 - \varepsilon, \tau_1)$, $\varepsilon > 0$ (the case $y < 0$ may be investigated similarly). Then according to (2) y''' is non-increasing and with respect to $y'''(\tau_1) = 0$ the relation $y''' > 0$ holds on J. As $y^{(i)}(\tau_1) = 0$, $i = 0, 1, 2, 3$ we have successively $y'' < 0$, $y' > 0$ and $y < 0$ on J which gives the contradictions with (2).

In the rest of the paper y will denote an oscillatory solution of (1), (2) of the first type defined on $[0, \infty)$. Let M_1, M_2 and M_3 be non-negative constants. Put

$$D_1(M_1, M_2, M_3) = \{(t, x_1, x_2, x_3, x_4) : t \geqq M_1, |x_1| \geqq M_1, |x_2| \geqq M_2, \\ |x_i| \leqq M_3 \text{ if } M_3 < \infty, |x_i| < \infty \text{ for } M_3 = \infty, i = 3, 4\},$$

$$D_2(M_1, M_2, M_3) = \{(t, x_1, x_2, x_3, x_4) : t \geqq M_1, |x_1| \geqq M_1, |x_2| \leqq M_2, \\ |x_3| \leqq M_2, |x_4| \leqq M_3\}.$$

Theorem 2. *The relations* $\limsup_{t \to \infty} |y^{(i)}(t)| = \infty$, $i = 0, 1$ *and*

$$|y'(t_k^2)| \geqq C\sqrt{t_k^2}, \qquad k \geqq 2$$

are valid where C is a positive constant.

Proof. According to Lemma 1, (5) and (6)

$$(11) \qquad |y'(t_k^2)|^2 = F(t_k^2) \geqq F(t_1^0) + F'(t_1^0)(t_k^2 - t_1^0) \geqq \\ \geqq F'(t_1^0)\left(1 - \frac{t_1^0}{t_1^2}\right)t_k^2 = C^2 t_k^2, \qquad k \geqq 2$$

and thus the statement of the theorem for $i = 1$ is valid. Let us prove by the indirect proof that it is valid also for $i = 0$. Thus suppose that

$$(12) \qquad |y(t)| \leqq M, \qquad t \in [0, \infty).$$

Put

$$J_{k+1} = [t_k^0, t_{k+1}^0],\ \Delta_k = t_k^0 - t_{k-1}^0,\ \Delta_k^{(1)} = t_k^0 - t_k^1,\ \Delta_k^{(2)} = t_k^1 - t_{k-1}^0,\ \Delta_k^{(3)} = \\ = t_k^0 - t_k^*,\ \Delta_k^{(4)} = t_k^* - t_k^1$$

where t_k^* is defined (uniquely, see (3)) by the relations

$$t_k^* \in (t_k^1, t_k^0),\ 2|y(t_k^*)| = |y(t_k^1)|.$$

It follows from (3), (5) and (6)

$$F(t_k^0) = y'^2(t_k^0) = 2\int_{t_k^1}^{t_k^0} y'(t)\,y''(t)\,dt \leqq 2|y''(t_k^0)|\ |y(t_k^1)| \leqq 2M\,|y''(t_k^0)|$$

$$F'(t) \geqq F'(t_k^0) = |y'(t_k^0)|\ |y''(t_k^0)| \geqq M_1 F(t_k^0)^{3/2}, \qquad t \in J_{k+1},\ M_1 = 1/(2M).$$

$$F'(t) \geqq M_1 F^{3/2}(t_k^0) = M_1\left(F^{3/2}(t) - \frac{3}{2}F^{1/2}(\xi)\,F'(\xi)\,(t - t_k^0)\right), \\ t \in J_{k+1}, \qquad \xi \in (t_k^0, t).$$

According to Lemma 1

$$(13) \qquad F(t) > 0, \qquad F'(t) > 0,$$

F and F' are non-decreasing on $[t_1^0, \infty)$ and thus for $k \geqq 1$

$$F'(t) \geqq M_1\left[F^{3/2}(t) - \frac{3}{2}F^{1/2}(t_{k+1}^0)F'(t)\,\Delta_{k+1}\right],$$

$$F'(t)\left[1 + \frac{3}{2}M_1F^{1/2}(t_{k+1}^0)\,\Delta_{k+1}\right] \geqq M_1F^{3/2}(t), \qquad t \in J_{k+1}. \tag{14}$$

