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ON OSCILLATION OF SOLUTIONS OF LINEAR DEVIATING DIFFERENTIAL EQUATION

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Abstract. A sufficient condition is given that all solutions of the equation $y^{(n)}(t) + p(t)\,y(g(t)) = 0$, $n \geqq 1$, are oscillatory if n is even or odd. It is assumed throughout this paper that $p(t)$, $g(t)$ are continuous on $[0, \infty)$, $p(t) > 0$, $g(t) < t$, $\lim_{t\to\infty} g(t) = \infty$, $g(t)$ is nondecreasing. The oscillatory behaviour of the equation involving retarded and advanced arguments is studied, too.

Key words. Deviating argument, linear equation of n-th order, oscillation of solutions, non-oscillatory solution of degree l.

1. Introduction

The purpose of this paper is to study the oscillatory behaviour of solutions of the linear differential equation with retarded argument

$$y^{(n)}(t) + p(t)\,y(g(t)) = 0, \qquad n \geqq 1, \tag{1}$$

and the asymptotic behaviour of solutions of the linear differential equation with advanced argument

$$y^{(n)}(t) + q(t)\,y(h(t)) = 0, \qquad n \geqq 2, \tag{2}$$

where $p(t)$, $q(t)$, $g(t)$ and $h(t)$ are continuous functions on $[0, \infty)$ such that $p(t) > 0$, $q(t) > 0$, $g(t) < t$, $h(t) > t$ and $\lim_{t\to\infty} g(t) = \infty$.

The oscillatory behaviour of the equation involving both retarded and advanced arguments

$$y^{(n)}(t) + p(t)\,y(g(t)) + q(t)\,y(h(t)) = 0, \qquad n \geqq 1, \tag{3}$$

will also be studied.

A solution $y(t)$ of the equation (1) (or (2), (3)) is called oscillatory if it has arbitrarliy large zeros, and it is called nonoscillatory otherwise.

Lemma 1. (Kiguradze). *Let $y(t)$ be a solution of the equation* (1) (*or* (2), (3)) *satisfying the condition*

$$y(t) > 0 \qquad for\ t \in [0, \infty),$$

and let $y^{(n)}(t) \leqq 0$ *for* $t \in [0, \infty)$.

Then there exist a $t_1 \in [0, \infty)$ *and an integer* $l \in \{0, 1, \dots, n-1\}$ *such that* $l + n$ *is odd and*

$$(4) \qquad y^{(i)}(t) > 0 \quad for\ t \in [t_1, \infty) \quad (i = 0, \dots, l-1),$$
$$(-1)^{i+l} y^{(i)}(t) > 0 \quad \text{for } t \in [t_1, \infty) \quad (i = l, \dots, n-1),$$

$$(5) \quad (t - t_1) \mid y^{(l-i)}(t) \mid \leqq (1 + i) \mid y^{(l-i-1)}(t) \mid \quad for\ t \in [t_1, \infty)\ (i = 0 \dots, l-1),$$
$$1 \leqq l \leqq n-1.$$

Analogous statement can be made if $y(t) < 0$ and $y^{(n)}(t) \geqq 0$ for $t \in [0, \infty)$.

An $y(t)$ which satisfies (4) is said to be a (nonoscillatory) solution of degree l (see Foster and Grimmer [1]).

2. Retarded equation

We consider the equation (1) with retarded argument where $p(t)$ and $g(t)$ are continuous on $[0, \infty)$, $p(t) > 0$, $g(t) < t$, $g(t)$ is nondecreasing and $\lim_{t\to\infty} g(t) = \infty$.

