Vasilios Anast. Staïkos; Panagiotis Ch. Tsamatos

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ON THE TERMINAL VALUE PROBLEM FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

V. A. STAIKOS and P. CH. TSAMATOS, Ioannina

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Abstract. The problem of the existence and uniqueness of solutions x of the equation

(E) $$x'(t) = f(t; x[\sigma_1(t)], \ldots, x[\sigma_k(t)])$$

under the "terminal" condition $\lim_{t\to\infty} x(t) = \xi$, where ξ is a vector in R^n or C^n, is studied.

Key words: differential equations with deviating arguments, terminal value problems, existence and uniqueness of solutions, asymptotic behavior of solutions.

1. Introduction

This paper is concerned with the existence and uniqueness of solutions x of the differential equation with deviating arguments

(E) $$x'(t) = f(t; x[\sigma_1(t)], \ldots, x[\sigma_k(t)])$$

which satisfy the "terminal" condition

(C) $$\lim_{t\to\infty} x(t) = \xi,$$

where $\xi \in K^n$ and K stands for the real line R or for the complex plane C. The function $f : [t_0, \infty) \times (K^n)^k \to K^n$ is a locally Caratheodory one and moreover the real-valued functions σ_i, $i = 1, 2, \ldots, k$ are supposed to be continuous on $[t_0, \infty)$ and, as usually, such that

$$\lim_{t\to\infty} \sigma_i(t) = \infty \qquad (i = 1, 2, \ldots, k).$$

The existence of solutions for the above "terminal-value" problem (E)–(C) is closely related to that of asymptotic equilibrium for the equation (E). Among numerous studies on the asymptotic equilibrium we choose to refer the recent ones due to Ladas and Lakshmikantham [5, 6] and Mitchell [8] for ordinary differential equations as well as Hallam, Ladas and Lakshmikantham [3] for functional differential equations. For the case of ordinary differential equations we mention the paper of Hallam [2] where the existence of solutions of the terminal value problem is treated by a comparison principle. Finally, for general interest and with respect to the differential equations with deviating arguments we refer also Kurzweil [4], Lim [7] and Pandolfi [9] for asymptotic properties of exponential type.

2. Existence

The technique used for the existence of solutions of the terminal value problem (E)–(C) is based on the well-known Schauder fixed point theorem.

The Schauder theorem. *Let E be a Banach space and X any nonempty convex and closed subset of E. If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point* (*i.e. there exists an $x \in X$ with $x = Sx$.*)

The concrete Banach space which appears in the following is the space $B_T([T_0, \infty), K^n)$, $T \geqq T_0$, of all continuous and bounded K^n-valued functions on the interval $[T_0, \infty)$ which are constant on $[T_0, T]$, endowed with the usual sup-norm $\| \ \|$. For subsets of the space $B_T([T_0, \infty), K^n)$ we need also the compactness criterion below (cf. Avramescu [1] and Staikos [10, 11]), which is a consequence of the well-known Arzelà–Ascoli theorem and it is based on the concept of "equiconvergence".

A set $\mathscr{F}$ of K^n-valued functions defined on the interval $[T_0, \infty)$ is said to be *equiconvergent at* ∞ if all functions in $\mathscr{F}$ are convergent in K^n at the point ∞ and, moreover, for every $\varepsilon > 0$ there exists an $A \geqq T_0$ such that, for all functions $f \in \mathscr{F}$

$$t \geqq A \Rightarrow | f(t) - \lim_{s \to \infty} f(s) | < \varepsilon.$$

Compactness criterion. *Let $\mathscr{F}$ be an equicontinuous and uniformly bounded subset of the Banach space $B_T([T_0, \infty), K^n)$. If $\mathscr{F}$ is equiconvergent at ∞, it is also relatively compact.*

Now, for every $r > 0$ we introduce the function a_r defined by the formula

$$a_r(t) = \max_{\substack{|z_i| \leqq r \\ i = 1, 2, \ldots, k}} | f(t; z_1, \ldots, z_k) |, \qquad t \geqq t_0$$

Theorem 1. *If for every $r > 0$,*

(C₁) $$\int^{\infty} a_r(t)\, dt < \infty,$$

then for every $\xi \in K^n$ *the terminal value problem* (E)$-$(C) *has at least one solution* x *on an interval of the form* $[T, \infty)$.