Let us divide $\{t_k^0\}_1^\infty$ into two subsequences $\{t_k^0\}_{k\in N_1}$, $\{t_k^0\}_{k\in N_2}$ in the following way: $N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 = N$, $k \in N_1$ if, and only if $\frac{3}{2}M_1F^{1/2}(t_k^0)\,\Delta_k \leqq 1$. Let $k \in N_1$. It follows from (14) that

$$2F'(t) \geqq M_1F^{3/2}(t), \qquad t \in J_k,$$

$$-\frac{1}{\sqrt{F(t_k^0)}} + \frac{1}{\sqrt{F(t_{k-1}^0)}} \geqq \frac{M_1}{4}\Delta_k.$$

As F is non-decreasing, then by adding of these inequalities for $k \in N_1$ we get

$$\frac{1}{\sqrt{F(t_s)}} \geqq \sum_{k\in N_1}\left(-\frac{1}{\sqrt{F(t_k^0)}} + \frac{1}{\sqrt{F(t_{k-1}^0)}}\right) \geqq \frac{M_1}{4}\sum_{k\in N_1}\Delta_k,$$

where $s = \min\{N_1\}$. Thus

$$\sum_{k\in N_1}\Delta_k < \infty. \tag{15}$$

Now, let $k \in N_2$ and let $N_2 = \{n_s\}$, $s \in N_3$, $N_3 = \{1, 2, \ldots, \bar{s}\}$ or $N_3 = N$. Then it follows from (14) and the definition of N_2 that

$$3M_1F^{1/2}(t_k^0)\,\Delta_kF'(t) \geqq M_1F^{3/2}(t), \qquad t \in J_k$$

holds and thus by integration on the interval J_k

$$6F^{1/2}(t_k^0)\,\Delta_k\left[-\frac{1}{\sqrt{F(t_k^0)}} + \frac{1}{\sqrt{F(t_{k-1}^0)}}\right] \geqq \Delta_k,$$

$$\frac{F(t_k^0)}{F(t_{k-1}^0)} \geqq \frac{49}{36} = \alpha > 1.$$

Thus according to (13)

$$F(t_{n_s}^0) \geqq F(t_{n_1}^0)\,\alpha^s, \qquad s \in N_3 \tag{16}$$

is valid and

$$\left(\frac{-y'(t)}{y(t)}\right)' = \frac{F(t)}{y^2} \geqq M_2\alpha^s, \qquad M_2 = \frac{F(t_{n_1}^0)}{\alpha M^2} > 0,$$

$$s \in N_3, \qquad t \in (t_{n_s-1}^0, t_{n_s}^0).$$

By integration on $[t_{n_s}^1, t]$, $t \leqq t_{n_s}^0$ we get

$$\frac{y'(t)}{y(t)} \leqq -M_2\alpha^s(t - t_{n_s}^1)$$

and by integration on $[t^1_{n_s}, t^*_{n_s}]$ the following inequality is valid

$$\frac{1}{\lambda} = \frac{y(t^2_{n_s})}{y(t^1_{n_s})} \leqq \exp\left\{-\frac{M_2\alpha^s}{2}\Delta^{(4)^2}_{n_s}\right\}.$$

From this

$$\alpha^s\Delta^{(4)^2}_{n_s} \leqq M_3^2, \qquad M_3 = \sqrt{\frac{2}{M_2}\ln 2} \tag{17}$$

$$\Delta^{(4)}_{n_s} \leqq M_3\alpha^{-s/2}, \qquad s \in N_3.$$

Next, as

$$\int_{\Delta_i^{(4)}} |\, y'(t)\,|\,\mathrm{d}t = \frac{|\, y(t_i^1)\,|}{2} = \int_{\Delta_i^{(3)}} |\, y'(t)\,|\,\mathrm{d}t$$

and with respect to the fact, that $|\, y'\,|$ is increasing on (t_i^1, t_i^0) (see (3)) we have $\Delta_i^{(4)} \geqq \Delta_i^{(3)}$. Therefore by virtue of (17) $\Delta^{(1)}_{n_s} = \Delta^{(3)}_{n_s} + \Delta^{(4)}_{n_s} \leqq 2M_3\alpha^{-s/2}$ and

$$\sum_{k \in N_2} \Delta_k^{(1)} < \infty \tag{18}$$

holds.