Theorem 1. *Suppose that for every* $l \in \{0, 1, \dots, n-1\}$ *such that* $n + l$ *is odd and for some* $d_l \in \{0, 1, \dots, n-l-1\}$ *it holds*

$$(6) \qquad \limsup_{t\to\infty} \int_{g(t)}^{t} [s - g(t)]^{n-l-d_l-1} [g(t) - g(s)]^{d_l} [g(s)]^l p(s) \, ds >$$
$$> l!(n - l - d_l - 1)!\, d_l!.$$

Then every solution of equation (1) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1) such that $y(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geqq 0$. Then with regard to Lemma 1 there exist $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \dots, n-1\}$ such that $n + l$ is odd and (4), (5) hold. For sufficiently large $t_2 \in [t_1, \infty)$ in view of (5) we have

$$(7) \qquad y(g(t)) \geqq \frac{[g(t) - t_1]^l}{l!} y^{(l)}(g(t)), \qquad t \geqq t_2, \qquad 0 \leqq l \leqq n-1.$$

From the equality

$$(8) \quad z^{(j)}(t) = \sum_{i=j}^{k-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} z^{(i)}(s) + \frac{(-1)^{k-j}}{(k-j-1)!} \int_t^s (u - t)^{k-j-1} z^{(k)}(u) \, du,$$

$s \geqq t \geqq t_2$, for $k = n - l$ we get

$$(9) \qquad z^{(j)}(t) = \sum_{i=j}^{n-l-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} z^{(i)}(s) +$$
$$+ \frac{(-1)^{n-l-j}}{(n-l-j-1)!} \int_t^s (u - t)^{n-l-j-1} z^{(n-l)}(u) \, du.$$

Choose $z(t) = y^{(l)}(t)$. Then for $j = d_l$, $d_l \in \{0, 1, \dots, n - l - 1\}$, from (9) with regard to (4) we have

$$(10) \qquad |z^{(d_l)}(g(t))| \geqq \frac{1}{(n - l - d_l - 1)!} \int_{g(t)}^{t} [u - g(t)]^{n-l-d_l-1} |z^{(n-l)}(u)| \, du.$$

From (9) for $u \in [g(t), t]$, $d_l \in \{0, 1, \dots, n - l - 1\}$, $j = 0$, we get

$$(11) \qquad |z(g(u))| \geqq \frac{[g(t) - g(u)]^{d_l}}{d_l!} |z^{(d_l)}(g(t))|.$$

From (10) in view of equation (1) we obtain

$$|z^{(d_l)}(g(t))| \geqq \frac{1}{(n - l - d_l - 1)!} \int_{g(t)}^{t} [u - g(t)]^{n-l-d_l-1} |y(g(u))| \, p(u) \, du.$$

From the last inequality using (7) and (11) we have

$$l!(n - l - d_l - 1)! \, d_l! \geqq \int_{g(t)}^{t} [u - g(t)]^{n-l-d_l-1} [g(t) - g(u)]^{d_l} [g(u) - t_1]^l \, p(u) \, du.$$

So for t sufficiently large we get a contradiction to (6). This completes the proof.

If for every $l \in \{0, 1, \dots, n - 1\}$ we take $d_l = 0$, we get the next corollary.

Corollary 1. Suppose that for every $l \in \{0, 1, \dots, n - 1\}$ such that $n + l$ is odd the following holds

$$(12) \qquad \limsup_{t \to \infty} \int_{g(t)}^{t} [s - g(t)]^{n-l-1} [g(s)]^l \, p(s) \, ds > l!(n - l - 1)!.$$

Then every solution of equation (1) is oscillatory.

Corollary 2. [5]. Let n be odd and let for some $d \in \{0, 1, \dots, n - 1\}$ hold

$$(13) \qquad \limsup_{t \to \infty} \int_{g(t)}^{t} [s - g(t)]^{n-d-1} [g(t) - g(s)]^d \, p(s) \, ds > (n - d - 1)! \, d!.$$

Then every bounded solution of equation (1) is oscillatory.

Proof. If $y(t)$ is a bounded nonoscillatory solution of equation (1) then $l = 0$ and we can apply Theorem 1.

We introduce the notation:

$$G(t) = \max \{s - g(s) : g(t) \leqq s \leqq t\}.$$

Corollary 3. Let n be even and let

$$(14) \qquad G(t) \leqq g(t) \qquad \text{for } t \geqq T, T \in [0, \infty),$$

hold and in addition

$$(15) \qquad \limsup_{t \to \infty} \int_{g(t)}^{t} [s - g(t)]^{n-2} g(s) \, p(s) \, ds > (n - 2)!.$$

Then every solution of equation (1) is oscillatory.