Proof. Let

$$K = \mathop{\times}_{i=1}^{n} B(\xi_i, r), \qquad B(\xi_i, r) = \{w \in K\,;\, |w - \xi_i| \leqq r\},$$

where $\xi = (\xi_1, \dots, \xi_n) \in K^n$ and $r > 0$. We set

$$T_0 = \min\{t_0, \min_{t \geqq t_0} \sigma_1(t), \dots, \min_{t \geqq t_0} \sigma_k(t)\}$$

and, by condition (C_1), we choose a $T \geqq T_0$ so that

$$\int_T^\infty a_r(t)\,\mathrm{d}t \leqq \frac{r}{2}. \tag{1}$$

Now, let the Banach space $E = B_T([T_0, \infty), K^n)$ and the subset X of E,

$$X = \left\{x \in E : \|x - \xi\| \leqq \frac{r}{2}\right\}.$$

The set X is obviously *nonempty* and, as it is easy to see, it is *convex* and *closed.*

In order to define a mapping S, which satisfies the assumptions of the Schauder theorem, we remark that for any function $x = (x_1, \dots, x_n) \in X$ and for every $t \geqq T_0$,

$$|x_i(t) - \xi_i| \leqq |x(t) - \xi| \leqq \frac{r}{2} < r.$$

Hence, because of the choice of T_0 we have

$$x[\sigma_i(t)] \in K, \qquad t \geqq T \qquad (i = 1, 2, \dots, k)$$

and therefore for every $t \geqq T$

$$|f(t; x[\sigma_1(t)], \dots, x[\sigma_k(t)])| \leqq a_r(t).$$

Thus, because of (C_1), we have that for every $t \geqq T$ the improper integral

$$\int_t^\infty f(s; x[\sigma_1(s)], \dots, x[\sigma_k(s)])\,\mathrm{d}s$$

exists in K^n and hence the formula

$$y(t) = \xi - \int_t^\infty f(s; x[\sigma_1(s)], \dots, x[\sigma_k(s)])\,\mathrm{d}s, \qquad t \geqq T$$

defines a mapping $S : X \to E$. This mapping is the required one. Namely, it satisfies the following.

a) $SX \subseteq X$.

In fact, taking into account (1), for every function $x \in X$ and for any $t \geqq T$ we have

$$|(Sx)(t) - \xi| = \left|\int_t^\infty f(s; x[\sigma_1(s)], \ldots, x[\sigma_k(s)])\,ds\right| \leqq \int_t^\infty a_r(s)\,ds \leqq \frac{r}{2}.$$

b) *The set SX is relatively compact.*
Let any $x \in X$. As in a), it follows that

$$|(Sx)(t)| \leqq |\xi| + \frac{r}{2} \qquad \text{for every } t \geqq T,$$

namely

$$\|Sx\| \leqq |\xi| + \frac{r}{2}.$$

Therefore, the set SX is a uniformly bounded subset of the space E. Moreover, it is equiconvergent at ∞, since

$$|(Sx)(t) - \xi| \leqq \int_t^\infty a_r(s)\,ds \qquad \text{for every } t \geqq T.$$

Also, it is easy to see that for every t_1, t_2 with $T_0 \leqq t_1 \leqq t_2$,

$$|(Sx)(t_1) - (Sx)(t_2)| \leqq \int_{t_1}^{t_2} a_r(s)\,ds,$$

which means that the set SX is equicontinuous. Thus, by the compactness criterion we conclude that SX is relatively compact.

c) *The mapping S is continuous.*

Let $x \in X$ and (x_ν) be an arbitrary sequence in X with $\|\ \|-\lim x_\nu = x$. Then we have

$$\lim x_\nu[\sigma_i(t)] = x[\sigma_i(t)], \qquad t \geqq T \qquad (i = 1, 2, \ldots, k).$$

Thus, by applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_\nu \int_t^\infty f(s; x_\nu[\sigma_1(s)], \ldots, x_\nu[\sigma_k(s)])\,ds = \int_t^\infty f(s; x[\sigma_1(s)], \ldots, x[\sigma_k(s)])\,ds.$$

So, for every $t \geqq T$ we have the pointwise convergenee

$$\lim (Sx_\nu)(t) = (Sx)(t).$$

It remains to prove that

$$\|\ \|-\lim Sx_\nu = Sx.$$

To this end, we consider any subsequence (u_μ) of (Sx_ν). Because of relative compactness of SX, there exists a subsequence (v_λ) of (u_μ) and a function $y \in E$, so that

$$\|\ \|-\lim v_\lambda = y.$$

Since the uniform convergence implies the pointwise one to the same limit function, we must always have

$$y = Sx.$$

We have thus proved that the mapping S satisfies the assumptions of the Schauder theorem and hence there exists a function $x \in X$ with $Sx = x$, namely

$$x(t) = \xi - \int_t^\infty f(s; x[\sigma_1(s)], \ldots, x[\sigma_k(s)])\,\mathrm{d}s, \qquad t \geqq T.$$

Obviously, x is a solution of the terminal value problem (E)–(C).