Let us investigate the intervals $[t^0_{k-1}, t^1_k]$. As according to (3) the function $|\, y'\,|$ is non-decreasing on $[t^0_{k-1}, t^2_k]$ and concave on (t^3_k, t_k), we have:

$$M \geqq |\, y(t_k^1)\,| = \int_{t^0_{k-1}}^{t^1_k} |\, y'(t)\,|\,\mathrm{d}t = \int_{t^0_{k-1}}^{t^3_k} \cdot/\cdot + \int_{t^3_k}^{t^1_k} \cdot/\cdot \geqq$$

$$\geqq |\, y'(t^0_{k-1})\,|\,(t^3_k - t^0_{k-1}) + \frac{1}{2}|\, y'(t^2_k)\,|\,(t^1_k - t^3_k),$$

$$2M \geqq |\, y'(t^0_{k-1})\,|\,\Delta_k^{(2)} = \sqrt{F(t^0_{k-1})}\;\Delta_k^{(2)}, \qquad k \in N_2.$$

From this, from (16) and (13)

$$\Delta^{(2)}_{n_s} \leqq \frac{2\sqrt{\alpha}M}{\sqrt{F(t^0_{n_1})}}\alpha^{-s/2}, \qquad s \in N_3$$

holds and thus $\sum_{i \in N_2} \Delta_i^{(2)} < \infty$. This inequality with (18) gives us $\sum_{i \in N_2} \Delta_i < \infty$. Thus with respect to $N = N_1 \cup N_2$ we can conclude that $\sum_{i \in N} \Delta_i < \infty$ which gives us the contradiction to the definition interval of y. The theorem is proved.

Lemma 3. *Let a constant $M > 0$ exist such that $|\, y^{(i)}(t_k^1)\,| \leqq M$, $k \in N_1 \subset N$, $N_1 = \{k_s\}_{s=1}^{\infty}$ holds, where $i = 2$ or $i = 3$. Then* $\lim_{s\to\infty} |\, y(t^1_{k_s})\,| = \infty$, $\lim_{s\to\infty} |\, y'(t^2_{k_s})\,| = \infty$,

$$\lim_{s\to\infty} (t^1_{k_s} - t^2_{k_s}) = \infty. \tag{19}$$

Proof. Denote $t^i_{k_s} = t^i_s$, $i = 1, 2$, $J_s = [t^2_s, t^1_s]$, $\Delta_s = t^1_s - t^2_s$.

The conclusion $\lim_{s\to\infty} |y'(t_s^2)| = \infty$ follows from Theorem 2. Then according to (3)

$$\infty \underset{s\to\infty}{\leftarrow} |y'(t_s^2)| = \int_{J_s} |y''(t)|\,dt \leqq |y''(t_s^1)|\,\Delta_s \leqq \Delta_s \int_{J_s} |y'''(t)|\,dt \leqq \Delta_s^2\,|y'''(t_s^1)|$$

and thus with respect to the assumptions of the theorem (19) is valid. The rest of the assertion follows from the estimate obtained by use of the fact that $|y'(t)|$ is concave on J_s (see (3)):

$$|y(t_s^1)| \geqq |y(t_s^1)| - |y(t_s^2)| = \int_{J_s} |y'(t)|\,dt \geqq \frac{1}{2}\,|y'(t_s^2)|\,\Delta_s \underset{s\to\infty}{\rightarrow} \infty.$$

The lemma is proved.

Theorem 3. *Let constants $\alpha \geqq 0$, $\beta \geqq 0$, $K \geqq 0$ and $K_1 > 0$ exist such that*

(20) $$|f(t, x_1, x_2, x_3, x_4)| \geqq a(t)\,|x_1|^\alpha\,|x_2|^\beta \qquad \text{holds in } D_1(K, K, K_1)$$

where $a \in L(R_+)$, $\liminf_{s\to\infty} \int_s^{s+1} a(t)\,t^{\alpha+\beta/2}\,dt = K_2 > 0$. Then

(21) $$\limsup_{t\to\infty} |y^{(i)}(t)| = \infty, \qquad i = 0, 1, 2.$$

Moreover, if $K_1 = \infty$, then

$$\limsup_{t\to\infty} |y'''(t)| = \infty$$

holds.

Proof. According to Theorem 2 the relation (21) is valid for $i = 0, 1$. We prove it for $i = 2$ by the indirect proof. Thus suppose that there exists a constant M such that

(22) $$|y''(t)| \leqq M, \qquad t \in [0, \infty).$$

Then it follows from Lemma 3 and Lemma 1 that

(23) $$\lim_{k\to\infty} |y'(t_k^0)| = \lim_{k\to\infty} |y'(t_k^2)| = \lim_{k\to\infty} |y(t_k^1)| = \infty,$$

(24) $$\lim_{k\to\infty} (t_k^1 - t_k^2) = \infty, \qquad k \in N$$

holds. Further, by use of (23), (22) and (3)

$$|y'(t_k^0)| = \int_{t_k^1}^{t_k^0} |y''(t)|\,dt \leqq M(t_k^0 - t_k^1),$$

(25) $$\lim_{k\to\infty} (t_k^0 - t_k^1) = \infty$$

holds.