Proof. In view of (14) the condition (15) implies (12) and we can apply

Corollary 4. Let n be odd, let (14) hold and in addition

$$\limsup_{t\to\infty} \int_{g(t)}^{t} [s - g(t)]^{n-1} p(s)\, ds > (n-1)!. \tag{16}$$

Then every solution of equation (1) is oscillatory.

Proof. In view of (14) the condition (16) implies (12) and we can apply Corollary 1.

Example 1. Every solution of the retarded differential equation

$$y'''(t) + y\left(t - \frac{3}{2}\pi\right) = 0,$$

with regard to the condition (16) is oscillatory. One such solution is $y(t) = \sin t$ But the corresponding ordinary differential equation has a nonoscillatory solution.

Example 2. Consider the differential equation with retarded argument

$$y'''(t) + \frac{\ln t}{t^3}\, y\left(\frac{2}{3}t\right) = 0, \qquad t > 1. \tag{17}$$

The well-known sufficient condition which guarantees that every solution of equation (17) is oscillatory or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, 2$,

$$\int^{\infty} [g(t)]^{2-\varepsilon} p(t)\, dt = \infty, \qquad \varepsilon > 0,$$

is not satisfied. The conditions (14), (16) are satisfied. So every solution of equation (17) is oscillatory.

Remark 1. Theorem 1 holds for the following differential inequality too

$$\{y^{(n)}(t) + p(t)\, y(g(t))\} \operatorname{sgn} y(g(t)) \leqq 0.$$

3. Advanced equation

In this section we are concerned with the differential equation (2) with advanced argument where $q(t)$ and $h(t)$ are continuous on $[0, \infty)$, $q(t) > 0$, $h(t) > t$, $h(t)$ is nondecreasing.

Theorem 2. *Suppose that the following condition is satisfied*

$$\limsup_{t\to\infty} \int_{t}^{h(t)} (s - t)\, s^{n-2} q(s)\, ds > (n-1)!. \tag{18}$$

Then equation (2) has no solution of degree $l \in \{2, \dots, n-1\}$.

Proof. Let $y(t)$ be a positive solution of equation (2) on $[t_0, \infty)$, $t_0 \geqq 0$, the degree of which is $l \in \{2, \ldots, n-1\}$, $n \geqq 3$. It is easy to see that equation (2) for $n = 2$ has only (nonoscillatory) solutions of degree $l = 1$. With regard to Lemma 1 from (8) for $j = l$, $k = n$, $t > t_0$, we have

$$y^{(l)}(t) \geqq \frac{1}{(n-l-1)!} \int_t^\infty (u-t)^{n-l-1} q(u)\, y(h(u))\, du.$$

We integrate the above inequality from t_0 to t, $t > t_0$,

$$y^{(l-1)}(t) \geqq \frac{(t-t_0)^{n-l}}{(n-l)!} \int_t^\infty q(u)\, y(h(u))\, du.$$

Repeating this procedure we get

$$y'(t) \geqq \frac{(t-t_0)^{n-2}}{(n-2)!} \int_t^\infty q(u)\, y(h(u))\, du.$$

We integrate the last inequality from t to $h(t)$, $t > t_0$,

$$y(h(t)) \geqq \frac{1}{(n-2)!} \int_t^{h(t)} q(u)\, y(h(u)) \int_t^u (s-t_0)^{n-2}\, ds\, du,$$

$$y(h(t)) \geqq \frac{1}{(n-1)!} \int_t^{h(t)} (u-t)(u-t_0)^{n-2} q(u)\, y(h(u))\, du.$$

Then

$$(n-1)! \geqq \int_t^{h(t)} (u-t)(u-t_0)^{n-2} q(u)\, du,$$

and for t sufficiently large we get a contradiction to (18). This proves the theorem.