3. Uniqueness

The uniqueness of solutions of the terminal value problem (E)–(C) is established under a generalized Lipschitz condition on f, but only for differential equations of advanced type.

Theorem 2. *If the equation* (E) *is of advanced type and for every compact subset B of the space $(K^n)^k$ the function f satisfies*

$$(\mathrm{C}_2) \qquad |f(t; x_1, \ldots, x_k) - f(t; y_1, \ldots, y_k)| \leqq L_B(t) \sum_{i=1}^{k} |x_i - y_i|$$

for every $t \geqq t_0$ and $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k)$ in B, where L_B is a real valued function with

$$(\mathrm{C}_3) \qquad \int^\infty L_B(t)\,\mathrm{d}t < \infty,$$

then the terminal value problem (E)–(C) *has at most one solution on an interval of the form $[T, \infty)$.*

Proof. We assume that x and y are solutions of the terminal value problem (E)–(C). Then, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \xi$ and hence for any $\varepsilon > 0$ and some $T_0 \geqq t_0$ we have

$$|x(t)| \leqq |\xi| + \varepsilon \quad \text{and} \quad |y(t)| \leqq |\xi| + \varepsilon \quad \text{for every } t \geqq T_0.$$

Thus, if

$$B = \{(z_1, \ldots, z_k) \in (K^n)^k : |z_i| \leqq |\xi| + \varepsilon\}$$

for $i = 1, 2, \ldots, k$ and every $t \geqq t_0$ we have

$$x(t) \in B,\ y(t) \in B,\ x[\sigma_i(t)] \in B \quad \text{and} \quad y[\sigma_i(t)] \in B.$$

We consider now the function L_B and because of (C_3), we choose a $T \geqq T_0$ so that

$$\int_T^\infty L_B(s)\,\mathrm{d}s \leqq \frac{1}{2k}.$$

Thus, taking also into account condition (C_2), for every $t \geqq T$

$$|x(t) - y(t)| \leqq \int_t^\infty |f(s; x[\sigma_1(s)], \dots, x[\sigma_k(s)] - f(s; y[\sigma_1(s)], \dots, y[\sigma_k(s)]| \, ds \leqq$$

$$\leqq \int_t^\infty L_B(s) \sum_{i=1}^k |x[\sigma_i(s)] - y[\sigma_i(s)]| \, ds \leqq kp(T) \int_t^\infty L_B(s) \, ds \leqq \frac{p(T)}{2},$$

where $p(T) = \sup \{| x(s) - y(s) | : s \in [T, \infty)\}$. Therefore $p(T) \leqq \frac{1}{2} p(T)$, that is $p(T) = 0$ and hence

$$x(t) = y(t) \qquad \text{for every } t \geqq T.$$

Remarks 1. Contrary to the initial value problem, the Lipschitz condition (C_2) is not sufficient for the uniqueness of solutions of the terminal value problem (E)–(C). In fact, the "smallness" condition (C_3) fails for the scalar differential equation

$$x'(t) + x(t^2) = 0, \qquad t \geqq 1$$

and though it is of advanced type, it has two solutions $x_1(t) = \frac{1}{t}$ and $x_2(t) = 0$ both of which satisfy the terminal condition $\lim_{t \to \infty} x(t) = 0$.

2. The uniqueness fails also in the case where the equation is not of advanced type. Counterexamples can be constructed in the cases of retarded or mixed type differential equations. For each of these cases we choose respectively to give the scalar equations

$$x'(t) + \frac{1}{t^{3/2}} x(\sqrt{t}) = 0, \qquad t \geqq 1$$

and

$$x'(t) + \frac{e^{1/2t}}{t^{3/2}} x(\sqrt{t}) \, e^{-|x(2t)|} = 0, \quad t \geqq 1,$$

both of which satisfy conditions (C_2) and (C_3) and have the solutions $x_1(t) = \frac{1}{t}$ and $x_2(t) = 0$ which satisfy the terminal condition $\lim_{t \to \infty} x(t) = 0$.

3. In the case where $\operatorname{Dom} f = (-\infty, t_0] \times (K^n)^k$ it makes sense the (left) terminal problem (E)–(C′), where

(C′) $$\lim_{t \to -\infty} x(t) = \xi.$$

The transformation $t \to 2t_0 - t$ leads to a (right) terminal problem of the form (E)–(C). Thus the existence theorem remains valid for the (left) terminal problem (E)–(C′), while the uniqueness one is valid for the retarded case.

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V. A. Staikos and P. Ch. Tsamatos
Department of Mathematics
University of Ioannina
Ioannina
Greece