By virtue of (24) there exists a sequence $\{t_k^*\}_{k_0}^{\infty}$, $t_k^* \in (t_k^2, t_k^1)$, $t_k^* - t_k^2 = 1$. Put $J_k = [t_k^2, t_k^*]$. According to (25), (22) and the fact that $|y'''|$ is non-decreasing on $[t_k^1, t_k^0]$ (see (3)) we have

$$M \geqq \int_{[t_k^1, t_k^0]} |y''(t)| \, dt \geqq |y'''(t_k^1)| (t_k^0 - t_k^1)$$

and thus by virtue of (25)

$$(26) \qquad \lim_{k\to\infty} |y'''(t_k^1)| = 0, \qquad \lim_{k\to\infty} |y'''(t_k^*)| = 0$$

holds. Further according to (3)

$$(27) \qquad |y''(t_k^*)| = \int_{J_k} |y'''(t)| \, dt \leqq |y'''(t_k^*)| [t_k^* - t_k^2] = |y'''(t_k^*)|$$

holds and from the relation

$$(28) \qquad |y'(t_k^2)| - |y'(t_k^*)| = \int_{J_k} |y''(t)| \, dt \leqq |y''(t_k^*)|,$$

(26), (27) and (23)

$$\lim_{k\to\infty} |y'(t_k^*)| = \infty$$

is valid. Let $\varepsilon > 0$ be an arbitrary number such that $\varepsilon \leqq \min\left(K_1, \frac{F'(t_1^0)}{K}\right)$ holds and let $k_0 \geqq 2$ be integer with the properties:

$$(29) \qquad t_k^2 \geqq K, |y'(t_k^*)| \geqq K, |y'''(t_k^*)| < \varepsilon, k \geqq k_0.$$

Then it follows from (29), (3), (26), (27) and from the fact

$$F'(t_1^0) \leqq F'(t_k^2) = -y'''(t_k^2) \, y(t_k^2) \leqq \varepsilon |y(t_k^2)|, k \geqq k_0,$$

that the following relation is valid:

$$(30) \qquad (t, y(t), y'(t), y''(t), y'''(t)) \in D_1(K, K, K_1), \qquad t \in J_k, k \geqq k_0.$$

There exists $k_1 \geqq k_0$ and $C_1 > 0$ such that (see Lemma 1, (6), Theorem 2)

$$|y'(t)| = |y'(t_k^2)| - \int_{t_k^2}^{t} |y''(t)| \, dt \geqq |y'(t_k^2)| - |y''(t)| \geqq$$
$$\geqq C\sqrt{t_k^2} - \varepsilon \geqq C\sqrt{t-1} - \varepsilon \geqq C_1\sqrt{t}, \qquad t \in J_k, k \geqq k_1,$$

$$(31) \quad |y(t)| \geqq \frac{-y(t)\,y'''(t)}{\varepsilon} \geqq \frac{1}{\varepsilon} F'(t) \geqq \frac{1}{\varepsilon}(F'(t_1^0) + F'(t_1^0)(t - t_1^0)) \geqq C_1 t, \qquad t \in J_k$$

is valid. From this and from (30) we can conclude that for suitable $k_2 \geqq k_1$

$$\varepsilon \geqq | y'''(t_k^*) | - | y'''(t_k^2) | = \int_{J_k} | y^{(4)}(t) | \, dt =$$
$$= \int_{J_k} | f(t, y(t), y'(t), y''(t), y'''(t) | \, dt \geqq$$
$$\geqq \int_{J_k} a(t) | y(t) |^\alpha | y'(t) |^\beta \, dt \geqq C_1^{\alpha+\beta} \int_{t_k^2}^{t_k^2+1} a(t) \, t^{\alpha+\beta/2} \, dt, \; k \geqq k_1,$$
$$\varepsilon \geqq \frac{1}{2} C_1^{\alpha+\beta} \liminf_{s\to\infty} \int_s^{s+1} a(t) \, t^{\alpha+\beta/2} \, dt = \frac{1}{2} C_1^{\alpha+\beta} K_2 > 0, \; k \geqq k_2.$$

As ε may be chosen arbitrarily small, this relation, in virtue of the assumptions of the theorem, gives the contradiction. Thus (21) is valid.