Remark 2. The Theorem 2 holds for the following differential inequality too

$$\{y^{(n)}(t) + q(t)\, y(h(t))\} \operatorname{sgn} y(h(t)) \leqq 0.$$

4. Equation witn retarded and advanced arguments

We shall consider the differential equation (3) with retarded and advanced arguments where $p(t)$, $q(t)$, $g(t)$ and $h(t)$ are continuous on $[0, \infty)$, $p(t) > 0$, $q(t) > 0$, $g(t)$ and $h(t)$ are nondecreasing, $g(t) < t$, $\lim_{t\to\infty} g(t) = \infty$ and $h(t) > t$.

Theorem 3. *Let n be even and let the following conditions hold*

$$\limsup_{t\to\infty} \int_t^{h(t)} (s-t)\, s^{n-2} q(s)\, ds > (n-1)!, \tag{19}$$

$$\limsup_{t\to\infty} \int_{g(t)}^t [s-g(t)]^{n-d-2} [g(t)-g(s)]^d g(s)\, p(s)\, ds > (n-d-2)!\, d!, \tag{20}$$

for some $d \in \{0, 1, \ldots, n-2\}$.

Then every solution of equation (3) is oscillatory.

Proof. Let $y(t)$ be a positive solution of equation (3). Let $y(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geqq 0$. Then from (3) we obtain

$$y^{(n)}(t) + q(t)\,y(h(t)) < 0, \tag{21}$$

$$y^{(n)}(t) + p(t)\,y(g(t)) < 0. \tag{22}$$

In view of the condition (19) and Theorem 2, the inequality (21) has no solution of degree $l \in \{2, \dots, n-1\}$. So $y(t)$ has degree $l = 1$ and it is a solution of (22), which is a contradiction to the condition (20). This proves the theorem.

In similar way we can prove the next theorem.

Theorem 4. *Let $n \geqq 3$ be odd. Let* (19) *and* (13) *hold. Then every solution of equation* (3) *is oscillatory.*

Remark 3. Let n = 1 and let hold

$$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s)\,ds > 1.$$

Then every solution of equation (3) is oscillatory.

Proof. Let $y(t)$ be a positive solution of (3). Then $y(t)$ is a solution of inequality

$$y'(t) + p(t)\,y(g(t)) < 0,$$

which is a contradiction to the condition (13) for $n = 1$.

Theorem 5. *Let n be even. Let* (19) *and the following condition holds*

$$\limsup_{t\to\infty} g(t) \int_{t}^{\infty} s^{n-2} p(s)\,ds > (n-1)!. \tag{23}$$

Then every solution of equation (3) *is oscillatory.*

Proof. Let $y(t)$ be a positive solution of equation (3). Let $y(g(t)) > 0$ for $t \in [t_0, \infty)$, $t_0 \geqq 0$. From (3) we get (21) and (22). The inequality (21) has no solution of degree $l \in \{2, \dots, n-1\}$. Then $y(t)$ has the degree $l = 1$ and it is a solution of (22). With regard to Lemma 1 and (22) from (8) for $j = 1$, $k = n$, $t > t_0$, we have

$$y'(t) \geqq \frac{1}{(n-2)!} \int_{t}^{\infty} (u-t)^{n-2} p(u)\,y(g(u))\,du.$$

Integrating the last inequality from T to t, $t > T \geqq t_0$, we obtain

$$y(t) \geqq \frac{1}{(n-1)!}(t-T) \int_{t}^{\infty} (u-T)^{n-2} p(u)\,y(g(u))\,du.$$

For $t > T$ such that $g(t) > T$ we get

$$y(g(t)) \geqq \frac{1}{(n-1)!}[g(t)-T] \int_{t}^{\infty} (u-T)^{n-2} p(u)\,y(g(u))\,du.$$

Since $y(t)$ is nondecreasing then we have

$$(n-1)! \geqq [g(t) - T] \int_t^\infty (u - T)^{n-2} p(u) \, du,$$

which is a contradiction to (23) for sufficiently large t.

Example 3. Consider the equation

$$y'''(t) + \frac{1}{2} y\left(t - \frac{3}{2}\pi\right) + \frac{1}{2} y\left(t + \frac{1}{2}\pi\right) = 0. \tag{24}$$

The conditions (19) and (13) are satisfied and so every solution of (24) is oscillatory by Theorem 4. The corresponding ordinary differential equation has a non-oscillatory solution.

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