Now we prove that $\limsup_{t\to\infty} | y'''(t) | = \infty$ by the indirect proof. Therefore suppose that

$$| y'''(t) | \leqq M_1, \qquad t \in [0, \infty).$$

According to Lemma 3 the relations (23) and (24) are valid and

$$| y''(t) | \leqq | y''(t_k^*) | = \int_{J_k} | y'''(t) | \, dt \leqq M_1, \qquad t \in J_k.$$

From this and from (28) and (23)

$$\lim_{k\to\infty} | y'(t_k^*) | = \infty$$

is valid. As for $\varepsilon = M_1$ and suitable k_1, C_1 the relation (31) is valid, we conclude that there exists an integer k_0 such that

$$(t, y(t), y'(t), y''(t), y'''(t)) \in D_1(K, K, M_1), \qquad t \in J_k, \; k \geqq k_0$$

holds. We get the contradiction in the same way as in the first part of the proof after the relation (31). The theorem is proved.

Theorem 4. *Let constants $\alpha \geqq 0$, $K_1 \geqq 0$ and $K_2 > 0$ exist such that for an arbitrary K, $0 < K < \infty$ the relation*

$$| f(t, x_1, x_2, x_3, x_4) | \geqq a_k(t) \, | x_1 |^\alpha \qquad \text{on } D_2(K_1, K, K_2)$$

holds where $a_k \in L(R^+)$ and $\liminf_{s\to\infty} \int_s^{s+1} a_k(t) \, t^\alpha \, dt > 0$.

Then

$$\limsup_{t\to\infty} | y^{(i)}(t) | = \infty, \qquad i = 0, 1, 2 \tag{32}$$

holds.

Proof. The assertion (32) follows from Theorem 2 for $i = 0, 1$. For $i = 2$ (32) may be proved by the indirect proof. Thus suppose that

$$| y''(t) | \leqq M, \qquad t \in [0, \infty). \tag{33}$$

According to Lemma 3 $\lim_{k\to\infty}(t_k^1 - t_k^2) = \infty$ holds and similarly as in the proof of Lemma 3

$$\lim_{t\to\infty} |y'''(t_k^1)| = 0 \tag{34}$$

is valid. Define the sequence $\{t_k^*\}$, $t_k^* \in (t_k^2, t_k^1)$, $t_k^1 - t_k^* = 1$ and let $J_k = [t_k^*, t_k^1]$. Further, according to (3), (5), (6) and Lemma 1 numbers $C > 0$, $C_1 > C$, $k_0 \geqq 2$ exist such that

$$|y'(t)| \leqq |y'(t_k^*)| = \int_{J_k} |y''(t)|\,dt \leqq M, \qquad t \in J_k, \tag{35}$$

$$Ct_k^* \leqq F(t_k^*) = y'^2(t_k^*) - y(t_k^*)\,y(t_k^*) \leqq M^2 + M\,|y(t_k^2)|,$$

$$|y(t)| \geqq |y(t_k^*)| \geqq C_1 t, \qquad t \in J_k,\ k \geqq k_0 \tag{36}$$

hold. Let ε, $0 < \varepsilon \leqq K_2$ be an arbitrary number. Then it follows from (33), (34), (35) and (36) that for a suitable $k_1 > k_0$ we have $(t, y(t), y'(t), y''(t), y'''(t)) \in D_2(K_1, M, K_2)$, $t \in J_k$, $\varepsilon \geqq |y'''(t_k^1)| - |y'''(t_k^*)| = \int_{J_k} |y^{(4)}(t)|\,dt = \int_{J_k} |f(t, y(t), y'(t), y''(t), y'''(t))|\,dt \geqq \int_{J_k} a_k(t)\,|y(t)|^\alpha\,dt = C_1^\alpha \int_{J_k} a_k(t)\,t^\alpha\,dt \geqq \frac{1}{2} C_1^\alpha \limsup_{s\to\infty} \int_s^{s+1} a_k(t)\,t^\alpha\,.$ $.\,dt = \text{const} > 0$, $k \geqq k_1$. As ε may be chosen arbitrarily small, we can conclude, that this relation gives us the contradiction. The theorem is proved.

Remark 2. The results of Theorems 2, 3 and 4 generalize the ones of [1] for the differential equation (1).

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